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SAGBI Bases of Algebras of Minors

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Introduction

The thesis deals with the problem of computation of SAGBI bases for the coordinate ring of Grassmannians and, more generally, algebras generated by minors of generic matrices.

The theory of Gröbner bases for ideals is well known and it is a central topic in commutative algebra. Gröbner bases were introduced in 1965 by Bruno Buchberger, who also provided an algorithm to compute them, based on the famous Buchberger criterion. We do a brief recap of monomial orders and Gröbner bases theory in Chapter 1.

In the 80s, Robbiano and Sweedler ([13]) presented bases for subalgebras which are the **S**ubalgebra **A**nalog to **G**röbner **B**ases for **I**deals, that is, SAGBI bases. Roughly speaking, a Gröbner basis is a special set of generators of an ideal which makes possible symbolic computations such that, for example, deciding if a polynomial belongs to the ideal. In the same way, a SAGBI basis is a special set of generators of a subalgebra. Although in many respects SAGBI bases theory is similar to the one of Gröbner bases, there is one major difference: unlike ideals, subalgebras of polynomial rings are not necessarily finitely generated. In particular, finite SAGBI bases for finitely generated algebras need not to exist. In Chapter 2, we give an overview of SAGBI bases trying to mirror Gröbner bases theory, with the aim of obtaining an analogue of the Buchberger criterion, namely the SAGBI criterion. Just as the cornerstones of Buchberger criterion are reduction and lifting of syzygies (i.e. the S-polynomials), the cornestones of SAGBI criterion are subduction and lifting of binomial relations. However, while the computation of S-polynomials is not an issue, the identification of the relevant binomial relations needed for SAGBI bases computation is more complicated. Finally, again in analogy with Buchberger criterion, the SAGBI criterion suggests an algorithm for the computation of SAGBI bases.

Consider a $d \times n$, $d \leq n$, matrix of indeterminates $X = (X_{ij})$. By a theorem of Bernstein, Sturmfels and Zelevinsky (see [3], [19]) the maximal minors of X are a universal Gröbner basis, namely a Gröbner basis for any monomial order. In view of this theorem, when moving to subalgebra setting the same question arises: are the maximal minors of X a universal SAGBI basis?

The starting point for answering this question is Chapter 3. There we introduce the algebra generated by maximal minors of X as the homogeneous coordinate ring of the Grassmannian variety. The Grassmannian is the set of all d-dimensional linear subspaces of a n-dimensional vector space over a field K. Through the Plücker embedding, the Grassmannian can be seen as projective variety with defining ideal given by the famous Plücker relations. We are interested in its coordinate ring, the so-called Plücker algebra. This ring is a graded algebra with straightening law. The special feature of such an algebra is that it is free over the ground ring with a basis whose multiplication table is compatible with a partial order on the algebra generators. Specializing this definition to the case of the Plücker algebra, it means that it has a K-basis consisting of products of comparable maximal minors of X, which we call standard bitableaux. We prove that they are linearly independent using the Robinson–Schensted–Knuth correspondence.

The existence of such a basis for the Plücker algebra assures us that the maximal minors are a SAGBI basis with respect to any diagonal monomial order, that is, a monomial order that chooses as initial monomial of a minor the product of the elements on the main diagonal. However, in Chapter 4 we show that there exists a lexicographic monomial order under which the maximal minors of a 3×6 matrix are not a SAGBI basis of the algebra they generate.

By another theorem of Sturmfels ([17]), the t-minors are a Gröbner basis of the ideal they generate under any diagonal monomial order. Despite the fact that algebras generated by lower size minors are much more complicated than Grassmannians and not yet fully explored, we wonder if the same statement applies when switching to SAGBI bases. We observe this in Chapter 4: a Krull dimension argument shows that the t-minors, $1 < t < d$, are not a SAGBI basis under any diagonal monomial order. Nevertheless, for a square matrix of size n, using the SAGBI criterion we prove that there exists a lexicographic monomial order under which the $n-1$-minors are a SAGBI basis.

A good portion of Chapter 4 is dedicated to the case of the 2-minors of a 3×3 matrix of indeterminates. If we consider a diagonal monomial order, the 2-minors are not a SAGBI basis. In order to obtain a SAGBI basis, one needs to add $X_{13} \cdot \det(X)$ and $X_{31} \cdot \det(X)$ to the set of the 2-minors. With an experimental approach, we observed that adding elements of the form $\det(X) \cdot X_{ij}$ to the set of the 2-minors always gives a SAGBI basis of the algebra generated by the 2-minors, regardless of the monomial order. Therefore the main purpose of the chapter is to prove that the 2-minors plus the set $\{X_{ij} \cdot \det(X)\}$ is a universal SAGBI basis of the algebra generated by the 2-minors. To prove this theorem, we use the characterization of feasible leading terms of a polynomial as vertices of its Newton polytope. We compute the vertices of the Newton polytope associated to the product of all 2-minors of X, then we derive the corresponding monomial order and finally we check that the set above is indeeed a SAGBI basis of the algebra generated by the 2-minors. In the end, we discuss some interesting consequences of this result, proving that there exists a single monomial order, up to symmetry, such that the 2-minors of a 3×3 are a SAGBI basis and that there is no monomial order under which the 2-minors of a matrix of size at least 3×4 are a SAGBI basis.

All computations have been implemented on CoCoA 5 or Macaulay2 and can be found in Appendix A.

Chapter 1

Preliminary Results

1.1 Monomial Orders

Let K be a field and consider $R = K[X_1, \ldots, X_n]$ the polynomial ring in n variables. Recall that a *monomial* of R is an element of the form $X^u = \prod_{i=1}^n X^{u_i}$, where $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$. A term is an element of the form $a\mu$, where $a \in K^*$ and μ is a monomial of R. We call $\mathrm{Mon}(R)$ the set of all the monomials of R. Note that $\mathrm{Mon}(R)$ is actually a K-basis of R: for every $f \in R$ there exists a unique finite subset $\mathrm{supp}(f)$ of $\mathrm{Mon}(R)$ such that

$$f = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu, \qquad a_\mu \in K,\, a_\mu \neq 0 \text{ for } \mu \in \mathrm{supp}(f).$$

The only thing that could be not unique in this representation is the order in which we write the terms. If we impose a total order on the set $\mathrm{Mon}(R)$, the representation above is uniquely determined if we require that the monomials are written in order, from the largest to the smallest.

Definition 1.1.1. A *monomial order* is a total order $\leq$ on the set $\mathrm{Mon}(R)$ that satisfies the following conditions:

1. $1 \leq \mu$ for every $\mu \in \mathrm{Mon}(R)$;
2. If $\mu_1, \mu_2, \mu_3 \in \mathrm{Mon}(R)$ and $\mu_1 \leq \mu_2$, then $\mu_1\mu_3 \leq \mu_2\mu_3$.

Note that, given condition 2., condition 1. of the previous definition is equivalent to require compatibility with division: if μ_1 divides μ_2, then $\mu_1 \leq \mu_2$.

Set now $\mu_1 = X_1^{u_1} \cdots X_n^{u_n}$ and $\mu_2 = X_1^{v_1} \cdots X_n^{v_n}$. We recall here the most important monomial orders:

(a) the *lexicographic order* (Lex): $\mu_1 <_{\mathrm{Lex}} \mu_2$ if $u_i < v_i$ for some i and $u_j = v_j$ for all $j < i$;

(b) the *degree lexicographic order* (DegLex): $\mu_1 <_{\text{DegLex}} \mu_2$ if $\deg(\mu_1) < \deg(\mu_2)$ or $\deg(\mu_1) = \deg(\mu_2)$ and $\mu_1 <_{\text{Lex}} \mu_2$.

(c) the *degree reverse lexicographic order* (DegRevLex): $\mu_1 <_{\text{DegRevLex}} \mu_2$ if $\deg(\mu_1) < \deg(\mu_2)$ or $\deg(\mu_1) = \deg(\mu_2)$ and $u_i > v_i$ for some i and $u_j = v_j$ for all $j > i$;

These monomial orders all satisfy $X_1 > \cdots > X_n$. One can choose a different total order of the indeterminates and consider the Lex, DegLex and DegRevLex orders induced by that order by changing suitably the definitions above.

An important observation arising from Definition 1.1.1:

Lemma 1.1.2. *Given a monomial order on R, there are no infinite descending chains in* $\text{Mon}(R)$.

Proof. Let us suppose $\mu_1 > \mu_2 > \ldots$ is such a chain. Then $\mu_i \notin (\mu_1, \ldots, \mu_{i-1})$, otherwise $m_j | m_i$ for some $j < i$ and this implies $m_j \leq m_i < m_j$, which is absurd. Therefore one has $(\mu_1, \ldots, \mu_{i-1}) \subsetneq (\mu_1, \ldots, \mu_i)$ for every i, that is a contradiction by Noetherianity of R. $\square$

From now on, we fix a monomial order $<$ on the monomials of R. Thus every polynomial $f \neq 0$ has a unique representation

$$f = a_1\mu_1 + \ldots a_k\mu_k,$$

where $a_i \in K^*$ and $\mu_1 > \cdots > \mu_k$.

Definition 1.1.3. Let $f \in R$, $f \neq 0$. The *initial monomial* of f with respect to $<$ is denoted by $\text{in}(f)$ and is, by definition, μ_1. The *initial term* is $\text{init}(f) = a_1\mu_1$ and the initial coefficient is $\text{inic}(f) = a_1$.

The compatibilty of monomial orders with multiplication, namely condition 2. of Definition 1.1.1, has as an immediate consequence that, for every nonzero polynomials $f, g \in R$,

$$\text{in}(fg) = \text{in}(f)\,\text{in}(g). \tag{1.1}$$

With respect to the sum of polynomials $f, g \in R$, assuming f, g and $f + g \neq 0$, one has

$$\text{in}(f + g) \leq \max\{\text{in}(f), \text{in}(g)\}.$$

1.2 Gröbner Bases

Now consider a K-vector subspace V of R, $V \neq 0$. Suppose that R is endowed with a monomial order and consider the following K-subspace of R

$$\text{in}(V) = \langle \text{in}(f) \mid f \in V, f \neq 0 \rangle.$$

Clearly, the set $\{\text{in}(f) \mid f \in V, f \neq 0\}$ is a K-basis of $\text{in}(V)$ since every set of monomials is linearly independent over K, and therefore it is a basis of the subspace it generates. Note that if V is an ideal of R or a K-subalgebra of R, the K-vector space $\text{in}(V)$ is respectively an ideal or a K-subalgebra as well thanks to 1.1. Therefore we can give the following:

Definition 1.2.1. Let I be an ideal of R. A set of nonzero polynomials $f_1, \ldots, f_m \in I$ is a *Gröbner basis* of I if the monomials $\text{in}(f_1), \ldots, \text{in}(f_m)$ generate $\text{in}(I)$ as an ideal.

Since R is Noetherian, $\text{in}(I)$ is a finitely generated ideal and hence every ideal I of R has a Gröbner basis.

For a detailed discussion regarding Gröbner bases we refer to [5], Section 1.2. Here, we just remind the most important concepts and algorithms.

Definition 1.2.2. Let $f_1, \ldots, f_m \in R$. A polynomial r is a *reduction of* $g \in R$ *modulo* $f_1, \ldots, f_m$ if there exist $q_1, \ldots, q_m \in R$ satisfying the following conditions:

(a) $g = q_1 f_1 + \cdots + q_m f_m + r$;

(b) $\text{in}(q_i f_i) \leq \text{in}(g)$ for all $i = 1, \ldots, m$;

(c) no monomial $\mu \in \text{supp}(r)$ is divisible by any $\text{in}(f_i)$, $i = 1, \ldots, m$.

The reduction plays the role of the Euclidean division when working with more than one variable. The idea, also suggested by the letters used, is that the q_i are the "quotients" of g modulo f_i and r is the "remainder". Thanks to this definition we have the first criterion to decide whether a set of polynomials in I is a Gröbner basis of I:

Proposition 1.2.3. *Let $f_1, \ldots, f_m \in R$, $I = (f_1, \ldots, f_m)$ and $J = (\text{in}(f_1), \ldots, \text{in}(f_m))$. Then the following are equivalent:*

(a) $f_1, \ldots, f_m$ form a Gröbner basis of I;

(b) every $g \in I$ reduces to 0 modulo $f_1, \ldots, f_m$;

(c) the monomials μ, $\mu \notin J$, are linearly independent in the K-vector space R/I.

An immediate but essential consequence of (a) $\Longrightarrow$ (b) of this proposition is that a Gröbner basis of I generates I.

Consider again $f_1, \ldots, f_m \in R$. Let $I = (f_1, \ldots, f_m)$, $J = (\text{in}(f_1), \ldots, \text{in}(f_m))$ and $F = R^m$. By the universal property of free R-modules, we have R-linear maps

$$\begin{aligned} \phi : F &\longrightarrow R & \qquad \psi : F &\longrightarrow R \\ e_i &\longmapsto f_i, & e_i &\longmapsto \text{init}(f_i). \end{aligned}$$

An element $s = (s_1, \ldots, s_m) \in \ker(\phi)$ is called a *syzygy* of $f_1, \ldots, f_m$. Clearly, $\operatorname{im}(\phi) = I$ and $\operatorname{im}(\psi) = J$. For $x = (x_1, \ldots, x_m) \in F$, we set

$$\operatorname{in}_\phi(x) = \max_i \operatorname{in}(x_i f_i).$$

Then we can assign $x \in F$ an initial term $\operatorname{init}_\phi(x) \in F$:

$$\operatorname{init}_\phi(x) = \begin{cases} \operatorname{init}(x_i) \text{ if } \operatorname{in}(x_i f_i) = \operatorname{in}_\phi(x) \\ 0 \qquad\quad \text{otherwise} \end{cases}.$$

Note that $s \in \ker(\phi)$ implies $\operatorname{init}_\phi(s) \in \ker(\psi)$, as one easily sees by reading the equation $s_1 f_1 + \cdots + s_m f_m = 0$ monomial by monomial. We instead say that $s \in \ker(\phi)$ *lifts* $t \in \ker(\psi)$ if $\operatorname{init}_\phi(s) = \operatorname{init}_\phi(t)$.

Let us take a closer look at $\ker(\psi)$. If we consider a pair of monomials of R, μ and ν, they have a well defined least common multiple. Clearly one has

$$\frac{\operatorname{lcm}(\mu,\nu)}{\nu}\nu - \frac{\operatorname{lcm}(\mu,\nu)}{\mu}\mu = 0,$$

and thus $\left(\frac{\operatorname{lcm}(\mu,\nu)}{\nu}, -\frac{\operatorname{lcm}(\mu,\nu)}{\mu}\right)$ is a syzygy of ν, μ, which we call the *divided Koszul syzygy*. For $f_1, \ldots, f_m$ we define $\kappa_{ij} \in F$ by

$$(\kappa_{ij})_k = \begin{cases} \frac{\operatorname{lcm}(\operatorname{in}(f_i),\operatorname{in}(f_j))}{\operatorname{init}(f_i)}, & \text{if } k = i \\ -\frac{\operatorname{lcm}(\operatorname{in}(f_i),\operatorname{in}(f_j))}{\operatorname{init}(f_j)}, & \text{if } k = j \\ 0 & \text{else} \end{cases},$$

for $k = 1, \ldots, m$. The elements κ_{ij} of F are the *divided Koszul syzygies* of $\operatorname{init}(f_1), \ldots, \operatorname{init}(f_m)$. It is not too difficult to see that the divided Koszul syzygies generate $\ker(\psi)$ as an R-module.

Theorem 1.2.4 (Buchberger Criterion)**.** *Let $f_1, \ldots, f_m \in R$ and $I = (f_1, \ldots, f_m)$. Then the following are equivalent:*

(a) $f_1, \ldots, f_m$ are a Gröbner basis of I;

(b) all κ_{ij}, $1 \le i < j \le m$, can be lifted to syzygies of $f_1, \ldots, f_m$;

(c) the S-polynomials

$$S_{ij} = \frac{\operatorname{lcm}(\operatorname{in}(f_i), \operatorname{in}(f_j))}{\operatorname{init}(f_i)} f_i - \frac{\operatorname{lcm}(\operatorname{in}(f_i), \operatorname{in}(f_j))}{\operatorname{init}(f_j)} f_j,$$

$1 \le i < j \le m$, reduce to 0 modulo $f_1, \ldots, f_m$.

The polynomial S_{ij} in the theorem is called the *S-polynomial* of f_i and f_j. The theorem suggests an algorithm, the *Buchberger algorithm*, in order to compute a Gröbner basis of an ideal $I = (f_1, \ldots, f_m)$:

1. Set $G = \{f_1, \ldots, f_m\}$ and $G' = \emptyset$.
2. For all i, j, $i < j$, apply the reduction algorithm modulo G to S_{ij}. If S_{ij} reduces to $h \neq 0$, replace G' by $G' \cup \{h\}$.
3. If $G' = \emptyset$, then G is the desired Gröbner basis.
4. If $G' \neq \emptyset$, replace G by $G \cup G'$ and go to 2.

This algorithm clearly terminates after finitely many steps since the ideal $(\mathrm{in}(g), g \in G)$ strictly increases if $G' \neq \emptyset$, and in R, a Noetherian ring, any strictly ascending chain of ideals is finite.

1.3 Initial Subspaces

At the start of Section 1.2, given a K-vector subspace of R, we defined the K-vector subspace $\mathrm{in}(V)$ as the subspace spanned by the initial monomials of all $f \in V, f \neq 0$. We also observed that the set $\{\mathrm{in}(f) \mid f \in V, f \neq 0\}$ is a K-basis of $\mathrm{in}(V)$. For simplicity, we give a name to the set above, namely $\mathrm{In}(V)$.

Proposition 1.3.1. *Let V be a K-vector subspace of R. Then:*

(a) $\mathrm{Mon}(R) \setminus \mathrm{In}(V)$ *is a K-basis of R/V;*

(b) for every $\mu \in \mathrm{In}(V)$ there exists a unique $f_\mu \in V$ satisfying the following conditions:

$$\text{(i) } \mathrm{in}(f_\mu) = \mu, \ \text{(ii) } \mathrm{inic}(f_\mu) = 1, \ \text{(iii) } \mathrm{supp}(f_\mu) \cap \mathrm{In}(V) = \{\mu\}.$$

(c) The set $\{f_\mu \mid \mu \in \mathrm{In}(V)\}$ is a K-basis of V;

(d) If V has finite dimension, then $\dim(V) = \dim(\mathrm{in}(V))$*;*

(e) if $\leq$ and $\preceq$ are monomial orders and $\mathrm{in}_\leq(V) \subseteq \mathrm{in}_\preceq(V)$*, then* $\mathrm{in}_\leq(V) = \mathrm{in}_\preceq(V)$.

Proof. (a). That $\mathrm{Mon}(R) \setminus \mathrm{In}(V)$ generates R/V follows from the same induction applied both in reduction and subduction algorithms (see Proposition 2.1.5): the residue class of $f \in R$ modulo V does not change if we subtract an element $g \in V$ from f. Therefore we can replace the largest monomial in $\mathrm{supp}(f) \cap \mathrm{In}(V)$ by smaller monomials. Since descending chains of monomials terminate, we get a representative of f modulo V that is supported in $\mathrm{Mon}(R) \setminus \mathrm{In}(V)$. The linear independence of $\mathrm{Mon}(R) \setminus \mathrm{In}(V)$ modulo V is immediate from the definition of $\mathrm{In}(V)$.

(b) Let $\mu \in \mathrm{In}(V)$, and let g be a reduction of μ modulo V as in (a). Then $\mathrm{supp}(g) \cap \mathrm{In}(V) = \emptyset$ by (a). If we set $f_\mu = \mu - g$, then $\mathrm{in}(f_\mu) = \mu$, $\mathrm{inic}(f_\mu) = 1$ and moreover $\mathrm{supp}(f_\mu) \cap \mathrm{In}(V) = \{\mu\}$. The unicity follows from the unicity of g, that is an immediate consequence of (a).

(c) Consider $f \in V$. After finitely many reductions as in (a), we must terminate at 0. Moreover in the reduction it is sufficient to use the f_μ and their K-multiples. Therefore the f_μ generate V, and they are linearly independent since they have different initial monomials.

(d) and (e) both follow immediately from (c). □

As we will see, this proposition will play a key role in Section 2.4.

Chapter 2

SAGBI Bases

2.1 Definition and First Properties

Let K be a field and $R = K[X_1, ..., X_n]$ the polynomial ring in n variables, endowed with a monomial order.

Now, let A be a K-subalgebra of R. Since in this Chapter we will always speak about K-algebras and K-subalgebras, we will sometime refer to them as algebras and subalgebras, omitting the prefix.

We want to consider the *initial algebra* $\mathrm{in}(A)$, that is, the subalgebra of R generated by the initial monomials of the nonzero polynomials $f \in A$. The theory is, in many ways, similar to that of Gröbner bases and initial ideals. However, while ideals in polynomial rings must be finitely generated, this is not true for subalgebras, and this fact makes a big difference. Even if A is a finitely generated subalgebra of R, it is not certain that $\mathrm{in}(A)$ is as well:

Example 2.1.1. Let $A \subseteq K[X, Y]$ be the subalgebra of R generated by $X + Y, XY, XY^2$. Our claim is to show that that $\mathrm{in}(A)$ is not finitely generated, regardless of the monomial order. Let us assume that $X > Y$ and observe that, since $X^2Y = (X + Y)XY - XY^2 \in A$, the algebra A is invariant under the exchange of X and Y and by symmetry we are done also in the case $Y > X$. Of course, taking the initial monomials of the generators, we have that X, XY, $XY^2 \in \mathrm{in}(A)$. Now, since we can obtain XY^3 as $XY^2(X + Y) - (XY)^2 \in A$, we deduce that $XY^3 \in \mathrm{in}(A)$. Similarly, we can see that $XY^k \in \mathrm{in}(A)$ for all $k \in \mathbb{N}$: in fact $XY^k = XY^{k-1}(X + Y) - (XY)(XY^{k-2}) \in A$ for all $k > 2$.

Now let B be the subalgebra of $K[X, Y]$ generated by the monomials $XY^k, k \in \mathbb{N}$. Clearly, $B \subseteq \mathrm{in}(A)$ and B is not a finitely generated subalgebra. What we are going to show is that $B = \mathrm{in}(A)$. Suppose that $B \subsetneq \mathrm{in}(A)$. Then, since $\mathrm{in}(A)$ is a strict monomial overalgebra of $B = K[X, XY, XY^2, XY^3, \ldots]$ contained in A, $\mathrm{in}(A)$ must contain Y^k for a certain $k > 0$. But, being Y^k the smallest monomial of degree k (we supposed $X > Y$), it would have to be in A in order to appear as an initial monomial (A is a graded subalgebra: it contains every homogeneous component of its elements). But Y^k is not in A for any $k > 0$, and so we deduce that $B = \text{in}(A)$. In fact, suppose $Y^k \in A$. Then we can write $Y^k = h(X+Y, XY, XY^2)$, as h ranges over polynomials in three variables. Setting X to 0, we obtain $h(Y, 0, 0) = Y^k$, and setting Y to 0 gives $h(X, 0, 0) = 0$. But since $h(Y, 0, 0) = Y^k$, it must be $h(X, 0, 0) = X^k$, and so we get $X^k = 0$, that is a contradiction.

While, as we have just seen, subalgebras of R (in particular initial algebras) are not necessarily finitely generated, they are always countably generated since R has a countable number of monomials. Therefore, as generators, it is enough to consider families $\mathcal{F} = \{f_i, i \in N\}$ of polynomials $f_i \in R$, where N is either the set $\{1, \dots, n\}$ for some $n \in \mathbb{N}$ or the set of all positive natural numbers.

Definition 2.1.2. A family $\mathcal{F} = \{f_i, i \in N\}$ of elements of A is called a *SAGBI basis* of A if the monomials $\text{in}(f_i), i \in N$, generate $\text{in}(A)$ as a subalgebra of R.

The acronym "*SAGBI*" stands for "Subalgebra analog to Gröbner bases of ideals". Therefore, we want to try to continue with similar arguments to that of Gröbner bases, starting with *reduction*. In order to do that, we need to replace R-linear combinations with polynomial expressions of the polynomials $f_i \in \mathcal{F}$.

Definition 2.1.3. Let $e = \{e_i, i \in N\}$ be an ordered family of natural numbers such that $e_i = 0$ for all but finitely many i. A *monomial in* $\mathcal{F}$ is an element of the form

$$\mathcal{F}^e = \prod_{i \in N} f_i^{e_i}.$$

Note that a monomial in $\mathcal{F}$ in general is not a monomial in R and that $\text{in}(\mathcal{F}^e) = (\text{in}(\mathcal{F}))^e$, where $\text{in}(\mathcal{F}) = \{\text{in}(f_i)\}_{i \in N}$ is the family of initial monomials of the elements of $\mathcal{F}$.

We can now define the analogue of reduction, called *subduction*.

Definition 2.1.4. Let $g \in R$. We say that $r \in R$ is a *subduction of g modulo $\mathcal{F}$* if there exist monomials $\mathcal{F}^{e_1}, \dots, \mathcal{F}^{e_m}$ and coefficients $a_i \in K$, such that:

(a) $g = a_1\mathcal{F}^{e_1} + \cdots + a_m\mathcal{F}^{e_m} + r$;

(b) $\text{in}(\mathcal{F}^{e_i}) \leq \text{in}(g)$ for all $i = 1, \dots, m$;

(c) no monomial $\mu \in \text{supp}(r)$ is of type $\text{in}(\mathcal{F}^e)$.

As it happens for reduction, that is, as we saw in the previous chapter, division with reminder in Gröbner bases theory, we have the following:

Proposition 2.1.5. *Let $\mathcal{F}$ be a family of polynomials in R and $g \in R$. Then g has a subduction modulo $\mathcal{F}$.*

Proof. Let us start with $r = g$ and suppose that there is a monomial μ in $\mathrm{supp}(g)$ such that $\mu = \mathrm{in}(\mathcal{F}^e)$ for some exponent vector e. We replace g with $g - a\mathcal{F}^e$, where a is chosen such that the term μ cancels. Iterating this operation, that is compatible with condition (2) of the definition, we are only introducing new monomials $\nu < \mu$. Since we cannot have infinite descending chains of monomials by definition of monomial order, we conclude.

□

We can now give a first characterization of SAGBI bases, once again mirroring Gröbner bases:

Proposition 2.1.6. *Let $\mathcal{F} = \{f_i, i \in N\}$ be a family of polynomials in A, $B = K[\mathrm{in}(\mathcal{F})]$ be the subalgebra of R generated by the initial monomials $\mathrm{in}(f_i), i \in N$. Then the following are equivalent:*

(a) $\mathcal{F}$ is a SAGBI basis of A;

(b) every $f \in A$ subduces to 0 modulo $\mathcal{F}$;

(c) the monomials $\mu \notin B$ are linearly independent in the K-vector space R/A.

If the equivalent conditions (a), (b), (c) hold, then:

(d) Every element of R has a unique subduction modulo $\mathcal{F}$;

(e) Moreover, the subduction depends only on A and the monomial order.

Proof. (a) $\Rightarrow$ (c). Suppose that (c) does not hold. Then there exists a polynomial $r \in A$ such that r is a linear combination of monomials $\mu \notin B$. But $r \in A$ and $\mathcal{F}$ is a SAGBI basis of A, then $\mathrm{in}(r)$ must belong to the subalgebra generated by $\mathrm{in}(\mathcal{F})$, which is B, and we get a contradiction.

(c) $\Rightarrow$ (b). Let r be a subduction of $f \in A$ modulo $\mathcal{F}$. We have:

$$f = a_1\mathcal{F}^{e_1} + \cdots + a_m\mathcal{F}^{e_m} + r.$$

Since f belongs to A and $\mathcal{F}$ is a family of elements of A, we deduce that also r belongs to A. But, by definition of subduction, no monomial of $\mathrm{supp}(r)$ is of type $\mathrm{in}(\mathcal{F})^e$ and so r is a linear combination of monomials $\mu \notin B$ that is 0 modulo A (since $r \in A$). Finally, by (c) we deduce that $r = 0$.

(b) $\Rightarrow$ (a). Let $f \in A$, $f \neq 0$. By (b), we have the following:

$$f = a_1\mathcal{F}^{e_1} + \cdots + a_m\mathcal{F}^{e_m},$$

with $\mathrm{in}(\mathcal{F}^{e_i}) \leq \mathrm{in}(f)$ for all $i = 1, \dots, m$. The monomial $\mathrm{in}(f)$ must appear on the right side of the equation, and so it must be $\mathrm{in}(f) = \mathrm{in}(\mathcal{F}^{e_i}) = (\mathrm{in}(\mathcal{F}))^{e_i}$ for at least one i. It follows that $\mathrm{in}(f)$ belongs to the subalgebra generated by $\mathrm{in}(\mathcal{F})$, and so we conclude by arbitrariness of f that $\mathcal{F}$ is a SAGBI basis of A.

(c) $\Rightarrow$ (d). Let $g \in R$ and let r_1 and r_2 both be subductions of g modulo $\mathcal{F}$.We have:

$$g = a_1\mathcal{F}^{e_1} + \cdots + a_m\mathcal{F}^{e_m} + r_1;$$
$$g = b_1\mathcal{F}^{h_1} + \cdots + b_s\mathcal{F}^{h_s} + r_2.$$

Then, $r_1 - r_2$ belongs to A and it is a linear combination of monomials that are not in B (again by definition of subduction). By (c), we deduce that $r_1 - r_2 = 0$ and so $r_1 = r_2$.
(e). In the proof of uniqueness we used only B, that is $K[\text{in}(\mathcal{F})]$, and $\mathcal{F}$ is a SAGBI basis of A. It follows that the subduction olny depends on A and the monomial order.

□

An immediate consequence of (a) $\Rightarrow$ (b) of Proposition 2.1.6 is the following:

Corollary 2.1.7. *A SAGBI basis of the K-subalgebra $A \subset R$ generates A as a K-algebra.*

2.2 The Lifting of Binomial Relations

Now, going on with the analogy with Gröbner bases theory, we want to discuss SAGBI analogs of Buchberger algorithm and lifting of syzygies.

Let A be the K-subalgebra of R generated by the family of polynomials $\mathcal{F} = \{f_i, i \in N\}$, and let us assume that the polynomials f_i are non zero and monic. Let us choose

$$P = K[Y_i, i \in N],$$

and consider the surjective algebra homomorphisms

$$\begin{aligned} \phi : P &\longrightarrow A & \qquad \psi : P &\longrightarrow K[\text{in}(\mathcal{F})] \\ Y_i &\longmapsto f_i, & Y_i &\longmapsto \text{in}(f_i). \end{aligned}$$

We want to pull back the monomial structure and order from R to P. A monomial in P is given by an exponent vector $e = (e_u)$, $u \in N$ of natural numbers e_u of which all but finitely many are 0. We set $Y^e = \prod_{u \in N} {Y_u}^{e_u}$.

Given two monomials $\zeta, \eta \in P$, the most natural thing to do would be setting $\zeta < \eta \iff \psi(\zeta) < \psi(\eta)$. However, this relation is not defined everywhere since we may have $\psi(\zeta) = \psi(\eta)$. Therefore, we will use ϕ to define a replacement of the initial monomial and term. Let $F \in P$, $F \neq 0$, $F = \sum_i a_i Y^{e_i}$ with $a_i \in K$ and $a_i \neq 0$ for all i. We set

$$\begin{aligned}\mathrm{in}_\phi(F) &= \max_i \mathrm{in}(\phi(Y^{e_i})),\\ \mathrm{init}_\phi(F) &= \sum_{\mathrm{in}(\phi(Y^{e_i}))=\mathrm{in}_\phi(F)} a_i Y^{e_i}.\end{aligned}$$

Note that $\mathrm{in}_\phi(F)$ is a monomial in R, but $\mathrm{init}_\phi(F)$ is a polynomial in P. Moreover, in general $\mathrm{in}_\phi(F) \neq \mathrm{in}(\phi(F))$: we could have $\mathrm{in}(\phi(F)) < \mathrm{in}_\phi(F)$ since there may be cancellations among the leading monomials of $\phi(a_i Y^{e_i})$, namely $\phi(\mathrm{init}_\phi(F))$ can be 0.

Let now $F \in \ker(\phi)$. Then $\mathrm{init}_\phi(F) \in \ker(\psi)$ and we can give the following

Definition 2.2.1. With the notation above, let $F \in \ker(\phi)$ and $G \in \ker(\psi)$. We say that F *lifts* G if $\mathrm{init}_\phi(F) = \mathrm{init}_\phi(G)$.

Example 2.2.2. Consider the matrix

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{pmatrix}$$

with entries in the polynomial ring $R = K[X_{ij} : i = 1, 2, j = 1, \dots, 4]$. We want to consider the subalgebra A generated by the maximal minors of X, that is, the six determinants

$$[i\ j] := \det \begin{pmatrix} X_{1i} & X_{1j} \\ X_{2i} & X_{2j} \end{pmatrix}, \quad i < j.$$

Let us choose the lexicographic monomial order such that $X_{uv} < X_{pq}$ if $u < p$ or $u = p$ and $v < q$. This is a *diagonal monomial order*: the initial monomial of the minor $[i\ j]$ is the product of the elements of the diagonal of the submatrix of X obtained by taking just the i-th and j-th columns, that is, in our case, $X_{1i}X_{2j}$. Let $P = K[Y_{ij} : 1 \le i < j \le 4]$, and consider

$$\begin{aligned}\phi : P &\longrightarrow A\\ Y_{ij} &\longmapsto [i\ j].\end{aligned}$$

The maximal minors satisfy the *Plücker Relation* (see Section 3.2):

$$[1\ 2][3\ 4] - [1\ 3][2\ 4] + [1\ 4][2\ 3] = 0,$$

so the polynomial $F = Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23} \in P$ is actually in $\ker(\phi)$. Then,

$$\begin{aligned}\mathrm{in}_\phi(F) &= \max_i \mathrm{in}(\phi(Y^{e_i})) = \max\{\mathrm{in}(\phi(Y_{12}Y_{34})), \mathrm{in}(\phi(Y_{13}Y_{24})), \mathrm{in}(\phi(Y_{14}Y_{23}))\}\\ &= \max\{\mathrm{in}([1\ 2][3\ 4]), \mathrm{in}([1\ 3][2\ 4]), \mathrm{in}([1\ 4][2\ 3])\}\\ &= \max\{X_{11}X_{22}X_{13}X_{24},\ X_{11}X_{23}X_{12}X_{24},\ X_{11}X_{24}X_{12}X_{23}\}\\ &= X_{11}X_{23}X_{12}X_{24}.\end{aligned}$$

We deduce that

$$\mathrm{init}_\phi(F) = -Y_{13}Y_{24} + Y_{14}Y_{23},$$

and so $G := -Y_{13}Y_{24} + Y_{14}Y_{23} \in \ker(\psi)$ is lifted by F.

Now, we can give a new characterization of SAGBI bases using the *lifting* definition we just introduced. Recall that $\mathcal{F}$ is a (countable) family of polynomials generating $A \subset R = K[X_1, ..., X_n]$ as a K-subalgebra.

Proposition 2.2.3. *The following are equivalent:*

(a) $\mathcal{F}$ is a SAGBI basis of A;

(b) for every $f \in A$, $f \neq 0$, there exists $F \in P$ such that $f = \phi(F)$ and $\mathrm{in}_\phi(F) = \mathrm{in}(f)$;

(c) every $G \in \ker(\psi)$ can be lifted to a polynomial $F \in \ker(\phi)$.

Proof. (a) $\Longleftrightarrow$ (b) is just the equivalence of Proposition 2.1.6 (a) and (b) in terms of the new notation we introduced. In fact, $f \in A$ subduces to 0 modulo $\mathcal{F}$ if and only if $f = \phi(F)$ for some $F \in P$ (since it has to be a combination of the elements of $\mathcal{F}$) and $\mathrm{in}_\phi(F) = \mathrm{in}(f)$ (so that there are not cancellations among the top monomials of $\phi(F)$).

(b) $\Rightarrow$ (c). Let $G \in \ker(\psi)$ and set $g = \phi(G) \in A$. By hypothesis, there exists $F \in P$ such that $g = \phi(F)$ and $\mathrm{in}_\phi(F) = \mathrm{in}(g)$. Since $G \in \ker(\psi)$, $\mathrm{in}_\phi(G) > \mathrm{in}(\phi(G)) = \mathrm{in}(g) = \mathrm{in}_\phi(F)$, and so $\mathrm{init}_\phi(G - F) = \mathrm{init}_\phi(G)$. Therefore $F - G \in \ker(\phi)$ lifts G.

(c) $\Rightarrow$ (b). Let $f \in A$. Therefore we can write $f = a_1\mathcal{F}^{e_1} + \cdots + a_m\mathcal{F}^{e_m}$, $a_i \in K$. Let F be the inverse image of f via ϕ. If $\mathrm{in}_\phi(F) > \mathrm{in}(f)$, then $G = \mathrm{init}_\phi(F) \in \ker(\psi)$, as we can see by evaluating the equation $f = a_1\mathcal{F}^{e_1} + \cdots + a_m\mathcal{F}^{e_m}$ in degree $\mathrm{in}_\phi(F)$. But, by hypothesis, G can be lifted to a polynomial $H \in \ker(\phi)$, and so we can write $f = \phi(F - H)$, with $\mathrm{in}_\phi(F - H) < \mathrm{in}_\phi(F)$. If $\mathrm{in}_\phi(F - H) = \mathrm{in}(f)$ we get the thesis, if not we iterate this process that will end after finitely many steps by definition of monomial order. □

In order to actually use Proposition 2.2.3, we need to understand how $\ker(\psi)$ looks like.

Before that, let us remark that the polynomial ring $R = K[X_1, \dots, X_n]$ is $\mathbb{Z}^n$-multigraded: taken a monomial in R, its multidegree will be its exponent vector. We want to pull back this multigrading to P via ψ: $\deg \zeta = \deg(\psi(\zeta))$ for every $\zeta \in \mathrm{Mon}(P)$. Clearly, $F \in P$ is ψ-multihomogeneous if and only if $F = \mathrm{init}_\phi(F)$. In particular, from this it follows:

1. Let $F \in \ker(\psi)$ be ψ-multihomogeneous and $\zeta \in \mathrm{Mon}(P)$. If F is lifted by G, then ζF is lifted by ζG;

2. Let F_1, $F_2 \in \ker(\psi)$ be ψ-multihomogeneous and lifted respectively by G_1, G_2. If $F_1 + F_2 \neq 0$, then $F_1 + F_2$ is lifted by $G_1 + G_2$.

We are now ready to prove the following

Proposition 2.2.4.

(a) The kernel of ψ is generated by binomials;

(b) Let $\mathcal{B}$ be a binomial system of generators of $\ker(\psi)$. Then the following are equivalent:

(i) every $\beta \in \mathcal{B}$ can be lifted to a polynomial $F_\beta \in \ker(\phi)$;

(ii) every $G \in \ker(\psi)$ can be lifted to a polynomial $F \in \ker(\phi)$.

(c) Under the equivalent conditions in (b), $\ker(\phi)$ is generated by the polynomials $F_\beta, \beta \in \mathcal{B}$.

Proof. (a). Let $M := \{\psi(\zeta) : \zeta \in \mathrm{Mon}(P)\}$. Then the K-algebra $B := \mathrm{im}(\psi)$ has M as a K-vector space basis. Taken $\mu \in M$, let us set $C_\mu := \{\zeta \in \mathrm{Mon}(P) : \psi(\zeta) = \mu\}$. As μ varies, the sets C_μ form a partition of $\mathrm{Mon}(P)$. Fixed μ, we choose a representative $\zeta_\mu \in C_\mu$. Finally, we set

$$\mathcal{B} := \bigcup_{\mu \in M} \{\eta - \zeta_\mu : \eta \in C_\mu\},$$

and we want to show that $\mathcal{B}$ generates $\ker(\psi)$.

Let $G \in \ker(\psi), G = \sum a_\eta \eta$ with $\eta \in \mathrm{Mon}(P)$. Let $\mu = \psi(\eta)$. Set $H = \sum a_\eta \zeta_\mu$. Then, $H = G - \sum a_\eta(\eta - \zeta_{\psi(\eta)})$. But, as μ varies, ζ_μ are linearly independent, since they are in bijection with a K-basis of $\mathrm{im}(\psi)$. Therefore H must be 0, and so G is a linear combination of the binomials in $\mathcal{B}$.

(b). (ii) $\Rightarrow$ (i) is trivial. For (i) $\Rightarrow$ (ii) let $G \in \ker(\psi)$, $G \neq 0$. Then also $\mathrm{init}_\phi(G) \in \ker(\psi)$. So (for proving the existence of a lifting) we can replace G by $\mathrm{init}_\phi(G)$, and therefore assume that G is ψ-multihomogeneous. Observe that binomials in $\ker(\psi)$ are also ψ-multihomogeneous.

We have $G = \sum_i a_i \zeta_i \beta_i$, with $a_i \in K$, $\zeta_i \in \mathrm{Mon}(P)$ and $\beta_i \in \mathcal{B}$. By multihomogeneity of G, we can assume that, for all i, $\zeta_i\beta_i$ has the same ψ-multidegree as G. Since all β_i are are liftable by hypothesis, it follows from (1) and (2) G is liftable as well.

(c). Let $F \in \ker(\phi)$. We already observed that $G := \mathrm{init}_\phi(F) \in \ker(\psi)$ and that G is ψ-multihomogeneous. From (b), we know that G can be lifted to $H \in \ker(\phi)$ that is a combination of the polynomials F_β. If $F = H$ we conclude, if not we consider $F - H \in \ker(\phi)$ and repeat. Since we cannot have infinite descending chains of monomials in R, this process will stop after finitely many times and we get F as a combination of the F_β. □

2.3 The SAGBI Criterion and Algorithm

We can now combine Proposition 2.2.3 and Proposition 2.2.4 to enunciate the analog of the Buchberger criterion, which we'll call the *SAGBI Criterion*:

Theorem 2.3.1. *Let A be the K-subalgebra of the polynomial ring R generated by the family $\mathcal{F}$, and let $\mathcal{B}$ be a binomial system of generators of* $\ker(\psi)$*. Then the following are equivalent:*

(a) $\mathcal{F}$ is a SAGBI basis of A;

(b) every element of $\mathcal{B}$ can be lifted to an element of $\ker(\phi)$*;*

(c) the polynomials $\phi(\beta)$, with $\beta \in \mathcal{B}$, subduce to 0 modulo $\mathcal{F}$.

Proof. The equivalence (a) $\iff$ (b) follows from (i) $\iff$ (ii) in Proposition 2.2.4 and then from (c) $\iff$ (a) in Proposition 2.2.3.

Since $\phi(\beta) \in A$, (a) $\Rightarrow$ (c) is covered by Proposition 2.1.6.

Finally, for (c) $\Rightarrow$ (b), we can reformulate the subduction of $\phi(\beta)$ as we did in (a) $\iff$ (b) in Proposition 2.2.3 and conclude as in (b) $\Rightarrow$ (c) in the same Proposition. □

An immediate consequence of the SAGBI criterion that is worth observing:

Corollary 2.3.2. *Let $g_1, \dots, g_r \in R = K[X_1, \dots, X_n]$ and let A be the subalgebra of R generated by $g_1, \dots, g_r$. If $\text{in}(g_1), \dots, \text{in}(g_r)$ are algebraically independent, then $g_1, \dots, g_r$ form a SAGBI basis of A.*

Proof. It is really enough to observe that if $\text{in}(g_1), \dots, \text{in}(g_r)$ are algebraically independent, then $\ker(\psi) = (0)$. Therefore there's no binomial relation to be lifted, and we conclude that $g_1, \dots, g_r$ form a SAGBI basis of A. □

Example 2.3.3. Let us look again at Example 2.2.2, and set A the K-algebra generated by the six maximal minors $[i\ j]$ of

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{pmatrix}.$$

We have:

$$\begin{array}{lll} \text{in}[1\ 2] = X_{11}X_{22} & \text{in}[1\ 3] = X_{11}X_{23} & \text{in}[1\ 4] = X_{11}X_{24} \\ \text{in}[2\ 3] = X_{12}X_{23} & \text{in}[2\ 4] = X_{12}X_{24} & \text{in}[3\ 4] = X_{13}X_{24} \end{array}$$

Observe that both in[1 2] and in[3 4] contain an indeterminate, respectively X_{22} and X_{13}, that does not appear in any other monomial. Then they cannot appear in any binomial relation in $\ker(\psi)$. Taking three of the remaining four monomials, we see with the same argument that they are algebraically independent. Therefore we have that $\dim(\mathrm{in}(A)) \geq 5$. Clearly $\dim(\mathrm{in}(A)) \leq 6$ and, since $-Y_{13}Y_{24}+Y_{14}Y_{23} \in \ker(\psi)$, we get that $\dim(\mathrm{in}(A)) < 6$. Therefore $\dim(\mathrm{in}(A)) = 5$, and this implies that $\ker(\psi)$ must be generated by a single binomial, namely the binomial we just mentioned. We know by Example 2.2.2 that it can be lifted to an element of $\ker(\phi)$, and by the SAGBI criterion we conclude that the maximal minors of X are a SAGBI basis of A.

As it happens for the Buchberger Criterion, which suggests an algorithm for the computation of Gröbner bases, the last Theorem suggests an algorithm for the computation of SAGBI bases. It starts from the finite family $\mathcal{F}_0$ that generates A as a K-subalgebra of R, then one proceeds as follows:

1. Set $i = 0$;
2. Set $\mathcal{F}' = \emptyset$ and compute a binomial system of generators $\mathcal{B}_i$ of $\ker(\psi_i)$, where $\psi_i : P_i \to K[\mathrm{in}(\mathcal{F}_i)]$, $P_i = K[Y_F : F \in \mathcal{F}_i]$, $\psi_i(Y_F) = \mathrm{in}(F)$;
3. For all $\beta \in \mathcal{B}_i$ compute the subduction r of $\phi_i(\beta)$ modulo $\mathcal{F}_i$, where $\phi_i : P_i \to K[\mathcal{F}_i]$, $\phi_i(Y_F) = F$, $F \in \mathcal{F}_i$. If $r \neq 0$, make r monic and add it to $\mathcal{F}'$;
4. If $\mathcal{F}' = \emptyset$, set $\mathcal{F}_j = \mathcal{F}_i$, $P_j = P_i$, $\mathcal{B}_j = \mathcal{B}_i$ for all $j \geq i$ and stop;
5. If $\mathcal{F}' \neq \emptyset$, set $\mathcal{F}_{i+1} = \mathcal{F}_i \cup \mathcal{F}'$, $i = i + 1$ and go to 2.

The similarity with Buchberger algorithm is clear: that starts from a system of generators $\mathfrak{g}$ of I, then one computes the S-polynomials and their reductions modulo $\mathfrak{g}$. In the SAGBI algorithm, the $\phi_i(\beta)$ play the role of the S-polynomials and, of course, reduction is replaced by subduction. Then, in the Buchberger algorithm, one adds the nonzero reductions to $\mathfrak{g}$ and repeats the process with new S-polynomials. We do the same, adding the nonzero subduction to $\mathcal{F}_i$ and repeating from step 2, starting with computing a new system of generators of $\ker(\psi_i)$.

However, we have to be careful: we know that the Buchberger algorithm will stop after finitely many steps by Noetherianity of R. We cannot expect the same for the SAGBI algorithm, since we can have infinite ascending chains of monomials subalgebras of R.

Proposition 2.3.4.

(a) $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$ *is a SAGBI basis of* A*;*

(b) If A *has a finite SAGBI basis, then the algorithm terminates.*

Proof. (a). We set $P = \bigcup_i P_i$ and $\mathcal{B} = \bigcup_i \mathcal{B}_i$. Then, the ϕ_i and ψ_i define homomorphisms ϕ, $\psi : P \to R$, such that $\mathcal{B}$ is a binomial system of generators of ker(ψ). In addition, the $\mathcal{F}_{i+1}$ are constructed such that all $\phi_i(\beta)$, $\beta \in \mathcal{B}_i$, subduce to 0 modulo $\mathcal{F}_i$. Therefore we get that all $\phi(\beta)$, $\beta \in \mathcal{B}$, subduce to 0 modulo $\mathcal{F}$, and we easily conclude using Theorem 2.3.1.

(b). Let us suppose that A has a finite SAGBI basis. We consider the (unique) minimal SAGBI basis of A, consisting of the elements $f \in A$ for which $\mathrm{in}(f)$ is irreducible in $\mathrm{in}(A)$. Using the SAGBI algortihm, we get a SAGBI basis $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$ of A. Therefore the irreducible monomials of $\mathrm{in}(A)$ must be contained in $\bigcup_i \mathrm{in}(\mathcal{F}_i)$. Since the irreducible monomials of $\mathrm{in}(A)$ are finite in number by hypothesis, there exists an i for which all of them belong to $\mathrm{in}(\mathcal{F}_i)$, and so $\mathcal{F}_i$ is a SAGBI basis of A. Then one has $\mathcal{F}' = \emptyset$ and the algortihm stops. □

2.4 Gradings and Initial Algebras

In general, the computation of SAGBI bases is a very complex operation. In particular, the algorithm we just introduced has limited practical usage due to its high complexity. Therefore, we now introduce another tool for the computation of SAGBI bases.

Recall that any vector $w = (w_1, \dots, w_n) \in \mathbb{N}^n_{>0}$ induces a $\mathbb{N}$-graded structure on the polynomial ring $R = K[X_1, \dots, X_n]$, which we call the *w-grading*. With respect to the w-grading, the indeterminate X_i has degree w_i, the monomial X^e has degree $\sum e_i w_i$ and the w-degree of a nonzero polynomial $f \in R$ is the largest w-degree of a monomial in $\mathrm{supp}(f)$. Then $R = \bigoplus_{i=0}^{\infty} R_i$, where R_i is the K-span of the monomials of w-degree i and $R_i R_j \subseteq R_{i+j}$. We say that the elements of R_i are w-homogeneous of w-degree i. Note that every R_i is finitely generated as a K-vector space: in fact, R_i is contained in the space of monomials of R with exponent $a_1, ..., a_n$ such that $\sum a_j w_j \leq i$, which are finite in number since $w_j > 0$ for every $j = 1, \dots, n$.

Let now A be a subalgebra of R. We say that A is w-graded (or w-homogeneous) if A is generated by w-homogeneous elements or, equivalently, A has a decomposition $A = \bigoplus_{i=0}^{\infty} A_i$, where $A_i = A \cap R_i$.

Finally, recall that the *Hilbert function* of A is defined as

$$\begin{aligned} HF_A : \mathbb{N} &\longrightarrow \mathbb{N} \\ i &\mapsto \dim_K(A_i), \end{aligned}$$

while the *Hilbert series* of A is the formal sum:

$$HS_A(t) = \sum_{i \in \mathbb{N}} HF_A(i) t^i \in \mathbb{Z}[[t]].$$

Note that, since R_i is finitely generated as a K-vector space for every $i \in \mathbb{N}$, the Hilbert function of A is well defined. For instance, the Hlbert series of the polynomial ring $R = K[X_1, \ldots, X_n]$ when all the indeterminates have degree 1 is

$$HS_R(t) = \frac{1}{(1-t)^n}.$$

Proposition 2.4.1. *Let $w \in \mathbb{N}^n_{>0}$ and R be a w-graded polynomial ring endowed with a monomial order, and suppose that A is a positively w-graded subalgebra of R with decomposition $A = \bigoplus_{i=0}^{\infty} A_i$. Then $\mathrm{in}(A) = \bigoplus_{i=0}^{\infty} \mathrm{in}(A_i)$. In particular A and $\mathrm{in}(A)$ have the same Hilbert function.*

Proof. The decomposition $A = \bigoplus_{i=0}^{\infty} A_i$ induces a decomposition of any polynomial $f \in A$ as $f = \sum_i f_i$, $f_i \in A_i$. Therefore A contains the homogeneous components of each of its elements, and, since $\mathrm{in}(f_i) \neq \mathrm{in}(f_j)$ if $i \neq j$ (monomials of different degrees are different), we get that $\mathrm{in}(f) = \max\{\mathrm{in}(f_i), f_i \neq 0\}$. Therefore $\mathrm{in}(A) \cap R_i = \mathrm{in}(A_i)$, obtaining the decomposition we wanted. We already know by Proposition 1.3.1 that $\dim_K(A_i) = \dim_K(\mathrm{in}(A_i))$, and therefore $HF_A(i) = HF_{\mathrm{in}(A)}(i)$ for all i. □

Now we can state the following Lemma, that is a simple but useful tool in the computation of SAGBI bases.

Lemma 2.4.2. *Let R be a polynomial ring endowed with a monomial order and a positive grading, and A a finitely generated graded K-subalgebra of R. Let $\mathcal{F}$ be a family of polynomials in A and set $B = K[\mathrm{in}(\mathcal{F})]$. Then:*

(a) $HF_B(i) \leq HF_A(i)$ for all $i \in \mathbb{N}$;

(b) $\mathcal{F}$ is a SAGBI basis of A if and only if $HF_B(i) = HF_A(i)$ for all $i \in \mathbb{N}$.

Proof. First of all note that from the previous Proposition we know that, for a graded subalgebra A of R, $HF_A(i) = HF_{\mathrm{in}(A)}(i)$ for all i. Of course we have the inclusions

$$B_i \subseteq \mathrm{in}(A)_i, \quad i \in \mathbb{N},$$

We get the equality if and only if $\dim_K(B_i) = \dim_K(\mathrm{in}(A)_i)$ for all i. Looking at the definition of the Hilbert function together with our first remark, we have the thesis. □

Example 2.4.3. Once again, consider Example 2.2.2. It turns out that the subalgebra A that we considered in the Example is a graded subalgebra of $K[X_{ij}]$. We know (from the Plücker relations) that $F = Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}$ is in $\ker(\phi)$, where as usual $\phi : P \longrightarrow A$, $\phi(Y_{ij}) = [i\ j]$. Therefore we have $HF_A(k) \leq HF_{P/(F)}(k)$. We also know that $\ker(\psi)$ is generated by the binomial $\beta = -Y_{13}Y_{24} + Y_{14}Y_{23}$. Set $B = K[\mathrm{in}[i\ j]] \subseteq \mathrm{in}(A)$, we get that $HF_{\mathrm{in}(A)}(k) \geq HF_B(k) = HF_{P/(\beta)}(k) = HF_{P/(F)}(k)$, where the last equality follows from the fact that $\beta = \mathrm{init}_\phi(F)$. Thus we have the chain

$$HF_B(k) \leq HF_{\mathrm{in}(A)}(k) = HF_A(k) \leq HF_{P/(F)}(k) = HF_{P/(\beta)}(k) = HF_B(k).$$

So $HF_B(k) = HF_{\text{in}(A)}(k)$ for all k and by the previous Lemma we conclude that $B = \text{in}(A)$, as we already knew from Example 2.3.3.

Now, we make the further assumption that A is a *standard graded* K-algebra: this means that $A = K[A_1]$, namely A is generated as a K-algebra by elements of degree 1. Actually, if $A = K[A_d]$ for some $d \in \mathbb{N}$, we can normalize the grading and thus thinking of all generators as elements of degree 1, obtaining a standard graded algebra as well. Therefore from now on, when speaking of standard grading, we will include also this case.

If A is standard graded, the following relevant Theorem (see [7], Theorem 4.1.3) holds

Theorem 2.4.4 (Hilbert). *Let A be a standard graded K-algebra of Krull dimension d. Then there exists a polynomial (called the Hilbert polynomial) $HP_A \in \mathbb{Q}[t]$ of degree $d-1$ such that:*

$$HF_A(i) = HP_A(i) \quad \forall i >> 0.$$

Starting from Hilbert's Theorem, one can show that if A is a standard graded K-algebra of Krull dimension d, then there exists a polynomial $h_A \in \mathbb{Z}[t]$ with $h_A(1) \neq 0$, called the h-polynomial of A, such that:

$$HS_A(t) = \frac{h_A(t)}{(1-t)^d}.$$

One may now ask what happens if A is not standard graded: we have to spend some words on the Hilbert series of A. If A is generated by elements of positive degree $e_1, \ldots, e_n$, it is known (see [16]) that the Hilbert series of A has a rational expression in t as follows:

$$HS_A(t) = \frac{P(t)}{\prod_{i=1}^{n}(1-t^{e_i})}, \qquad P(t) \in \mathbb{Z}[t].$$

Moreover, if d denotes the Krull dimension of R, then d is the order to which $t = 1$ is a pole of $HS_A(t)$ (see [2], Chapter 11). In other words, d is the unique integer for which $\lim_{t \to 1}(1-t)^d HS_A(t)$ is nonzero and noninfinite.

Thus, if A is finitely generated and positively $\mathbb{N}$-graded (either standard or not), the Hilbert function of A uniquely determines the Krull dimension of A. Therefore, combining this statement with Proposition 2.4.1, we immediately obtain the following:

Theorem 2.4.5. *Let R be a polynomial ring endowed with a monomial order and a positive grading, and A a finitely generated graded K-subalgebra of R. If* $\text{in}(A)$ *is a finitely generated K-algebra, then A and* $\text{in}(A)$ *have the same Krull dimension.*

2.5 Weight Orders

Let us consider as usual the polynomial ring $R = K[X_1, \dots, X_n]$. We have seen that, given a vector $w \in \mathbb{N}^n_{>0}$, we may define on R a $\mathbb{N}$-grading. We now introduce a partial order on the monomials of R given by w.

Definition 2.5.1. A *weight vector* on R is a vector w in $\mathbb{N}^n$ with positive entries.

Consider now $f \in R$. We can write $f = \sum c_\alpha \mathbf{X}^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\mathbf{X}^\alpha = {X_1}^{\alpha_1} \dots {X_n}^{\alpha_n}$. For any weight vector $w \in \mathbb{N}^n$, we write $\mathrm{in}_w(f)$ for the sum of all terms $c_\alpha \mathbf{X}^\alpha$ appearing in f such that the dot product $\langle \alpha, w \rangle$ is maximized. For a K-vector subspace V of R, in accordance with our usual notation, we denote $\mathrm{in}_w(V)$ the w-initial subspace generated by $\{\mathrm{in}_w(f),\ f \in V\}$. A simple example to understand how this all works:

Example 2.5.2. Consider $f = X^2 - XY + Z^2 \in K[X, Y, Z]$. For the weight vector $w = (2, 1, 2)$ one has $\mathrm{in}_w(f) = X^2 + Z^2$. Changing w to $(2, 2, 1)$, we get $\mathrm{in}_w(f) = X^2 - XY$. Finally for $w = (4, 2, 1)$, $\mathrm{in}_w(f) = X^2$.

At this point, one immediately deduces that the weight vector alone does not define a monomial order: there are monomials μ, ν, for example $X_1^{w_2}$ and $X_2^{w_1}$, such that $w(\mu) = w(\nu)$. This happened also in our previous Example: for $w = (2, 1, 2)$, $w(X^2) = w(Z^2) = 4$. Therefore, we need a refinement of the partial order given by w, that will be obtained using a monomial order $<$ on R (called the tie-breaker) as follows:

$$\mu <_w \nu \iff \begin{cases} w(\mu) < w(\nu) \text{ or} \\ w(\mu) = w(\nu) \text{ and } \mu < \nu. \end{cases}$$

A standard choice for the tie-breaker is the DegRevLex order. For example, this is the first choice of many computer packages, including Macaulay2 and CoCoA. Taking back Example 2.5.2, using DegRevLex with $X > Y > Z$ one has $\mathrm{in}_w(f) = X^2$ for all the w we considered. The next proposition will show us how the initial spaces related to $<$ and $<_w$ are connected to each other.

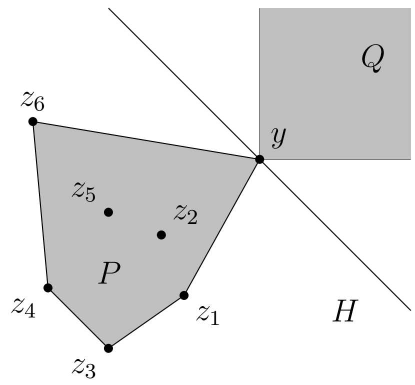
Proposition 2.5.3. *Let $w \in \mathbb{N}^n_{>0}$ and $<$ a monomial order on R. For every K-subspace V of R one has:*

(a) $\mathrm{in}_{<_w}(V) = \mathrm{in}_{<_w}(\mathrm{in}_w(V)) = \mathrm{in}_<(\mathrm{in}_w(V))$;

(b) if either $\mathrm{in}_w(V) \subset \mathrm{in}_<(V)$ *or* $\mathrm{in}_<(V) \subset \mathrm{in}_w(V)$, *then* $\mathrm{in}_w(V) = \mathrm{in}_<(V)$.

Proof. For (a) it is enough to analyze the selection of the initial monomial for each space. On the left, one just selects the initial monomial with respect to $<_w$. In the center, we first select the monomials that have the highest weight, look at the weight again and then pick the biggest monomial with respect to $<$: that's

Figure 2.1: the polytopes P and Q



the same as on the left. Finally, on the right we do the exact same thing but without looking at thw weights a second time, which is obviously superfluous.

(b) Without loss of generality, suppose that $\text{in}_w(V) \subset \text{in}_<(V)$. Then we have $\text{in}_<(\text{in}_w(V)) \subset \text{in}_<(\text{in}_<(V)) = \text{in}_<(V)$. By (a), we know that $\text{in}_<(\text{in}_w(V)) = \text{in}_{<_w}(V)$. Therefore we obtain $\text{in}_{<_w}(V) \subset \text{in}_<(V)$. Since they are both monomial orders, by Proposition 1.3.1 we get the equality. $\square$

Summing up, we have seen that from every weight vector on R, although with a little help, we can get a term order $<_w$. Now, a natural question is: can every monomial order on R be represented by a weight vector?

Consider $f \in R$. Let us start by characterizing those monomials $\mu \in \text{supp}(f)$ that can appear as initial monomials with respect to some monomial order.

Recall that the *Newton polytope* P_f of f is the polytope in $\mathbb{R}^n$ spanned by the exponent vectors of the monomials $\mu \in \text{supp}(f)$. It is a key tool to give the characterization we wanted and to go from monomial orders to weight vectors:

Lemma 2.5.4. *Let $f \in R$ and $\mu \in \text{supp}(f)$, $\mu = X_1^{y_1} \cdots X_n^{y_n} = \boldsymbol{X}^y$. Then the following are equivalent:*

(a) there exists a monomial order $<$ on R such that $\mu = \text{in}_<(f)$;

(b) the exponent vector y of μ is a vertex of P_f and $P_f \cap (y + \mathbb{R}^n_+) = \{y\}$;

(c) there exists a weight vector w on R such that $\mu = \text{in}_w(f)$.

Proof. (a) $\Rightarrow$ (b). Let $\text{supp}(f) \setminus \mu = \{\nu_1, \cdots, \nu_k\}$ and let z_i, $i = 1, \ldots, k$, be the exponent vector of ν_i. We set $P = P_f$ and $Q = y + \mathbb{R}^n_+$. Therefore our goal is to show that $P \cap Q = \{y\}$: see Figure 2.1. Since $P \cap Q$ is the intersection of rational polyhedra (see [6] for general facts about polyhedral geometry), if $P \cap Q \neq \{y\}$ then $P \cap Q$ must contain a rational point x. After multiplication with a positive common denominator of the coordinates of x, we can assume that $x \in \mathbb{N}^n$: P and

Q are replaced by homothetic images. Rewriting in terms of monomials, we are doing nothing but raising all monomials to the same power. Let now ξ be the monomial with exponent vector x, $\xi = \mathbf{X}^x$. Since $x \in y + \mathbb{R}^n_+$ and $x \neq y$, we get that $\mu < \xi$.

So, by (a), now we know that $\xi > \mu > \nu_i$ for all $i = 1, \dots, k$. On the other hand, since $x \in P$, x is a rational convex combination of $z_0 = y, z_1, \dots, z_k$: $x = \sum_{i=0}^k (a_i/b_i) z_i$, with $a_i, b_i \in \mathbb{Z}_+$ and $\sum_{i=0}^k a_i/b_i = 1$. Therefore we obtain the following binomial relation:

$$\xi^b = \mu^{a'_0} \nu_1^{a'_1} \cdots \nu_k^{a'_k},$$

where $b = \prod_{i=0}^k b_i$ and $a'_i = a_i \prod_{j \neq i} b_j$. The fact that $\sum_{i=0}^k a_i/b_i = 1$ implies that $\sum a'_i = b$, and therefore the numbers of factors on the left hand side of our binomial relation equals the number of factors on the right hand side. This contradicts the fact that ξ is larger than every factor on the right. Therefore $P \cap Q = \{y\}$. The same exact argument shows that y is not a convex combination of $z_1, \dots, z_k$. Hence y is a vertex of P.

(b) $\Rightarrow$ (c). With our notation, (b) can be riformulated as: P and Q intersect in the common face $\{y\}$. By [6], Theorem 1.32, there exists a (rational) hyperplane H such that $y \in H$, $P \smallsetminus \{y\}$ is in the interior of one of the two halfspaces defined by H and $Q \smallsetminus \{y\}$ lies in the interior of the other halfspace. H is an rational hyperplane in $\mathbb{R}^n$, and so is defined by a vector $w \in \mathbb{Z}^n$ and $w_0 \in \mathbb{Z}$ via the equation $w_1 u_1 + \cdots + w_n u_n = w_0$. Without loss of generality, we have $\sum_{i=1}^n w_i u_i \geq w_0$ for all $u = (u_1, \dots, u_n) \in Q$. Then the definition of Q implies that $w_i > 0$ for all $i = 1, \dots, n$. In fact, fix $u \in Q \smallsetminus \{y\}$: we know that $\sum w_i y_i = w_0$ and $u = y + v$ for some $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$. Therefore, since $u \neq y$, $\sum w_i u_i = \sum w_i (y_i + v_i) = w_0 + w_i v_i > w_0$. By arbitrariness of $u \in Q$, we get that $w_i v_i > 0$ for all $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ and hence $w_i > 0$ for all $i = 1, \dots, n$. So, since $y \in H \cap P$ and $\sum_{i=1}^n w_i u_i < w_0$ for all $u \in P \smallsetminus \{y\}$, w is our desired weight vector.

Finally, (c) $\Rightarrow$ (a) is covered by refinement of the weight w that we discussed above. □

The significant implication of the previous lemma is (a) $\Rightarrow$ (c): it tells us that, given a monomial order $<$ on R and set $f \in R$, there exists a weight vector on R that, on f, acts like $<$. As we have seen in the proof of the lemma, this weight vector depends on f. Clearly, what we would like to have is a single weight vector w on R that, at least on a fixed K-subalgebra A of R, does the same job as $<$, in order to relate $\mathrm{in}_w(A)$ and $\mathrm{in}_<(A)$. This is possible with some finiteness assumptions on $\mathrm{in}_<(A)$:

Proposition 2.5.5. *Let $<$ be a monomial order on R and let A be a K-subalgebra of R. If A admits a finite SAGBI basis with respect to $<$ then there exists a weight vector w on R such that* $\mathrm{in}_<(A) = \mathrm{in}_w(A)$.

In order to prove this, we need one last simple preliminary lemma:

Lemma 2.5.6. *Let $<$ be a monomial order on R and consider the finite set $\{(\mu_1, \nu_1), \dots, (\mu_k, \nu_k)\}$, where (μ_i, ν_i) is a pair of monomials such that $\mu_i > \nu_i$ for all $i = 1, \dots, k$. Then there exists a weight vector w on R such that $w(\mu_i) > w(\nu_i)$ for all $i = 1, \dots, k$.*

Proof. One can assume $\mu = \mu_1 = \cdots = \mu_k$: we can multiply both μ_i and ν_i by $\prod_{j \neq i} \mu_j$ without loss of generality, as follows from the monotonicity of $<$ and the linearity of weights.

Now we set $f = \mu + \nu_1 + \cdots \nu_k$ and conclude just using the implication (a) $\Rightarrow$ (c) of Lemma 2.5.4. $\square$

We are now ready to prove Proposition 2.5.5:

Proof (Proposition 2.5.5). Let $\mathcal{F}$ be a finite SAGBI basis of A. Consider the set U of pairs of monomials $(\mathrm{in}_<(f), \mu_f)$, where $f \in \mathcal{F}$ and μ_f is any noninitial monomial appearing in f. Since U is finite, by Lemma 2.5.6 there exists a weight vector w on R such that, for every $f \in \mathcal{F}$, $w(\mathrm{in}_<(f)) > w(\mu_f)$. This means that $\mathrm{in}_w(f) = \mathrm{in}_<(f)$ for every $f \in \mathcal{F}$. We now show that w is the desired weight vector. By construction, generators of $\mathrm{in}_<(A)$, namely $\{\mathrm{in}_<(f)\}_{f \in \mathcal{F}}$, belong to $\mathrm{in}_w(A)$ so that $\mathrm{in}_<(A) \subseteq \mathrm{in}_w(A)$. But now, by Proposition 2.5.3(b), we conclude that $\mathrm{in}_w(A) = \mathrm{in}_<(A)$. $\square$

Chapter 3

The Grassmannian

3.1 Plücker Embedding and Algebra

Fix a field K. We want to consider the d-dimensional subspaces of K^n, $d \leq n$. Taken one of these, it can be expressed as the row span of some $d \times n$ matrix Θ with entries in K. Clearly, Θ must have rank d, since its d rows span a d-dimensional vector space. This means that there are d columns of Θ forming a square matrix with nonzero determinant.

Definition 3.1.1. The determinant of a square $r \times r$ submatrix, $r \leq d$, of the $d \times n$ matrix Θ is called a *minor* of size r. If r is as large as possible, namely $r = d$, we say that the minor is *maximal*.

Definition 3.1.2. The *Grassmannian* $G(d, n)$ is the set of all d-dimensional linear subspaces of K^n.

It follows from the discussion above that a point of $G(d, n)$ can be represented by a (non-unique) $d \times n$ matrix with entries in K and of rank d.

Now, suppose that V is a d-dimensional subspace of K^n. We write $\mathfrak{i}(V)$ for the vector in $\mathbb{P}^{\binom{n}{d}-1}$ of all $d \times d$ minors (in some prescribed order) of the matrix representing V.

Example 3.1.3. Let $d = 2$ and $n = 5$, and let V be the linear subspace of K^5 spanned by $(1, 1, 1, 1, 1)$ and $(a_1, a_2, a_3, a_4, a_5)$ for some $a_i \in K$, not all identical. Then $\mathfrak{i}(V)$ is the point in $\mathbb{P}^9$ whose 10 homogeneous coordinates are $a_i - a_j$ for $1 \leq i < j \leq 5$.

Note that $\mathfrak{i}(V)$ does not depend on the chosen basis of V. In fact, if Θ and Θ' have the same row span, there exists a $d \times d$ invertible matrix Λ such that $\Theta = \Lambda\Theta'$. If we denote Θ_σ the $d \times d$ submatrix of Θ with columns indices $\sigma_1, \ldots, \sigma_d$, we have $\Theta_\sigma = \Lambda\Theta'_\sigma$ and so $\det(\Theta_\sigma) = \det(\Lambda)\det(\Theta'_\sigma)$ for every $\sigma \subset [n]$, $|\sigma| = d$.

Therefore we can define the following map

$$\mathfrak{i} : G(d,n) \longrightarrow \mathbb{P}(K^{\binom{n}{d}}),$$

where $\mathbb{P}(K^{\binom{n}{d}})$ denotes the projectivization of $K^{\binom{n}{d}}$, that is, $\mathbb{P}^{\binom{n}{d}-1}$.

Lemma 3.1.4. *The map* $\mathfrak{i}$ *is injective.*

Proof. Consider $V_1, V_2 \subset K^n$ two d-dimensional subspaces and assume $\mathfrak{i}(V_1) = \mathfrak{i}(V_2)$. Set now M_{V_1}, M_{V_2} the matrices of rank d that represent V_1 and V_2 respectively. Without loss of generality we may assume that the first d columns of M_{V_1} and M_{V_2} are linearly independent. By performing linear operations of the rows of M_{V_1} and M_{V_2}, we transform both M_{V_i} in $\tilde{M}_{V_i}$ whose leftmost $d \times d$ submatrix is the identity. Now, any entry of $\tilde{M}_{V_i}$ not in the first k columns is either a maximal minor of M_{V_i} or its negative. By hypothesis, $\mathfrak{i}(V_1) = \mathfrak{i}(V_2)$, and therefore we get $\tilde{M}_{V_1} = \tilde{M}_{V_2}$. This implies $V_1 = V_2$. □

This allows us to give the following

Definition 3.1.5. The inclusion of the Grassmannian in $\mathbb{P}(K^{\binom{n}{d}})$ is called the *Plücker embedding.*

Actually, the Grassmannian in its Plücker embedding is a projective subvariety of $\mathbb{P}(K^{\binom{n}{d}})$:

Proposition 3.1.6. *The image of the Grassmannian $G(d,n)$ through the Plücker embedding is Zariski closed.*

We will prove this statement in the special case of $G(2,4)$ later in Section 3.5. For the proof of the general statement, see [11], Theorem 5.4.

Clearly, the Plücker embedding is determined by the maximal minors of Θ, to whom one can refer as *Plücker coordinates*:

Definition 3.1.7. The *Plücker coordinates* of Θ are the maximal minors $\det(\Theta_\sigma)$ for $\sigma \subset [n]$, $|\sigma| = d$, $\sigma = \sigma_1 < \cdots < \sigma_d$, where Θ_σ denotes the $d \times d$ submatrix of Θ with columns indices $\sigma_1, \ldots, \sigma_d$.

We can think of the Plücker coordinate indexed by σ as the *generic (maximal) minor* $det(X_\sigma)$ of the $d \times n$ generic matrix $X = (X_{ij})$ of variables. As in the last definition, X_σ is a $d \times d$ submatrix with row indices $1, \ldots, d$ and column indices $\sigma_1, \ldots, \sigma_d$. Therefore, the Plücker coordinates are elements of the polynomial ring $K[X] := K[X_{ij}]$, $i = 1, \ldots, d$, $j = 1, \ldots, n$.

Definition 3.1.8. The *Plücker algebra* is the subalgebra $G(X)$ of $K[X]$ generated by the Plücker coordinates $det(X_\sigma)$.

The coordinate ring of the Grassmannian in its Plücker embedding is hence given by the Plücker algebra. In fact, if a variety Y is given as the closure of the image of a polynomial map associated to polynomials $f_1, \ldots, f_n$, it is well known that its coordinate ring is isomorphic to $K[f_1, \ldots, f_n]$. Therefore in our setting we get that the coordinate ring of the Grassmannian in its Plücker embedding is isomorphic to $G(X)$.

From now on, by $G(d, n)$ we will denote the Grassmannian seen as a subvariety of $\mathbb{P}^{\binom{n}{d}-1}$, namely as the image of the Plücker embedding.

3.2 Plücker Relations

We have just seen that $G(d, n)$ is a subvariety of $\mathbb{P}^{\binom{n}{d}-1}$. Next, we want to find its defining ideal.

As in the previous section, let $X = (X_{ij})$ be a $d \times n$ matrix of indeterminates and let $K[X]$ denote the polynomial ring over a field K generated by these indeterminates. Now define a second polynomial ring $K[\mathbf{p}]$ by introducing a variable p_σ for each subset σ of $\{1, \ldots, n\}$, $|\sigma| = d$. It comes natural to define the ring homomorphism

$$\begin{aligned} \phi_{d,n} : K[\mathbf{p}] &\longrightarrow K[X] \\ p_\sigma &\longmapsto \det(X_\sigma). \end{aligned}$$

The map $\phi_{d,n}$ gives a presentation for the Plücker algebra as a quotient of $K[\mathbf{p}]$. Therefore, the defining ideal of $G(d, n)$ is the kernel of this map.

In the next lemma we describe explicitely some relations that are satisfied by maximal minors: the *Plücker relations*. We use the notation of Example 2.2.2, writing $[j_1, \ldots, j_m]$ for the m-minor obtained by taking just the j_i-th columns, $i = 1, \ldots, m$.

Lemma 3.2.1. *For every $d \times n$ matrix, $d \leq n$, with elements in a commutative ring and for all indices $a_1, \ldots, a_k$, $b_l, \ldots, b_d$, $c_1, \ldots, c_s \in \{1, \ldots, n\}$ such that $s = d - k + l - 1 > d$ and $t = d - k > 0$ one has*

$$\sum_{\substack{i_1 < \cdots < i_t \\ i_{t+1} < \cdots < i_s \\ \{1, \ldots, s\} = \{i_1, \ldots, i_s\}}} sign(i_1, \ldots, i_s)[a_1, \ldots, a_k, c_{i_1}, \ldots, c_{i_t}][c_{i_{t+1}}, \ldots, c_{i_s}, b_l, \ldots, b_d] = 0,$$

where $sign(i_1, \ldots, i_s)$ denotes the sign of the permutation of $\{1, \ldots, s\}$ represented by the sequence $(i_1, \ldots, i_s)$.

Proof. First of all, note that the minors in the formula are actually maximal minors. In fact, $k+t = d$ and $(s-t)+(d-l+1) = ((d-k+l-1)-t)+(d-l+1) = 2d - k - t = 2d - t + t - d = d$.

Since every commutative ring is a $\mathbb{Z}$-algebra, by the universal property of polynomial rings it suffices to prove this for a matrix X of indeterminates over $\mathbb{Z}$. Next we can replace $\mathbb{Z}$ by $\mathbb{Q}$, and finally the ring $\mathbb{Q}[X]$ with his ring of fractions $\mathbb{Q}(X)$. Consider the $\mathbb{Q}(X)$-module C generated by the columns of X. As a $\mathbb{Q}(X)$-module it has rank d. Let $\alpha : C^s \to \mathbb{Q}(X)$ be given by

$$\alpha(y_1, \dots, y_s) =$$

$$\sum_{\pi \in Sym(1,\dots,s)} sign(\pi) det(X_{a_1}, \dots, X_{a_k}, y_{\pi(1)}, \dots, y_{\pi(t)}) \cdot det(y_{\pi(t+1)}, \dots, y_{\pi(s)}, X_{b_l}, \dots, X_{b_m}),$$

where X_j denotes the j-th column of X and $Sym(1, \dots, s)$ the group of permutations of $\{1, \dots, s\}$. It is straightforward to check that α is a multilinear form on C^s. When two of the vectors y_i coincide, every term in the expansion of α, which does not vanish already, is canceled by a term of the opposite sign: thus α is alternating. Since $s >$ rank C, α, being alternating, is the zero map.

Now, we fix a subset $\{i_1, \dots, i_t\}$ of $\{1, \dots, s\}$, with $i_1 < \cdots < i_t$. Then, for all π such that $\pi(\{1, \dots, t\}) = \{i_1, \dots, i_t\}$ the summand corresponding to π in the expansion of α equals

$$sign(i_1, \dots, i_s) det(X_{a_1}, \dots, X_{a_k}, y_{i_1}, \dots, y_{i_t}) \cdot det(y_{i_{t+1}}, \dots, y_{i_s}, X_{b_l}, \dots, X_{b_m}),$$

where $i_{t+1}, \dots, i_s$ are chosen as above. Therefore, each of this terms appears $t!(s-t)!$ times in the expansion of α, and canceling the factor $t!(s-t)!$, as we can do since the relation holds in $\mathbb{Q}(X)$, we obtain the desired formula. □

Note that we have just shown that the ideal of $K[\mathbf{p}]$ generated by the Plücker relations is contained in $\ker(\phi_{d,n})$, since the maximal minors satisfy these relations. We want to show the opposite containment to get the equality. This will require some more work.

3.3 Algebras with Straightening Law

With respect to the monomial basis of $K[X]$, a minor of X is a very complicated expression. Therefore, what we would like to have for our purposes is a new basis of $K[X]$ which contains the minors and as many of their products as possible.

The first step for the construction of such a basis will be the introduction of a special class of algebras:

Definition 3.3.1. Let A be a B-algebra and $\Pi \subset A$ a finite subset with a partial order $\leq$, namely what we call a *poset*. We say that A is a *graded algebra with straightening law (on* Π*, over* B*)* if the following hold:

(H_0) $A = \bigoplus_{i \geq 0} A_i$ is a graded B-algebra such that $A_0 = B$, Π consists of homogeneous elements of positive degree and generates A as a B-algebra;

(H_1) the products $\xi_1 \cdots \xi_m, m \in \mathbb{N}, \xi_i \in \Pi$, such that $\xi_1 \leq \cdots \leq \xi_m$ are B-linearly independent. These products are called *standard monomials*;

(H_2) (Straightening law) for all incomparable $\xi, \nu \in \Pi$ the product $\xi\nu$ has a representation

$$\xi\nu = \sum a_\mu \mu, \quad a_\mu \in B,\ a_\mu \neq 0, \quad \mu \text{ standard monomial},$$

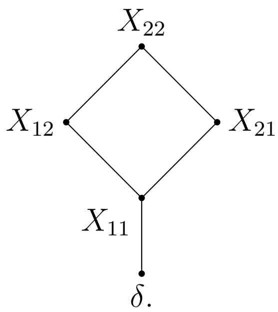
satisfying the following condition: every μ contains a factor $\zeta \in \Pi$ such that $\zeta \leq \xi$ and $\zeta \leq \nu$ (it is allowed that $\xi\nu = 0$, $\sum a_\mu \mu$ being empty).

The notation *algebra with straightening law* will be abbreviated by *ASL*, and the relations in (H_2) will be referred to as *straightening relations*.

Let us give a simple but useful example below:

Example 3.3.2. Let X be a 2×2 matrix, and δ its determinant. Given a ring B, we consider the B-algebra $B[X]$, with as usual $B[X] = B[X_{ij}]$, $i = 1, 2,\ j = 1, 2$.

We (partially) order the set Π of minors with $(\preceq)$ as in the previous section:



Clearly, since the minors of X are homogeneous elements of $B[X]$ of positive degree, (H_0) is satisfied. Regarding (H_2), the only incomparable minors are X_{12} and X_{21}, and the straightening law consists of the single relation $X_{12}X_{21} = X_{11}X_{22} - \delta$. It remains to prove that the standard monomials are linearly independent: one has a bijective degree preserving correspondence between the ordinary monomials and the standard monomials of $B[X]$:

$$X_{11}^i X_{12}^j X_{21}^k X_{22}^l \longleftrightarrow \begin{cases} X_{11}^i X_{12}^{j-k} \delta^k X_{22}^l & \text{if } j \geq k, \\ X_{11}^i X_{21}^{k-j} \delta^j X_{22}^l & \text{if } k > j. \end{cases}$$

Note that this correspondence is actually degree preserving since δ has degree 2, and that it is a bijection, sending the monomial $X_{12}X_{21}$ to δ. Then, the standard monomials must be linearly independent, and finally $B[X]$ is an ASL on Π.

Proposition 3.3.3. *Let A be a graded ASL over B on Π. Then the standard monomials form a B-basis of A.*

Proof. For $\xi \in \Pi$, let $u(\xi)$ be the maximum length of a chain of elements in Π ascending from ξ: $u(\xi) = |\{\delta \in \Pi : \xi \leq \delta\}|$, and define the weight of a monomial in A as

$$w(\mu) = \sum_{i=1}^{m} w(\xi_i) = \sum_{i=1}^{m} 3^{u(\xi_i)}, \qquad \mu = \xi_1 \cdots \xi_m.$$

Given ν incomparable with ξ, we have that $w(\xi\nu) < w(\mu)$ for all standard monomials μ appearing in the standard representation of $\xi\nu$. In fact, set $\alpha = \max\{u(\xi), u(\nu)\}$. From (H_2) we know that for every μ in the straightening relations it exists a factor $\zeta \leq \xi$, $\zeta \leq \nu$, and so we get $\alpha < u(\zeta)$. It follows that $w(\xi\nu) \leq 3^\alpha + 3^\alpha < 3^{\alpha+1} \leq 3^{u(\zeta)} \leq w(\mu)$.

Because of (H_1), it suffices to prove that every monomial in A (namely an expression of type $\mu = \xi_1 \cdots \xi_m$, with factors that need not be distinct) is a combination of standard monomials. If all the factors $\xi_1, \ldots, \xi_m$ of μ are comparable, μ is a standard monomial (up to reordering the factors). Otherwise, two of the factors are incomparable, and by (H_2) we can replace their product by the right side of the corresponding straightening relation. It produces a linear combination of monomials of the same degree as μ but with a greater weight. If any of the resulting monomials is non standard, we repeat the process, but we must terminate eventually because there are only finitely many monomials of a given degree. □

The proof of the preceding proposition shows that the standard representation of an element of A can be obtained by successive applications of the straightening relations. As a consequence the straightening relations generate the defining ideal of A:

Proposition 3.3.4. *Let A be a graded ASL over B on Π, and T_ξ, $\xi \in \Pi$, a family of indeterminates over B. For each monomial $\mu = \xi_1 \cdots \xi_m$, $\xi_i \in \Pi$, let $T_\mu = T_{\xi_1} \cdots T_{\xi_m}$. Then the kernel of the B-algebras epimorphism*

$$\phi : B[T_\xi : \xi \in \Pi] \longrightarrow A, \quad T_\xi \longmapsto \xi,$$

is generated by the elements $T_\xi T_\nu - \sum a_\mu T_\mu$ representing the straightening relations.

Proof. Let I be the ideal in $B[T_\xi : \xi \in \Pi]$ generated by the elements $T_\xi T_\nu - \sum a_\mu T_\mu$ representing the straightening relations. Clearly $I \subseteq \ker(\phi)$.

Conversely, let $f \in \ker(\phi)$, $f = \sum b_\mu T_\mu$, $b_\mu \in B$. If all the monomials μ are standard monomials (and therefore linearly independent), then $b_\mu = 0$ for all μ and $f \in I$. Otherwise, the preceding proof shows that there exists $g \in I$ such that $f - g = \sum b_\zeta T_\zeta$, ζ standard. It follows that $0 = \phi(f - g) = \sum b_\zeta \zeta$. Then (H_1) assures that all b_ζ are 0, and hence $f \in I$. □

3.4 Standard Bitableaux

Throughout the last Section, our discussion has been very general. We want to specialize it to the Plücker algebra $G(X)$. What we are going to do is introduce new combinatorial objects called the *bitableaux*, that, as we will see, will take the place of standard monomials.

First of all, let us recall the setting of this chapter: we have a $d \times n$, $d \le n$, matrix of indeterminates

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{d1} & \cdots & X_{dn} \end{pmatrix}.$$

The indeterminates live in $K[X] = K[X_{ij}]$, $i = 1, \dots, d$, $j = 1, \dots, n$, where K is a fixed field.

We introduce a notation for the t minors of X:

$$[a_1 \dots a_t | b_1 \dots b_t] = det(X_{a_i b_j} : i = 1, \dots, t, j = 1, \dots, t).$$

If $t = d$, since there's no need to specify the rows from which the minor is obtained, we find once again the notation of example 2.2.2:

$$[b_1, \dots, b_d] = [1, \dots, d | b_1 \dots b_d].$$

Now that we have notation for minors, we want to introduce a way of representing products of minors.

Our symbol for a product $\delta_1 \cdots \delta_w$ of minors will be Δ, and we assume that the sizes $|\delta_i|$ (namely the number of rows of the submatrix of X whose determinant is δ_i) are nonincreasing: this means $|\delta_1| \ge \cdots \ge |\delta_w|$. The *shape* of Δ is the sequence $(|\delta_1|, \dots, |\delta_w|)$.

A product of of minors is also called a *bitableau*. The choice of this term is motivated by a graphical description of a product Δ as a pair of *Young tableaux* (see [20] for a more precise discussion).

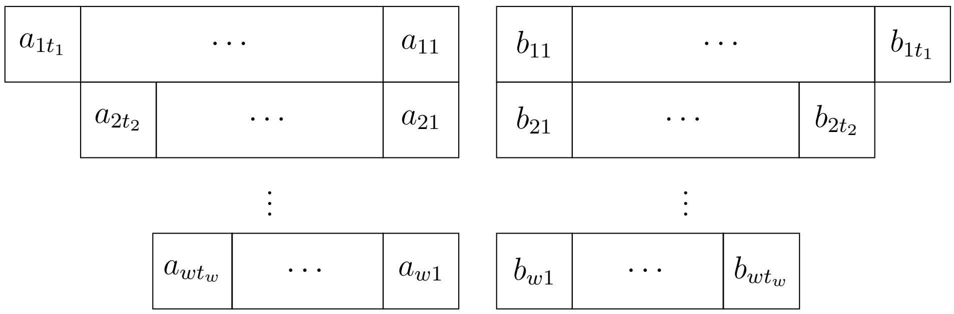
Definition 3.4.1. Let $\lambda = (\lambda_1 \ge \cdots \ge \lambda_k \ge 0)$. A *Young tableaux* of shape λ is a left-justified shape of k rows of boxes of length $\lambda_1, \dots, \lambda_k$ that assigns a positive integer to each box.

According to the notation we introduced above, λ is called the *shape* of the tableau.

Example 3.4.2. A bitableau of shape $(3, 3, 2)$:

<table>
<tr><td>4</td><td>3</td><td>1</td><td></td><td>2</td><td>3</td><td>5</td></tr>
<tr><td>3</td><td>2</td><td>1</td><td></td><td>1</td><td>4</td><td>6</td></tr>
<tr><td></td><td></td><td>2</td><td></td><td>3</td><td></td><td></td></tr>
</table>

As we said before, every product of minors is represented by a bitableaux, and, conversely, every bitableau stands for a product of minors. More explicitly, the general bitableau



represents the product of minors

$$\Delta = \delta_1 \cdots \delta_w, \qquad \delta_i = [a_{i_1} \dots a_{i_{t_i}} | b_{i_1} \dots b_{i_{t_i}}], \ i = 1, \dots, w.$$

As one may have noticed, for symmetric reasons we read the indices of the left tableau from right to left. For instance, the bitableau in Example 3.4.2 represents the product of minors

$$[1\ 3\ 4 \,|\, 2\ 3\ 5][1\ 2\ 3 \,|\, 1\ 4\ 6][2 \,|\, 3].$$

Now we want to introduce the notion of *standard bitableaux*. In order to do that, we have to consider a partial order on the set of the minors of X, defined as follows:

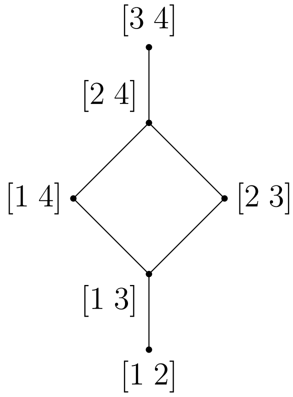
$$\begin{gathered}[a_1 \dots a_t | b_1 \dots b_t] \preceq [c_1 \dots c_u | d_1 \dots d_u] \Longleftrightarrow \\ t \geq u \text{ and } a_i \leq c_i, \ b_i \leq d_i, \quad i = 1, \dots, u.\end{gathered} \tag{$\preceq$}$$

Note that, respect to this order, $[\,|\,]$ is maximal among all minors.

If we consider just the maximal minors, we can more simply describe the order as

$$[a_1, \dots, a_d] \preceq [b_1, \dots, b_d] \Longleftrightarrow a_1 \leq b_1, \dots, a_d \leq b_d.$$

Example 3.4.3. We draw the diagram of the partial order on the maximal minors of a 2×4 generic matrix.



We can now go back to our main discussion and give the following

Definition 3.4.4. A product $\Delta = \delta_1 \cdots \delta_w$ of minors is called a *standard bitableau* if

$$\delta_1 \preceq \cdots \preceq \delta_w.$$

In other words, what we are requiring is that:

- $|\delta_1| \geq \cdots \geq |\delta_w|$, namely the sizes of δ_i are nonincreasing. We already made this assumption at the start of the section;
- the indices in the columns of our bitableau are nondecreasing from top to bottom.

Example 3.4.5. The bitableau in example 3.4.2 is not standard: already the first two indices of the first column of the right size are decreasing from top to bottom.

Here's an example of a standard bitableau:

3	2	1		1	3	5
4	2	1		2	4	6
		2		3		

It represents the product $[1\ 2\ 3 \,|\, 1\ 3\ 5][1\ 2\ 4 \,|\, 2\ 4\ 6][2 \,|\, 3]$.

3.5 The Grassmannian as an ASL

Finally, this section pulls together all of our previous work: we will show that the Plücker algebra $G(X)$, that we saw is isomorphic to the coordinate ring of the Grassmannian, is an ASL with straightening relations resulting from the Plücker relations. This way, thanks to Proposition 3.3.4, our claim will be proved:

the ideal generated by the Plücker relations is indeed the defining ideal of the Grassmannian.

Let K be a field and $X = (X_{ij})$ a $d \times n$, $d \leq n$, matrix of indeterminates over K. From Section 3.1 we know that the Llücker algebra $G(X)$ is generated by the Plücker coordinates, namely the maximal minors of X, ordered partially with the order $\preceq$, which we introduced in the previous Section. We recall here the simpler version of $\preceq$ for the maximal minors:

$$[a_1, \dots, a_d] \preceq [b_1, \dots, b_d] \Longleftrightarrow a_1 \leq b_1, \dots, a_d \leq b_d.$$

This Section goal will be to prove the following

Theorem 3.5.1. *The Plücker algebra $G(X)$ is a graded ASL on the set of Plücker coordinates of X.*

Let us take a look at what we already know. Condition (H_0) is obviously satisfied: the maximal minors of X are homogeneous elements of positive degree d and generate $G(X)$ as a K-algebra. It remains to prove conditions (H_1) and (H_2).

Before going on, note that (standard) monomials, when considering Π as the set of maximal minors ordered with $\preceq$, actually are nothing but (standard) bitableaux in which every factor has size d.

What we are going to do now is proving (H_2) assuming that (H_1) holds. As anticipated at the start of the Section, we will not describe the straightening relations in (H_2) explicitely, but they will result from the Plücker relations. We recall here the Plücker relations for a $d \times n$ matrix, $d \leq n$: for all indices $a_1, \dots, a_k$, $b_l, \dots, b_d$, $c_1, \dots, c_s \in \{1, \dots, n\}$ such that $s = d - k + l - 1 > d$ and $t = d - k > 0$ one has

$$\sum_{\substack{i_1 < \dots < i_t \\ i_{t+1} < \dots < i_s \\ \{1,\dots,s\} = \{i_1,\dots,i_s\}}} sign(i_1, \dots, i_s)[a_1, \dots, a_k, c_{i_1}, \dots, c_{i_t}][c_{i_{t+1}}, \dots, c_{i_s}, b_l, \dots, b_d] = 0.$$

We already met a non trivial Plücker relation, it occurred for a 2×4 matrix in Example 2.2.2:

$$[1\ 2][3\ 4] - [1\ 3][2\ 4] + [1\ 4][2\ 3] = 0.$$

It corresponds to $k = 1$, $a_1 = 1$, $l = 3$, $(c_1, c_2, c_3) = (2, 3, 4)$. Note that the first two bitableaux are standard, but the last is not. One can easily see this in two ways. First, using the graphical characterization of standard bitableaux: they must have non decreasing columns from top to bottom. When speaking of products of maximal minors, as we do not write the rows index since they are always $[1 \dots d]$, we do not even draw the right half of the corresponding bitableau. Moreover, since in this case one always selects d columns, these (bi)tableaux will be "squared". In our example we get the three tableaux

1	2
3	4

1	3
2	4

1	4
2	3

and one immediately sees that the last one is the only nonstandard one. Otherwise, one can use the definition of *standard monomial* in (H_2) and check if the maximal minors in each product are comparable. In our case, looking at the diagram in Example 3.4.3, one sees that [1 4] and [2 3] are incomparable, and so the product [1 4][2 3] is not standard.

Therefore, since the other two bitableaux are standard, solved for [1 4][2 3] the Plücker relation above is a straightening relation for this case. Note that it's actually a straightening relation: [1 2] and [1 3] are less than both [1 4] and [2 3]. Since just [1 4] and [2 3] are incomparable in the entire poset of maximal minors, this is the only straightening relation that occurs speaking of 2×4 matrices.

We now deal with a more complicated case:

Example 3.5.2. Consider a generic 3×6 matrix, and the Plücker relation corresponding to (after reordering the indices in ascending order) $k = 1$, $a_1 = 1$, $l = 3$, $b_3 = 5$, $(c_1, \dots, c_4) = (4, 6, 2, 3)$:

$$[1\ 4\ 6][2\ 3\ 5]+[1\ 2\ 4][2\ 5\ 6]-[1\ 3\ 4][2\ 5\ 6]+[1\ 2\ 6][3\ 4\ 5]-[1\ 3\ 6][2\ 4\ 5]-[1\ 2\ 3][4\ 5\ 6] = 0.$$

The first product is the worst one, with the both last two columns decreasing. For the fourth and the fifth one, the incomparability results just from the last column. The other three instead are all standard. We start from the two bitableaux with just one incomparability position. They are straightened by the Plücker relations

$$[1\ 2\ 6][3\ 4\ 5] - [1\ 2\ 3][4\ 5\ 6] + [1\ 2\ 4][3\ 5\ 6] - [1\ 2\ 5][3\ 4\ 6] = 0$$
$$[1\ 3\ 6][2\ 4\ 5] - [1\ 2\ 3][4\ 5\ 6] + [1\ 3\ 4][2\ 5\ 6] - [1\ 3\ 5][2\ 4\ 6] = 0,$$

corresponding respectively to $k = 2$, $a_1 = 1$, $a_2 = 2$, $l = 4$, $(c_1, \dots, c_4) = (6, 3, 4, 5)$ and $k = 2$, $a_1 = 1$, $a_2 = 3$, $l = 4$, $(c_1, \dots, c_4) = (6, 2, 4, 5)$.

After substitution we finally obtain the straightening relation for [1 4 6][2 3 5]:

$$[1\ 4\ 6][2\ 3\ 5] = -[1\ 2\ 3][4\ 5\ 6] - [1\ 2\ 5][3\ 4\ 6] + [1\ 3\ 5][2\ 4\ 6].$$

This stepwise method where at each step one increases the number of comparable positions works in general for obtaining the straightening relations:

Lemma 3.5.3. *Let* $[a_1, \dots, a_d]$,$[b_1, \dots, b_d]$ *be maximal minors of* X, $a_i \leq b_i$ *for* $i = 1, \dots, k$, $a_{k+1} > b_{k+1}$ *(note that* k *may be* 0*). We set*

$$l = k + 2, \quad s = d + 1, \quad (c_1, \dots, c_s) = (a_{k+1}, \dots, a_d, b_1, \dots, b_{k+1}).$$

Then, in the Plücker relation corresponding to these data, all the terms

$$[f_1, \ldots, f_d][e_1, \ldots, e_d] \neq 0 \quad \textit{and different from} \quad [a_1, \ldots, a_d][b_1, \ldots, b_d]$$

have the following properties (after arranging the indices in ascending order):

$$\textit{(i)}\ [f_1, \ldots, f_d] \preceq [a_1, \ldots, a_d] \quad \textit{and} \quad \textit{(ii)}\ f_1 \le e_1, \ldots, f_{k+1} \le e_{k+1}.$$

Proof. After reordering the indices, without loss of generality we have $b_1 < \cdots < b_{k+1} < a_{k+1} < \cdots < a_d$. Looking at the Plücker relations, we see that $[f_1, \ldots, f_d]$ arises from $[a_1, \ldots, a_m]$ by replacing some of the a_i with smaller indices. This implies (i) and $f_i \le e_i$ for $i = 1, \ldots, k$. It remains to prove just $f_{k+1} \le e_{k+1}$ in (ii). Note that $f_{k+1} \in \{a_1, \ldots, a_k, b_1, \ldots, b_{k+1}\}$, so $f_{k+1} \le b_{k+1}$, and $e_{k+1} \in \{a_{k+1}, \ldots, a_d, b_{k+1}, \ldots, b_d\}$, so $e_{k+1} \ge b_{k+1}$. Hence $f_{k+1} \le e_{k+1}$ and we have the thesis. □

Now we can return to the proof of Theorem 3.5.1. In particular, what we wanted to prove first was the (H_2) condition of the ASL definition.

By induction on k, from Lemma 3.5.3 it follows that every product $\alpha\beta$ of maximal minors can be expressed by a linear combination of standard bitableaux (involving just maximal minors) $\delta\epsilon$: the fact that $\delta\epsilon$ is standard, that is, $\delta \preceq \epsilon$, is assured by point (ii) of the Lemma. Moreover, $\delta \preceq \alpha$ from point (i) of the Lemma.

We have to show that starting from this representation we can derive condition (H_2). One could try to obtain a representation satisfying the latter just by Lemma 3.5.3, but likely a standard bitableau violating the condition in (H_2) will appear:

Example 3.5.4. Consider the bitableau [1 5 6][2 3 4]. Clearly, it is nonstandard since the last two positions are decreasing or, equivalently, since the two minors are incomparable. Now we apply Lemma 3.5.3 in order to straighten the product. We have $k = 1$, $l = 3$, $s = 4$ and $(c_1, \ldots, c_4) = (5, 6, 2, 3)$, and, from the $(c_1, \ldots, c_4)$ permutation $(5, 3, 2, 6)$, after reordering the columns one obtains the standard bitableau [1 3 5][2 4 6] that violates the condition in (H_2). In fact, neither [1 3 5] or [2 4 6] are less or equal (always using $\preceq$ as partial order) to [2 3 4].

Therefore, we will embark on a different path and for now we will assume that (H_1) holds. When a product $\alpha\beta$ of incomparable minors is given (that is, the corresponding bitableau is nonstandard), one first straightens it in the order $\alpha\beta$, obtaining, as from the discussion above, a linear combination of standard bitableaux $\delta_i\epsilon_i$ in which $\delta_i \preceq \alpha$ for every i. Then one straightens it in the order $\beta\alpha$, obtaining a standard representation (up to sign) $\sum_j \zeta_j \nu_j$ in which $\zeta_j \preceq \beta$ always. By linear independence in (H_1), the two representations coincide and so (H_2) follows.

Now, always taking condition (H_1) for granted, it is finally time for us to enunciate the following

Corollary 3.5.5. *The ideal generated by the Plücker relations with $s = d+1$ and $a_1 \leq a_d$, $b_1 \leq b_d$ is the defining ideal of the Grassmannian $G(d,n)$ in its Plücker embedding.*

This follows easily from Theorem 3.5.1 and Proposition 3.3.4. In fact, in Section 3.2, we defined the ring homomorphism

$$\begin{aligned} \phi_{d,n} : K[\mathbf{p}] &\longrightarrow K[X] \\ p_\sigma &\longmapsto \det(X_\sigma), \end{aligned}$$

by introducing a new set of indeterminates $\mathbf{p} = \{p_\sigma\}$, one for every maximal minor $det(X_\sigma)$, where X is a $d \times n$ generic matrix. Always in Section 3.2, we proved that the ideal generated by the Plücker relations is contained in $\ker(\phi_{d,n})$. What Proposition 3.3.4 assures us, since by Theorem 3.5.1 the $G(X)$ is actually an ASL on the set of maximal minors, is that $\ker(\phi_{d,n})$ is generated by the elements representing the straightening relations. Since, as we showed above, the straightening relations are obtained by iterated applications of the Plücker relations (with the given parameters) one gets the thesis.

Now that we know what $\ker(\phi_{d,n})$ is, it's finally time for us to show that the the Grassmannian in its Plücker embedding is Zariski closed. As we already said, instead of proving it in the general case, we just show the special case in which $n = 4$ and $d = 2$. With the notation above, consider $\ker(\phi_{2,4}) \subset K[p_{\sigma_1}, \dots, p_{\sigma_6}]$. We know that it is generated by the elements of $K[\mathbf{p}]$ representing the straightening relations. From Section 3.5, we know that when $n = 4$ and $d = 2$ we have a single straightening relation:

$$[1\ 2][3\ 4] - [1\ 3][2\ 4] + [1\ 4][2\ 3] = 0.$$

If we set $\sigma_1 = \{1,2\}$, $\sigma_2 = \{1,3\}$, $\sigma_3 = \{1,4\}$, $\sigma_4 = \{2,3\}$, $\sigma_5 = \{2,4\}$, $\sigma_6 = \{3,4\}$, we get

$$\ker(\phi_{2,4}) = (p_{\sigma_1}p_{\sigma_6} - p_{\sigma_2}p_{\sigma_5} + p_{\sigma_3}p_{\sigma_4}).$$

Now, let $a = (a_1, \dots, a_6)$ be a point in $\mathbb{P}^5$ that satisfies the relation generating $\ker(\phi_{2,4})$: $a_1a_6 - a_2a_5 + a_3a_4 = 0$. Without loss of generality we can assume $a_1 = 1$. We want to show that a is the vector of maximal minors of a 2×4 matrix A of rank 2, so that a is a point in $G(2,4)$. Consider

$$A = \begin{pmatrix} 1 & 0 & -a_4 & -a_5 \\ 0 & 1 & a_2 & a_3 \end{pmatrix}.$$

Clearly $\text{rank}(A) = 2$ since the leftmost 2×2 submatrix of A is the identity. The vector of maximal minors of A is

$$(\det(A_{\sigma_i}))_{i=1,\dots,6} = (1, a_2, a_3, a_4, a_5, -a_3a_4 + a_2a_5) = (1, a_2, a_3, a_4, a_5, a_6) = a.$$

Hence a is a point in $G(2,4)$. By arbitrariety of a, we conclude that $G(2,4)$ is Zariski closed.

3.6 Linear Independence of Standard Bitableaux

To make sure that the good results we have given are true, we still have to prove (H_1), namely that products of comparable maximal minors (that, as we saw, one can represent as squared standard bitableaux) are linearly independent over K. We will prove something more, that is, the linear independence of all standard bitableaux, that clearly will imply the linear independence of the "squared" ones.

The tool we are going to use will be the *Robinson–Schensted–Knuth correspondence*, that sets up a bijection between standard bitableaux and monomials in the ring $K[X]$.

The passage from bitableaux to monomials is done through the *deletion* algorithm.

Definition 3.6.1. Let $A = (a_{ij})$ be a standard tableau of shape $(s_1, s_2, \dots)$ and let p be an index such that $s_p > s_{p+1}$. *Deletion* constructs a standard tableau B and a number x as follows:

1. Define the sequence $k_p, k_{p-1}, \dots, k_1$ by setting $k_p = s_p$ and choosing k_i for $i < p$ to be the largest integer $\leq s_i$ such that $a_{ik_i} \leq a_{i+1,k_{i+1}}$;
2. Define B to be the standard tableau obtained from A by
   (i) removing a_{ps_p} from the p-th row;
   (ii) replacing the entry a_{ik_i} of the i-th row by $a_{i+1,k_{i+1}}$, $i = 1, \dots, p-1$.
3. Set $x = a_{1k_1}$.

Let us look at an example to understand how this algorithm actually works:

Example 3.6.2. Consider A as the following tableau of shape $(4, 3, 3, 2)$

1	3	4	9
1	4	5	
2	6	8	
2	7		

and choose $p = 4$. We now have to define k_i for $i = 1, \dots, 4$.

$(i = 4)$ Since $p = 4$, k_4 is forced to be $s_4 = 2$;

$(i = 3)$ k_3 is defined as the largest integer $\leq s_3 = 3$ such that $a_{3k_3} \leq a_{4k_4} = a_{42} = 7$. Hence $k_3 = 2$;

$(i = 2)$ Again, k_2 has to be $\leq s_2 = 3$ and the largest integer such that $a_{2k_2} \leq a_{3k_3} = a_{32} = 6$. Thus $k_2 = 3$.

$(i = 1)$ $k_1 \leq s_1 = 4$ and is the largest integer such that $a_{1k_1} \leq a_{2k_2} = 5$. We conclude that $k_1 = 3$.

Now we can go to step 2. (whose affected entries are marked in gray): we remove $a_{42} = 7$ from the fourth row and for $i = 1, 2, 3$ we replace a_{ik_i} by $a_{i+1,k_{i+1}}$. So $4 = a_{1k_1}$ is replaced by $a_{2k_2} = 5$, $5 = a_{2k_2}$ by $a_{3k_3} = 6$ and finally $6 = a_{3k_3}$ by $a_{pk_p} = 7$. So we obtain the standard tableau B:

1	3	5	9
1	4	6	
2	7	8	
2			

In the end, in step 3. we simply set $x = a_{1k_1} = a_{13} = 4$.

Basically what we did is to push the entry a_{ps_p} to the row $p - 1$, such that it pushes itself the largest entry $\leq a_{ps_p}$ to the row above and so on. This way, we lose the largest entry of the first row $\leq a_{2k_2}$ which is, however, encoded in the variable x.

Note that the deletion algorithm actually produces a standard tableau with the same shape of A but with the p-th row shorter by one entry. The condition $s_p > s_{p+1}$ assures us that removing one entry from the p-th row will not be in contrast with the definition of tableau, and the fact that one always push to the row above an entry larger than the one that is going to replace ensures the resulting tableau is standard.

By reversing the process, one can easily construct an inverse to deletion:

Definition 3.6.3. *Insertion* takes a standard tableau $A = (a_{ij})$ of shape $(s_1, s_2, \dots)$ and an integer x, and constructs a standard tableau B and an index p as follows:

1. Set $i = 1$ and $B = A$;

2. If $s_i = 0$ or $x > a_{is_i}$, then add x at the end of the i-th row of B, set $p = i$ and terminate;

3. Otherwise, let k_i be the smallest j such that $x \leq a_{js_i}$, replace $b_{k_is_i}$ with x, set $x = a_{k_is_i}$ and $i = i + 1$. Then go to 2.

As before, one easily sees that the output of insertion algorithm is a standard tableau with the same shape of A, except that the p-th row of B is longer by one entry.

Now we are ready to start constructing the Robinson–Schensted–Knuth (RSK, for short) correspondence. If one starts from bitableaux, the RSK correspondence is constructed from the deletion algorithm.

Let $\Sigma = (A \,|\, B) = (a_{ij} \,|\, b_{ij})$ a non empty standard bitableau. Then the *RSK-step* costructs a pair of integers (ℓ, r) and a standard bitableau Σ' as follows:

1. Choose the largest entry ℓ in the left tableau of Σ; suppose that $\{(i_1, j_1), \ldots, (i_u, j_u)\}$, $i_1 < \cdots < i_u$, is the set of indices (i, j) such that $\ell = a_{ij}$. Set $p = i_u$ and $q = j_u$. We call the pair (p, q) the *pivot position*;

2. let A' be the standard tableau obatined by removing a_{pq} from A;

3. apply deletion to the pair (B, p), obtaining a standard tableau B' and an element r;

4. set $\Sigma' = (A' \mid B')$.

Now it is possible to define RSK recursively: let Σ be a nonempty standard bitableau of shape $s_1, \ldots, s_p$. Set $k = s_1 + \cdots + s_p$ and define the two line array

$$RSK(\Sigma) = \begin{pmatrix} \ell_1 & \cdots & \ell_k \\ r_1 & \cdots & r_k \end{pmatrix},$$

where the pair (ℓ_i, r_i) is constructed, together with the bitableau Σ_{i-1}, by applying the RSK-step to Σ_i, starting from $\Sigma_k = \Sigma$.

Again, we give an example to clarify how the algorithm works.

Example 3.6.4. Consider Σ as the bitableau

5	4	3	1		1	2	3	6
		6	2		4	5		

We have $k = 4 + 2 = 6$.

The largest entry of the left tableau is $6 = \ell_6$ (marked in gray), and obviously the pivot position is $(p, q) = (2, 2)$ (remember that, when speaking of bitableaux, we read the indices of the left tableau from right to left). Now, applying steps 2 and 3 (namely deletion to (B, p); the affected entries are marked in gray as well) to Σ=Σ_6, we get the bitableau Σ_5:

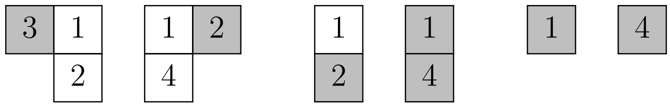
5	4	3	1		1	2	5	6
			2		4			

and the integer $r_6 = 3$.

Now we apply the algorithm again, this time to Σ_5: $\ell_5 = 5$, and the pivot position is $(1, 4)$. Since $p = 1$, deletion here does nothing but removing the entry a_{1s_1}, and we get $r_5 = 6$ and Σ_4 as

4	3	1		1	2	5
		2		4		

Reiterating the algorithm until $i = 1$ we obtain $r_4 = 5$, $r_3 = 2$, $r_2 = 1$ and $r_1 = 4$:



Therefore

$$RSK(\Sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 5 & 6 & 3 \end{pmatrix}.$$

Note that in both rows of $RSK(\Sigma)$ indices may appear several times. However, they satisfy two properties:

(a) $\ell_i \leq \ell_{i+1}$ for all $i = 1, \dots, k-1$;

(b) if $\ell_i = \ell_{i+1}$, then $r_i \geq r_{i+1}$.

(a) is clear since by definition the algorithm chooses $\ell_{i+1} \geq \ell_i$. If $\ell_i = \ell_{i+1}$, $p_{i+1} > p_i$: this means that the pivot position corresponding to ℓ_{i+1} lies below the one corresponding to ℓ_i, and so deletion applied to (B, p_{i+1}) in the right tableau produces an integer less or equal to the one obtained in the next step.

By now, what we constructed through RSK is a correspondence between standard bitableaux and two-line arrays with properties (a) and (b). Note that this correspondence is actually bijective since deletion algorithm has an inverse: at step i, to construct the left tableau, one applies insertion algorithm to (Σ_{i-1}, r_i) (starting from $[\,|\,], r_1)$. Simultaneously one constructs the right tableau by placing ℓ_i in the position which is added to the left tableau by the i-th insertion.

It remains just to explain how to go from two-line arrays satisfying (a) and (b) to monomials in $K[X]$. This is simply done as follows:

$$\begin{pmatrix} \ell_1 & \cdots & \ell_k \\ r_1 & \cdots & r_k \end{pmatrix} \longleftrightarrow X_{\ell_1 r_1} \cdots X_{\ell_k r_k}.$$

Clearly, if we are given a monomial, by arranging its factors in the following order:

$$X_{ij} \geq X_{st} \Longleftrightarrow i \leq s \text{ or } i = s \text{ and } j \geq t,$$

there is always a unique way to represent it as a two rowed array satisfying the conditions above. Therefore, we actually established a bijection between monomials and standard bitableaux. Looking at the specific case we are interested in, we get

Theorem 3.6.5. *Let X be a $d \times n$ matrix of indeterminates. Then the map RSK is a bijection between the monomials of $K[X]$ and the set of standard bitableaux where the right tableau has entries in $\{1, \dots, d\}$ and the left one in $\{1, \dots, n\}$.*

Hence, we finally proved the linear independence of standard bitableaux and, consequently, condition (H_1) for the Grassmannian: in fact, we needed to prove that products of comparable maximal minors, that are nothing but the standard monomials appearing in condition (H_1), are linearly independent over K. As we told at the start of the current section, products of comparable maximal minors can be represented as squared standard bitableaux, which turned out to be linearly independent over K by Theorem 3.6.5. Hence we proved (H_1) for the Grassmannian, and this finally finishes the proof of Theorem 3.5.1.

Chapter 4

Minors as SAGBI Bases

4.1 Diagonal Monomial Orders

Let K be a field and $X = (X_{ij})$ a $d \times n$, $d \leq n$, matrix of indeterminates. We denote by $\mathcal{M}$ the set of maximal minors of X, and consider the subalgebra $K[\mathcal{M}]$ of $K[X] = K[X_{ij} : i = 1, \dots, d,\ j = 1, \dots, n]$. From Chapter 3, we know that $K[\mathcal{M}]$ is the homogeneous coordinate ring of the Grassmannian in its Plücker embedding, namely the Plücker algebra, which we denoted $G(X)$. Throughout this chapter, we will maintain this notation.

First of all note that, applying Proposition 3.3.3 to the Grassmannian, we get

Corollary 4.1.1. *The standard bitableaux in $G(X)$ form a K-basis of $G(X)$.*

We want to study when $\mathcal{M}$ is a SAGBI basis of $G(X)$. The first step is obviously to define a monomial order on the ring $K[X]$. A natural and easy choice could be the lexicographic order with

$$X_{11} > \cdots > X_{1n} > X_{21} > \cdots > X_{2n} > \cdots > X_{d1} > \cdots X_{dn}.$$

It is a *diagonal* monomial order. We already met this definition in Example 2.2.2: it means that the product of the indeterminates in the main diagonal is the initial monomial of each minor.

We introduce a specific notation for diagonals, given the important role they play in this Section. If $\delta = [a_1 \dots a_t \,|\, b_1 \dots b_t]$ is a minor of X, we let

$$\langle \delta \rangle = \langle a_1 \dots a_t \,|\, b_1 \dots b_t \rangle = \prod_{i=1}^{t} X_{a_i b_i},$$

and for a bitableau $\Delta = \delta_1 \cdots \delta_w$ we set

$$\langle \Delta \rangle = \langle \delta_1 \cdots \delta_w \rangle = \prod_{j=1}^{w} \langle \delta_j \rangle.$$

If one chooses a diagonal monomial order on $K[X]$, clearly one has $\text{in}(\delta) = \langle\delta\rangle$ and $\text{in}(\Delta) = \langle\Delta\rangle$.

Proposition 4.1.2. *Let X be a $d \times n$ matrix of indeterminates and let $K[X]$ be endowed with a diagonal monomial order. Then the following hold:*

(a) Let the standard bitableaux Σ, Σ' be products of maximal minors of X . If $\Sigma \neq \Sigma'$, then $\langle\Sigma\rangle \neq \langle\Sigma'\rangle$;

(b) if $f \in G(X)$, $f \neq 0$, then there exists a standard bitableau $\Sigma \in G(X)$ such that $\text{in}(f) = \langle\Sigma\rangle$.

Proof. (a). If Σ and Σ' have different shapes (that is, one of them has more factors than the other), clearly also $\langle\Sigma\rangle$ and $\langle\Sigma'\rangle$ will have a different number of factors and we are done.

Let now $\Sigma = \delta_1 \cdots \delta_r$, $\Sigma' = \delta'_1 \cdots \delta'_r$ and suppose $\langle\Sigma\rangle = \langle\Sigma'\rangle =: \mu$. Then one can factorize μ as $\mu_1 \cdots \mu_d$, where μ_i collects all indeterminates from row i appearing in μ. Let X_{ic_i} be the factor of μ_i with the smallest column index. Then, since $\delta_1 \preceq \delta_j$ for all $j \neq 1$, δ_1 is forced to be $[c_1 \dots c_d]$ and so is δ'_1. Then we are done by induction on d.

(b). We know by Corollary 4.1.1 (actually, just from Lemma 3.5.3) that f is a K-linear combination of standard bitableaux $\Sigma_i \in K[\mathcal{M}]$. From (a) it follows that, for every $i \neq j$, $\langle\Sigma_i\rangle \neq \langle\Sigma_j\rangle$ and so the largest monomial among the $\text{in}(\Sigma_i) = \langle\Sigma_i\rangle$ is the initial monomial of f. □

Note that this Proposition implies directly the linear independence of standard bitableaux in $K[\mathcal{M}]$. In fact, suppose that there exists a K-linear combination of Σ_i that is equal to 0. Since K is a field, we can write Σ_1 as a K-linear combination of Σ_i, $i \geq 2$. At this point, as in the proof of (b), one obtains $\text{in}(\Sigma_1) = \text{in}(\Sigma_i)$ for some $i \neq 1$ and thus $\langle\Sigma_1\rangle = \langle\Sigma_i\rangle$, that is a contradiction by (a).

Another immediate consequence is the following

Theorem 4.1.3. *Let K be a field and X a $d \times n$, $d \leq n$, matrix of indeterminates. Consider a diagonal monomial order on the polynomial ring $K[X]$. Then the set $\mathcal{M}$ of maximal minors of X is a SAGBI basis of $G(X)$.*

Proof. Let $f \in G(X)$. By Proposition 4.1.2(b), we know that there exists a standard bitableau $\Sigma = \delta_1 \cdots \delta_w$, $\delta_i \in \mathcal{M}$, such that $\text{in}(f) = \langle\Sigma\rangle = \prod_{i=1}^{w}\langle\delta_i\rangle$. Since $K[X]$ is endowed with a diagonal monomial order, $\langle\delta_i\rangle = \text{in}(\delta_i)$, and so $\text{in}(f)$ belongs to the subalgebra $K[\text{in}(\mathcal{M})]$. We conclude by arbitrariety of f. □

At this point, there are at least two questions that come naturally: are the maximal minors always a SAGBI basis of $G(X)$, regardless of the monomial order? And can we generalize Theorem 4.1.3 to minors that are not maximal? Let us answer the second question first.

4.2 Minors of Arbitrary Size

Let us put ourselves in the assumptions of Theorem 4.1.3: K is a field, X a $d \times n$ matrix, $d \leq n$, and $K[X]$ is endowed with a diagonal monomial order. Now we introduce a notation for the algebra of minors

$$A_t = K[\mathcal{M}_t],$$

where $\mathcal{M}_t$ denotes the set of minors of X of size t. We already discussed the case $t = d$, and there is nothing to say about the case $t = 1$, in which $A_1 = K[X]$ and $\mathcal{M}_1 = \{X_{ij} : i = 1, \ldots, d,\ j = 1, \ldots, n\}$. For all the other cases, we have the following

Proposition 4.2.1. *If* $t < d$, *then* $\dim(A_t) = \dim(K[X]) = dn$.

Proof. We know that the Krull dimension of an affine domain over K (namely an integral domain that is finitely generated as a K-algebra) is the transcendence degree of its fraction field over K (see [10], Theorem 5.6). With the usual notation $K(X) = frac(K[X])$, we have the fields extension

$$\begin{array}{c} K(X) \\ | \\ K(A_t) \\ | \\ K \end{array}$$

Therefore, by additivity of transcendence degrees, we have

$$\operatorname{tr.deg}_K(K(X)) = \operatorname{tr.deg}_{K(A_t)}(K(X)) + \operatorname{tr.deg}_K(K(A_t)),$$

that we reformulate as

$$\dim(K[X]) = \operatorname{tr.deg}_{K(A_t)}(K(X)) + \dim(A_t).$$

So our claim is $\operatorname{tr.deg}_{K(A_t)}(K(X)) = 0$. In other words, we have to prove that the indeterminates X_{ij} are algebraic over $K(A_t)$. The general case follows from the one in which $d = n = t + 1$. Therefore let us suppose X is a $(t+1) \times (t+1)$ generic matrix. The *adjoint* matrix $\tilde{X}$ is the $(t+1) \times (t+1)$ matrix whose entry at position (i, j) is the t-minor

$$(-1)^{i+j}[1 \ldots i-1\ i+1 \ldots t+1 \,|\, 1 \ldots j-1\ j+1 \ldots t+1] \in A_t.$$

By construction, $X\tilde{X}$ is the diagonal matrix with $\det(X)$ in all diagonal entries, it follows that $\det(\tilde{X}) = \frac{\det(X)^{t+1}}{\det(X)} = \det(X)^t$. Since $\tilde{X}$ has entries in A_t, we get that $\det(\tilde{X}) \in A_t$ and therefore it follows that $\det(X)$ is algebraic over $K(A_t)$. Now set $D = \det(X)$, $L = K(A_t)$ and $E = L[D]$. Since D is algebraic over L, E is a field. Now, X^{-1} has entries in E, $\det(X^{-1}) \neq 0$ and therefore the entries of $X = (X^{-1})^{-1}$ are in E. Therefore $K(X) = E$. □

Note that, if $d = n$ and $t = n - 1$, A_t is isomorphic to a polynomial ring. In fact, there are nd t-minors $m_1, \dots, m_{nd}$, so, from universal property of K-algebras, we know that

$$A_t = K[m_1, \dots, m_{nd}] \cong K[y_1, \dots, y_{nd}]/\mathfrak{p},$$

where $K[y_1, \dots, y_{nd}]$ is a polynomial ring and $\mathfrak{p}$ is prime since A_t is an integral domain. Since A_t has dimension nd by the previous proposition, A_t and $K[y_1, \dots, y_{nd}]$ have the same dimension. Therefore $ht(\mathfrak{p})$ is forced to be 0: otherwise one would have $\dim(A_t) \leq \dim(K[y_1, \dots, y_{nd}]) - 1$. Since (0) is prime in $K[y_1, \dots, y_{nd}]$, $\mathfrak{p} = (0)$ and we get $A_t \cong K[y_1, \dots, y_{nd}]$.

Example 4.2.2. Set $d = n = 3$ and $t = 2$. As we have just seen, in this case A_2 is isomorphic to a polynomial ring of dimension 9. We deduce from this that the 2-minors are not a SAGBI basis with respect to a diagonal monomial order. By contradiction, assume that the diagonals of the 2-minors generate $\text{in}(A_2)$, namely the 2-minors are a SAGBI basis of A_2. Since A_2 is a (standard, if we give degree 1 to the 2-minors) graded algebra and, by our assumption, $\text{in}(A_2)$ is finitely generated over K, we know from Theorem 2.4.5 that $\dim(A_2) = \dim(\text{in}(A_2))$. Therefore the diagonals of the 2-minors need to be algebraically independent: they are nine and the algebra they generate has Krull dimension 9. But this is not true, since we can find a nontrivial relation between the diagonals, for example:

$$\langle 1\ 2\,|1\ 3\rangle\langle 1\ 3\,|2\ 3\rangle - \langle 1\ 3\,|1\ 3\rangle\langle 1\ 2\,|2\ 3\rangle = X_{11}X_{23}X_{12}X_{33} - X_{11}X_{33}X_{12}X_{23} = 0.$$

Alternatively, one can note that the diagonals of the 2-minors do not involve all the nine indeterminates, but only seven of them: in fact, the variables on the antidiagonal corners of X, that is, X_{13} and X_{31}, do not appear in any of the diagonals. Hence the algebra generated by the diagonals of the 2-minors is a subalgebra of a polynomial ring in seven indeterminates, and so it cannot have Krull dimension 9. We conclude that the 2-minors of a 3×3 matrix are not a SAGBI basis of A_2 with respect to the diagonal monomial order.

Using the same Krull dimension argument of Example 4.2.2, one shows

Theorem 4.2.3. *Let X be a $d \times n$ matrix of indeterminates. Then the set of t minors of X, $1 < t < d$, is not a SAGBI basis of A_t with respect to any diagonal monomial order.*

This gives us the answer to our question: it is not possible to generalize Theorem 4.1.3 to minors of arbitrary size, it fails already when speaking of small matrices like we have seen in the previous example.

At this point, one might wonder if there exists a monomial order for which the t-minors of X, $1 < t < d$, are a SAGBI basis of A_t. Consider the case in which X is a $n \times n$ matrix of indeterminates and $t = n-1$. We start by seeking a monomial order that can work for the case $n = 3$ and $t = 2$. As a first step, note that in order not to run into the same problem of Example 4.2.2, we need a monomial order involving all nine indeterminates in the leading terms of the 2-minors: this way we can hope that $K[\mathrm{in}(\mathcal{M}_2)]$ has Krull dimension 9. To achieve this goal, we would like to identify a term order with the following properties:

1. every indeterminate should appear at least in one initial term of a 2-minor;
2. most of the indeterminates appear in exactly one initial term so that the latter cannot be involved in any algebraic relation among initial terms.

A good candidate could be Lex where the bigger indeterminates are the one appearing in the main diagonal of X, for example Lex with

$$X_{11} > X_{22} > X_{33} > \cdots , \qquad (\tau)$$

where the order in which all the other indeterminates appear is irrelevant.

With respect to this monomial order, we have:

$$\begin{array}{lll} \mathrm{in}[1\,2\,|\,1\,2] = X_{11}X_{22} & \mathrm{in}[1\,2\,|\,2\,3] = X_{13}X_{22} & \mathrm{in}[1\,2\,|\,1\,3] = X_{11}X_{23} \\ \mathrm{in}[1\,3\,|\,1\,2] = X_{11}X_{32} & \mathrm{in}[1\,3\,|\,1\,3] = X_{11}X_{33} & \mathrm{in}[1\,3\,|\,2\,3] = X_{12}X_{33} \\ \mathrm{in}[2\,3\,|\,1\,2] = X_{22}X_{31} & \mathrm{in}[2\,3\,|\,1\,3] = X_{21}X_{33} & \mathrm{in}[2\,3\,|\,2\,3] = X_{22}X_{33} \end{array}$$

Note that six of the nine initial terms involve an indeterminate that does not appear in any of the others, as we wanted. Hence none of them can satisfy any relation of algebraic dependence with any of the other eight. To prove that $K[\mathrm{in}(\mathcal{M}_2)]$ has Krull dimension 9, it remains to show that $X_{11}X_{22}$, $X_{11}X_{33}$ and $X_{22}X_{33}$ are algebraically independent. Note that $S = K[X_{11}X_{22}, X_{11}X_{33}, X_{22}X_{33}] \subseteq K[X_{11}, X_{22}, X_{33}]$ is an integral extension of rings. In fact:

- X_{11} satisfies the polynomial $t^2X_{22}X_{33} - X_{11}X_{22} \cdot X_{11}X_{33} \in S[t]$;
- X_{22} satisfies the polynomial $t^2X_{11}X_{33} - X_{11}X_{22} \cdot X_{22}X_{33} \in S[t]$;
- X_{33} satisfies the polynomial $t^2X_{11}X_{22} - X_{11}X_{33} \cdot X_{22}X_{33} \in S[t]$.

Therefore, by property of integral extensions:

$$\dim(K[X_{11}X_{22}, X_{11}X_{33}, X_{22}X_{33}]) = \dim(K[X_{11}, X_{22}, X_{33}]) = 3.$$

Hence we can conclude that $X_{11}X_{22}$, $X_{11}X_{33}$ and $X_{22}X_{33}$ are algebraically independent. Moreover, we have that $\dim(K[\mathrm{in}(\mathcal{M}_2)]) = 9$ and, since there are no binomial relations among the initial terms to be lifted, by Theorem 2.3.1, namely the SAGBI criterion, we conclude that the 2-minors are a SAGBI basis of A_3 with respect to the monomial order τ.

The same strategy works for the general case: consider a $n \times n$ matrix X and the set $\mathcal{M}_{n-1}$ of the $n-1$-minors. For the same reasons as above, choose as monomial order Lex with $X_{11} > X_{22} > \cdots > X_{nn} > \cdots$, where again the order in which the remaining indeterminates appear is irrelevant. We want to show that the initial terms of the $n-1$-minors are algebraically independent, so we can conclude again by the SAGBI criterion that the latter are a SAGBI basis of A_{n-1} with respect to the monomial order we introduced above. As it happened for $n = 3$, all indeterminates (except the one on the main diagonal of X) appear just one time in the initial terms of the $n-1$-minors: in fact, X_{ij}, $i \neq j$, appears in an initial monomial just when it appears with $\prod_{k\neq i,j} X_{kk}$. Therefore, what remains to be shown is that the products $X_{11} \cdots \hat{X}_{ii} \cdots X_{nn}$, as i varies in $\{1, \dots, n\}$, are algebraically independent. For simplicity, since we are working just with variables on the main diagonal of X, let us denote X_{ii} as X_i. We will show that the following rings extension

$$S = K[X_1 \cdots \hat{X}_i \cdots X_n \mid i = 1, \dots, n] \subseteq K[X_1, \dots, X_n]$$

is an integral extension. Fix $j \in \{1, \dots, n\}$, and consider X_j. Then X_j satisfies the polynomial

$$t^{n-1}(X_1 \cdots \hat{X}_j \cdots X_n)^{n-2} - \prod_{i\neq j} X_1 \cdots \hat{X}_i \cdots X_n \in S[t].$$

Therefore every X_j, $j = 1, \dots, n$, is algebraic over S: this implies that $\dim(S) = \dim(K[X_1, \dots, X_n]) = n$ and so the products $X_1 \cdots \hat{X}_i \cdots X_n$, as i varies in $\{1, \dots, n\}$, are algebraically independent, proving our claim.

With our discussion we have shown

Theorem 4.2.4. *Consider a $n \times n$ generic matrix of indeterminates X. Then the $n-1$-minors of X are a SAGBI basis of A_{n-1} with respect to the Lex order associated to the total order $X_{11} > X_{22} > X_{33} > \cdots$, where the order in which the other indeterminates appear is irrelevant.*

Now that we proved Theorem 4.2.4, we want to go back to Example 4.2.2 and find an actual SAGBI basis for A_2 with respect to a diagonal monomial order. In order to examine this case properly, we need a lemma for the proof of which we refer to [8], Lemma 10.10.

Lemma 4.2.5. *Let $u, v \in \mathbb{N}$, $0 \leq u \leq v-2$, and suppose that* $\operatorname{char} K = 0$. *Then $I_u I_v \subseteq I_{u+1} I_{v-1}$.*

In the terminology of the lemma, given an integer t, I_t denotes the ideal of $K[X]$ generated by the t-minors of the matrix X. The first non trivial case is the one in which $u = 1$ and $v = 3$: the lemma states that the product of a 1-minor, namely and indeterminate, and a 3-minor must be in I_2^2.

Recall the setting of Example 4.2.2 more precisely: X is a 3×3 matrix of indeterminates and the polynomial ring $K[X]$ is endowed with a diagonal monomial order. We are interested in finding a SAGBI basis for A_2, that is, the algebra generated by the 2-minors of X. We observed that the nine diagonals of the 2-minors only involve seven indeterminates out of nine: X_{13} and X_{31} are missing. This was the reason why the 2-minors could not form a SAGBI basis of A_2.

Now, since X is a 3×3 matrix, Lemma 4.2.5 assures us that $\det(X)\cdot X_{ij}$ is in I_2^2 for every $i, j \in \{1, 2, 3\}$. Since $\det(X)\cdot X_{ij}$ is an homogeneous polynomial of degree 4, it actually belongs to $K[\mathcal{M}_2] = A_2$. Therefore the products of $\text{in}(\det(X))$, namely the main diagonal of X, with X_{13} and X_{31} both belong to $\text{in}(A_2)$, but they cannot be written as combinations of the diagonals of the 2-minors for the reason explained above. We now want to prove that adding these two products to the set of the 2-minors gives a SAGBI basis of A_2. Instead of doing it using the SAGBI Criterion as in Theorem 4.2.4, this time we will use Hilbert series. Let us give a name to the diagonals of the nine 2-minors (and thus to the nine 2-minors themselves)

$$\begin{array}{lll} \langle m_1\rangle = X_{11}X_{22} & \langle m_2\rangle = X_{12}X_{23} & \langle m_3\rangle = X_{21}X_{32} \\ \langle m_4\rangle = X_{22}X_{33} & \langle m_5\rangle = X_{11}X_{23} & \langle m_6\rangle = X_{21}X_{33} \\ \langle m_7\rangle = X_{12}X_{33} & \langle m_8\rangle = X_{11}X_{32} & \langle m_9\rangle = X_{11}X_{33}, \end{array}$$

and let us set $m_{10} = \det(X) \cdot X_{13}$ and $m_{11} = \det(X) \cdot X_{31}$. Therefore we have

$$\langle m_{10}\rangle = X_{11}X_{22}X_{33}X_{13} \quad \langle m_{11}\rangle = X_{11}X_{22}X_{33}X_{31}.$$

We call $\mathcal{F}$ the family $\{m_1, \ldots, m_{11}\}$ and we want to present $K[\text{in}(\mathcal{F})]$ as a quotient of a polynomial ring. As we did in Section 2.2, we introduce the algebras homomorphism

$$\begin{aligned} \psi : K[Y_1, \ldots, Y_{11}] &\longrightarrow K[\text{in}(\mathcal{F})] \\ Y_i &\longmapsto \langle m_i\rangle. \end{aligned}$$

Note that the grading structure induced on $K[Y_1, \ldots, Y_{11}]$ by the m_i is not standard. In fact, working with normalized degree and so giving degree 1 to the 2-minors of X, $m_1, \ldots, m_9$ have degree 1 while m_{10} and m_{11} have degree 2. Therefore $\deg(Y_1) = \cdots = \deg(Y_9) = 1$ and $\deg(Y_{10}) = \deg(Y_{11}) = 2$.

We are interested in finding the kernel of ψ, that we know is generated by binomials by Proposition 2.2.4. Clearly Y_{10} and Y_{11} cannot appear in any binomial relation in $\ker(\psi)$ since $\langle m_{10}\rangle$ and $\langle m_{11}\rangle$ contain an indeterminate, respectively X_{13} and X_{31}, that does not appear in any other monomial among the diagonals of the m_i. To understand $\ker(\psi)$ we look for relations of the type

$$\langle m_i\rangle\langle m_j\rangle - \langle m_k\rangle\langle m_l\rangle,$$

where $\{i, j, k, l\} \in \{1, \ldots, 9\}$. Therefore we need to find products of four indeterminates that "cross" well with each other, for example

$$X_{11}X_{23}X_{12}X_{33}.$$

In fact, pairing the first two indeterminates and the last two, one obtains the product between $\langle m_5 \rangle$ and $\langle m_7 \rangle$, while pairing the first with the last and the second with the third one obtains a different product of diagonals, namely $\langle m_9 \rangle \langle m_2 \rangle$. Thus

$$Y_5 Y_7 - Y_9 Y_2 \in \ker(\psi).$$

Applying this argument, one sees that there's only one other relation of the type above, that is

$$\langle m_8 \rangle \langle m_6 \rangle - \langle m_3 \rangle \langle m_9 \rangle.$$

Therefore we conclude that

$$(Y_5 Y_7 - Y_9 Y_2,\ Y_8 Y_6 - Y_3 Y_9) \subseteq \ker(\psi).$$

Actually one can prove that the equality holds, and this gives us the presentation of $K[\text{in}(\mathcal{F})]$ we wanted:

$$K[\text{in}(\mathcal{F})] \cong K[Y_1, \ldots, Y_{11}] / (Y_5 Y_7 - Y_9 Y_2,\ Y_8 Y_6 - Y_3 Y_9).$$

We now want to compare the Hilbert series of A_2 and $K[\text{in}(\mathcal{F})]$, in order to use Lemma 2.4.2. We already observed that when $d = n = 3$ A_2 is isomorphic to a polynomial ring in nine indeterminates of normalized degree 1, therefore its Hilbert series is

$$HS_{A_2}(t) = \frac{1}{(1-t)^9}.$$

Set now $S = K[Y_1, \ldots, Y_{11}]$, $B = K[\text{in}(\mathcal{F})]$, $\beta_1 = Y_5 Y_7 - Y_9 Y_2$ and $\beta_2 = Y_8 Y_6 - Y_3 Y_9$. We need to find the Hilbert series of B, or equivalently the one of $S/(\beta_1, \beta_2)$. In order to compute the latter, we first need to show that β_1, β_2 is a S-regular sequence. Clearly, since S is an integral domain and β_1 is nonzero, β_1 cannot be a zero divisor in S and therefore is a S-regular element. Now, from the fact that B is an integral domain we deduce that the ideal $(\beta_1, \beta_2) \subset S$ is prime. Thus we can assume that both β_1 and β_2 are irreducible and therefore prime since S is a UFD. This implies that β_2, being nonzero in $S/(\beta_1)$ that is a domain, cannot be a zero divisor in $S/(\beta_1)$. This proves that β_1, β_2 is a S-regular sequence.

Recall now that S is a polynomial ring in eleven variables, of which the first nine have degree 1 and the last two have degree 2. Therefore the Hilbert series of S has the following form:

$$HS_S(t) = \frac{1}{(1-t)^9 (1-t^2)^2}.$$

Since $\deg(\beta_1) = \deg(\beta_2) = 2$, multiplying the expression above by $(1-t^2)^2$ gives us the Hilbert series of $B = S/(\beta_1, \beta_2)$:

$$HS_B(t) = \frac{(1-t^2)^2}{(1-t)^9 (1-t^2)^2} = \frac{1}{(1-t)^9}.$$

Since $HS_B(t) = HS_A(t)$, we conclude by Lemma 2.4.2 that $\mathcal{F}$ is a SAGBI basis of A_2.

It makes sense to extend this argument in order to find a SAGBI basis of A_2 for $d = n = 3$ regardless of the monomial order. In fact, we observed that for every $i, j \in \{1, 2, 3\}$ $\det(X) \cdot X_{ij}$ belongs to A_2: if it happens, like in Example 4.2.2, that the diagonal of the 2-minors do not involve all of the nine indeterminates, we can make up the missing ones using the expression above. Clearly it is not enough to involve all indeterminates to be a SAGBI basis, but as we observed again in Example 4.2.2 it is a necessary condition. Our idea is that there may be two possible scenarios:

1. The set $\mathcal{M}_2$ is a SAGBI basis of A_2. We are sure that such a situation is feasible thanks to Theorem 4.2.4;
2. The set $\mathcal{M}_2$ is not a SAGBI basis of A_2 and the elements that we need to add to $\mathcal{M}_2$ in order to obtain a SAGBI basis of A_2 are of the form $\det(X) \cdot X_{ij}$ for some $i, j \in \{1, 2, 3\}$.

In order to support this, we did some work on CoCoA, using the characterization of leading monomials given by Lemma 2.5.4. First, we calculated the product F of all 2-minors of a 3×3 matrix of indeterminates X: this is useful to understand which monomials in the support of the 2-minors can appear as initial monomials with respect to some monomial order. After that:

1. We computed all monomials in $\operatorname{supp}(F)$, up to the action of $S_3 \times S_3$ on X. We found eight of them:

$$v_1 = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix} \quad v_2 = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 4 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 1 & 4 \end{pmatrix} \quad v_4 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

$$v_5 = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 3 \end{pmatrix} \quad v_6 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad v_7 = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad v_8 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

   We want to understand which of these eight are vertices of the Newton polytope P_F.

2. We deleted v_6, v_7 and v_8 since they appear more than once in the product F, thus they can't be vertices P_F.
3. We checked that the remaining five are actually vertices of P_F, and then we

computed the corresponding weight vectors:

$$\begin{aligned} w_1 &= [2,1,0,0,0,0,0,1,2], \\ w_2 &= [2,1,0,0,0,1,0,2,2], \\ w_3 &= [2,1,0,1,2,2,0,0,2], \\ w_4 &= [2,1,1,1,2,1,1,1,2], \\ w_5 &= [1,0,0,2,0,2,1,1,2]. \end{aligned}$$

Note that v_1 is the vertex corresponding to the diagonal monomial order of Example 4.2.2, while v_4 is the vertex corresponding to the lexicographic monomial order that we used to prove Theorem 4.2.4, namely the Lex one induced by the total order $X_{11} > X_{22} > X_{33} > \dots$ (where the order of the remaining inderminates is irrelevant). Moreover, observe that none other vertex except v_4 involves all nine variables. Since we know that this is a necessary condition to be a SAGBI basis, we just proved the following

Theorem 4.2.6. *Up to row and column permutations, there exists only one monomial order on $K[X]$ that makes the 2-minors of a 3×3 matrix of indeterminates a SAGBI basis of A_2.*

Now, with a quick test on CoCoA or Macaulay2 considering the monomial orders induced by w_1, w_2, w_3 and w_6 one sees that adding the set $\{\det(X) \cdot X_{ij} \mid i, j = 1, 2, 3\}$ to $\mathcal{M}_2$ gives a SAGBI basis of A_2 in all these four cases. Something interesting that we observed during this test:

- With respect to the monomial order induced by w_1 and w_2, the SAGBI basis has 11 elements.
- With respect to the monomial order induced by w_3, the SAGBI basis has 10 elements.
- With respect to the monomial order induced by w_5, the SAGBI basis has 12 elements.

Therefore the elements that we need to add to $\mathcal{M}_2$ in order to obtain a SAGBI basis of A_2 are exactly the elements $\{\det(X) \cdot X_{ij}\}$ corresponding to the indeterminates that are not involved in any initial monomial of the 2-minors.

Going back to our main discussion, we can finally state the following:

Theorem 4.2.7. *A universal SAGBI basis for the algebra generated by the set $\mathcal{M}_2$ of the 2-minors of a 3×3 generic matrix of indeterminates X is given by the set*

$$\mathcal{M}_2 \cup \{\det(X) \cdot X_{ij} \mid i, j = 1, 2, 3\}.$$

We now discuss an interesting consequence of Theorem 4.2.6, proving the following theorem

Theorem 4.2.8. *There is no monomial order on $K[X]$ such that the set $\mathcal{M}_2$ of the 2-minors of a $d \times n$ matrix, $d \geq 3$, $n \geq 4$, is a SAGBI basis of A_2.*

Proof. If there existed a monomial order $<$ such that the 2-minors of X were a SAGBI basis, it would have to work well when restricting to the 3×3 submatrices of X. By Theorem 4.2.6, this means that $<$, on every 3×3 submatrix of X, up to rows and columns permutations should behave as Lex where the highest indeterminates are those on the main diagonal. It is enough to prove the theorem for a 3×4 matrix. Let

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \end{pmatrix}.$$

Suppose that, when considering the first three columns, X_{11}, X_{22} and X_{33} are the three variables that appear 4 times in the leading terms of the 2-minors. Now consider the submatrix X_{124} obtained by taking the first, second and fourth column of X. Then X_{11} and X_{22} again have to appear four times, since in the previous case they appeared respectively with X_{22}, X_{32} and X_{11}, X_{31} that all belong to the submatrix we are considering. Therefore, since the indeterminates that appear four times have to be all in different rows and columns, X_{34} is forced to appear four times: with X_{11}, X_{22}, X_{12} and X_{21}. Now consider X_{134}. Again, for the same reason above, X_{11} and X_{33} have to appear four times and therefore X_{24} is forced to appear four times as well: with X_{11}, X_{31}, X_{12} and X_{32}. If we now go back to X_{124}, we see that both X_{24} and X_{34} have to appear four times, that is a contradiction since they are in the same column. □

4.3 Universality and Maximal Minors

We now want to answer our first question: are maximal minors a SAGBI basis of $G(X)$ for every monomial order? Namely, are the maximal minors a *universal* SAGBI basis for $G(X)$?

Let us start from the simplest situation: $d = 2$. This case is covered by the following

Proposition 4.3.1. *Up to row and column permutations there is only one monomial order, the diagonal one, for the 2-minors of a $2 \times n$ generic matrix of indeterminates X.*

Proof. First of all, note that the case $n = 2$ is trivial. Now, we proceed by induction on n. We call a variable X_{ij} *dominant* if every 2-minor containing X_{ij} has as its initial monomial the one containing X_{ij}. By induction, we just need to prove that there is a dominant variable. Assume there's not such a variable, and, without loss of generality, make the further assumption that the initial monomial of [1 2] is $X_{11}X_{22}$. Since X_{22} is not dominant, there exists a minor $[2\ c]$, $c \neq 1$, with initial monomial $X_{12}X_{2c}$. Therefore we can assume $c = 3$. Since X_{23} is not dominant, it exists a minor $[3\ d]$, $d \neq 2$, such that it has $X_{13}X_{2d}$ as initial monomial. Observe also that $d \neq 1$ is impossible. In fact, suppose $d = 1$. We know that

$$X_{11}X_{22} > X_{12}X_{21}, \quad X_{12}X_{23} > X_{13}X_{22}, \quad X_{13}X_{21} > X_{11}X_{23},$$

and this leads to a contradiction: the product of the left sides equals the one of the right sides. Therefore, we can assume $d = 4$ and proceed like that, shifting by one column at a time. In the end, since X_{2n} is not dominant, there exists a minor $[n\ m]$, $m \neq n-1$, such that it has $X_{1n}X_{2m}$ as initial monomial. But every $m < n-1$ yelds a contradiction as it happened before. Hence we conclude. □

Therefore, combining the previous Proposition with Theorem 4.1.3, we get our answer for the case $d = 2$.

Corollary 4.3.2. *The set $\mathcal{M}_2$ of 2-minors of a $2 \times n$ generic matrix of indeterminates X is a universal SAGBI basis of $G(X)$.*

Unfortunately, this is truly an exceptional case. Indeed, as one may have noticed, the proof of Proposition 4.3.1 strongly uses the fact that $d = 2$.

Let us go one step further and consider the case of $G(3, 6)$. Let us think of our 3×3 usual matrix of indeterminates X as two 3×3 blocks A and B one next to each other, that is

$$X = \left(A \mid B\right).$$

Consider now the main diagonal of A, where the indeterminates X_{11}, X_{22} and X_{33} appear, and set

$$X_{11} > X_{22} > X_{33}.$$

Doing the same for the block B we obtain

$$X_{14} > X_{25} > X_{36},$$

and finally we combine them:

$$X_{11} > X_{22} > X_{33} > X_{14} > X_{25} > X_{36}.$$

The monomial order we are going to consider is Lex induced by the chain of inequalities above, to be continued with all the other missing indeterminates in some prescribed order. Seen on the blocks A and B separately, this is the monomial order that we constructed in order to prove Theorem 4.2.4. Altough it worked well in that case, our computation on Macaulay2 (see Appendix A.2) shows that with respect to this monomial order the set $\mathcal{M}_3$ of 3-minors of X is not a SAGBI basis of $G(3, 6)$. Therefore we get a negative answer to our question:

Proposition 4.3.3. *Consider a 3×6 matrix of indeterminates X. Then there exists monomial order on $K[X]$ such that the set $\mathcal{M}_3$ of the maximal minors of X is not a SAGBI basis of $G(X)$.*

A proof of this result was given in Corollary 5.6 of [15], but the monomial order there considered failed to be lexicographic. In fact, the following weight matrix was considered

$$W = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

This means that the entry X_{ij} of the usual 3×6 matrix of indeterminates has weight W_{ij}. Let us give a look at how the leading terms of the maximal minors of X are distributed (see [19]):

$$v(W) = \begin{pmatrix} 8 & 2 & 8 & 2 & 0 & 0 \\ 2 & 8 & 0 & 0 & 8 & 2 \\ 0 & 0 & 2 & 8 & 2 & 8 \end{pmatrix}.$$

In order to read this matrix one has to fix a column, for example the first one. Of the ten minors involving the first column, eight contain X_{11} in their leading term and two contain, again in their leading term, X_{21}. None of the ten leading terms involve X_{31}.

If the monomial order induced by W on $K[X]$ was lexicographic, there would be a variable X_{ij} such that X_{ij} is the highest in our order. Then X_{ij} must appear in the leading term of every of the ten maximal minors of X involving the i-th column. Therefore the i-th column of $v(W)$ must contain a 10 in the j-th row, forcing the other two entries to be zero. As we can see, the matrix $v(W)$ does not contain any column with two zero entries, and therefore the monomial order induced by W on $K[X]$ cannot be lexicographic.

For instance, the matrix of distribution of the leading terms with respect to our lexicographic monomial order is

$$\begin{pmatrix} 10 & 0 & 0 & 6 & 2 & 2 \\ 0 & 10 & 0 & 2 & 6 & 2 \\ 0 & 0 & 10 & 2 & 2 & 6 \end{pmatrix},$$

and one sees that there's at least a column with two zero entries.

Appendix A

Experiments on CoCoA and Macaulay2

A.1 A Universal SAGBI basis for A_2

A.1.1 Preliminary Experiments

We report the Macaulay2 code we used to experimentally support our idea in Section 4.2 that led to Theorem 4.2.7. In this case, we used as weight matrix a random 3×3 matrix with non-negative entries.

```
needsPackage "SubalgebraBases"

-- we construct the polynomial ring endowed with the
   monomial order induced by a random 3x3 matrix
ordR = random(ZZ^3,ZZ^3)
ordR'= transpose(ordR)
r = reshape(ZZ^1,ZZ^9,ordR')
R = QQ[X_{1,1}..X_{3,3},MonomialOrder => {Weights =>
   flatten entries(r)}]

-- we construct the matrix of indeterminates X and
   check if the 2-minors of X are already a SAGBI basis
M = transpose(genericMatrix(R,X_{1,1},3,3))
I = minors(2,M)
L = flatten entries(gens I)
isSAGBI L

-- we construct our (universal) SAGBI basis and check
   if it actually is a SAGBI basis
indet = gens R
```

```
d = det M
indet' = d*indet
SagbiUniv = join(indet',L)
isSAGBI SagbiUniv
```

Here we report the code that does exactly the same thing on CoCoA 5. Note that CoCoA doesn't have a boolean function to check if a certain set is a SAGBI basis, while as we saw above Macaulay has function isSAGBI.

```
-- we construct the polynomial ring endowed with the
   monomial order induced by a random vector of nine
   entries
S::=QQ[X[1..3,1..3]];
ordR:= MakeTermOrdMat(RowMat([random(0,10)|k in 1..9])
   );
R:=NewPolyRing(QQ, IndetSymbols(S), ordR, 1);

-- we construct the matrix of indeterminates X and
   check if the 2-minors of X are already a SAGBI basis
L:=indets(R);
M:=MakeMatByRows(3,3,L);
two_min:=minors(M,2);
SB:=sagbi(two_min);
len(SB);

-- we construct our (universal) SAGBI basis and check
   if it actually is a SAGBI basis
d:=det(M);
univ_SB:=concat(two_min, d*L);
LT_sagbi:=[LT(f) | f in SB];
LT_sagbi_univ:=[LT(f) | f in univ_SB];
IsSubset(LT_sagbi, LT_sagbi_univ);
```

A.1.2 Computation of the Newton Polytope

After these experiments we proved Theorem 4.2.7. Here we report the CoCoA computation we used:

```
-- we construct the polynomial ring and the generic matrix
   X
Use R::=QQ[x[1..3, 1..3]], Lex;
X:=Mat([[x[I,J] | J In 1..3] | I In 1..3]);
```

```
-- we compute the product F of the 2-minors of X and we
   record in FF all the possible ways in which monomials in
    the support of F can form while calculating the product
F:=Product(Minors(X,2));
MM:=Minors(X,2);
FF:=[1];

Foreach A In MM Do
SA:=Support(A);
FF:=Concat(SA[1]*FF, SA[2]*FF);
EndForeach;

FS:=MakeSet(FF);

-- we construct the list JF of the monomials that appear
   only once in the product. The vertices of the Newton
   polytope are a subset. There are 102 monomials in JF,
   while the support of F has 156. Clearly JF is a subset
   of Support(F)
JF:=[ A In FS | Len([ B In FF | B=A])=1];
Len(JF);
Len(Support(F));

-- we check which monomials are in the support of F up to
   rows and columns permutations of X
W:=[exponents(A) | A In Support(F)];
WW:=[];

Foreach B In W Do
Append(ref WW, Mat([ [ B[I+3*(J-1)] | I In 1..3] | J In
   1..3]));
EndForeach;

SS:=Permutations(1..3);
EWW:=[];

While WW<>[] Do
B:=WW[1];
Append(ref EWW, B);
ORB:=[];
Foreach P In SS Do
Foreach Q In SS Do
```

```
Append(ref ORB,  Mat([ [ B[P[I], Q[J] ] | J In 1..3] | I
   In 1..3]));
EndForeach ;
EndForeach;
ORB:=MakeSet(ORB);
WW:=[ C In WW | Not(C IsIn ORB)] ;
PrintLn Len(WW), "  ", Len(ORB);
EndWhile;

Foreach A In EWW Do PrintLn A; EndForeach;

-- we compute vertices of the Newton polytope of F with
   random weights. After some computations we found 102,
   but 102 was the cardinality of JF and therefore we found
    all of them.
W:=Mat([ exponents(A) | A In Support(F)]);

VERT:=[];
PESI:=[];

For YU:=1 To 10000 Do
U:=Mat([ [Rand(0,2) | K In 1..9]]);
WU:=Transposed(W*Transposed(U));
WU:=GetRow(WU,1);
MAX:=Max(WU);
YY:=[ Y In 1..NumRows(W)  | WU[Y]=MAX];
If len(YY)=1 Then If  Not(YY[1] IsIn VERT ) Then Append(
   ref VERT, YY[1]);  Append(ref PESI, U); Print "-" ;
   EndIf;
EndIf;
EndFor;

Len(VERT);

-- we print all 102 vertices with the associated weight
   vectors
W:=[ W[K] |K In VERT];

WW:=[];

Foreach B In W Do
```

```
Append(ref WW, Mat([ [ B[I+3*(J-1)] |  I In 1..3] | J In
    1..3]));
EndForeach;

For I:=1 To Len(WW) Do
PrintLn ;
PrintLn I,")", "====================";
PrintLn WW[I];
PrintLn PESI[I];
EndFor;

-- we print just the vertices up to rows and columns
   permutations of X
SS:=Permutations(1..3);
EWW:=[];

While WW<>[] Do
B:=WW[1];
Append(ref EWW, B);
ORB:=[];
Foreach P In SS Do
Foreach Q In SS Do
Append(ref ORB,  Mat([ [ B[P[I], Q[J] ] | J In 1..3] | I
   In 1..3]));
EndForeach ;
EndForeach;
ORB:=MakeSet(ORB);
WW:=[ C In WW | Not(C IsIn ORB)] ;
PrintLn Len(WW), "   ", Len(ORB);
EndWhile;

Foreach A In EWW Do PrintLn A; EndForeach;
```

A.2 A Counterexample for $G(3,6)$

The following is the Macaulay2 code we used in order to show that the maximal minors are not a SAGBI basis of $G(3,6)$.

```
needsPackage "SubalgebraBases"

w_1 = {1}
w_2 = {0,0,0,0,0,0,0,1}
```

```
w_3 = {0,0,0,0,0,0,0,0,0,0,0,0,0,0,1}
w_4 = {0,0,0,1}
w_5 = {0,0,0,0,0,0,0,0,0,0,0,1}
w_6 = {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1}

-- we construct the polynomial ring endowed with the
   monomial order given by the 9x9 matrix whose the
   first nine rows are the w_i
R = QQ[X_{1,1}..X_{3,6},MonomialOrder =>{Weights =>
   w_1, Weights => w_2, Weights => w_3, Weights => w_4,
    Weights => w_5, Weights => w_6, Lex}]

-- we construct the 3x6 matrix of indeterminates X and
    check if the 3-minors are a SAGBI basis
M = transpose(genericMatrix(R,X_{1,1},6,3))
I = minors(3,M)
L = flatten entries(gens I)
isSAGBI L
```

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