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Spectral deviation of Toeplitz operators acting on reproducing kernel Hilbert spaces

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Notation

$\mathbb{N}_0$, $\mathbb{N}$	Natural numbers, respectively with or without 0
1_X	Indicator function on the set X
I_X	Identity function on the set X The dependence on X may be omitted when evident from the context.
d	Euclidean distance
d_{gr}	Graph distance
$\mathcal{C}(X)$	Continuous complex-valued functions on the topological space X
$\mathcal{C}^\infty(\mathbb{R}^d)$	Infinitely differentiable complex-valued functions
$\mathcal{S}(\mathbb{R}^d)$	Schwartz functions
$\mathcal{S}'(\mathbb{R}^d)$	Tempered distributions
$\mathcal{K}(X)$	Compact operators from the normed space X to itself
$\\|\cdot\\|_{\mathrm{op}}$	Operator norm
$R(T)$	Range of the linear operator T
$B_r(z)$	Open ball, of radius $r > 0$ and center $z \in X$, of a metric space X
C_q	Constant depending only on the parameter q Its specific value may change from line to line.
$A \lesssim B$	$A \leq cB$ for some constant $c > 0$

Introduction

This dissertation aims to extend the threshold-robust spectral bound proved in [28] for the concentration operators of the short-time Fourier transform to Toeplitz operators acting on reproducing kernel Hilbert spaces.

Given a compact set Ω and a normalised window function $g \in L^2(\mathbb{R}^d)$, the concentration (or localisation) operator L_Ω is a partial reconstruction of functions in $L^2(\mathbb{R}^d)$ from a localisation on Ω of their short-time Fourier transform:

$$L_\Omega f(t) = \int_\Omega V_g f(x, w)\, g(t - x) e^{2\pi i t \cdot w}\, dx\, dw, \qquad t \in \mathbb{R}^d,$$

where the integral converges in the quadratic norm. Since L_Ω is compact, self-adjoint, positive semi-definite, and contractive, given a threshold $\delta \in (0, 1)$, it is possible to count the number of eigenvalues greater than δ.

The localisation operators were introduced and studied by Daubechies [8, 9, 11], Ramanathan and Topiwala [32], and have found many applications to time-frequency analysis, quantum mechanics, differential equations, and signal processing. They are also called (generalised) anti-Wick operators, specially when considered, in greater generality, with respect to symbols [3], so that the truncated integral on Ω is replaced by a weighted integral.

The study of the eigenvalue counting function for the concentration operators, and, more in general, for for operators that localise both in time and frequency, is at the core of many physical and signal models, and measurement and estimation methods [23]. For example, the spectral profile of localisation operators describes the number of prominent degrees of freedom in problems where functions are assumed to be compactly supported in time and/or in frequency [26]. Many methods regarding applications require threshold-robust formulas for the spectral deviation for such operators, that in the case of the concentration operator L_Ω reads as

$$\left| \#\{\lambda \in \sigma(L_\Omega) : \lambda > \delta\} - |\Omega| \right| .$$

For instance, Landau's method to study the discrepancy of sampling and interpolating sets [25] and the quantification of the performance of randomised linear algebra methods [22] applied to Toeplitz operators rely on threshold-robust formulas.

In [28], Marceca and Romero provide an estimate for the eigenvalue counting function of L_Ω which is robust and almost sharp in δ. In particular, the cases of a Gelfand-Shilov-type condition and of a polynomial decay of the STFT $V_g g$ of the window function g are treated. Their proof relies on the fact that the time-frequency concentration operator is unitarily equivalent to the restriction of the Toeplitz operator

$$T_\Omega F = P\big(1_\Omega \cdot PF\big)\,, \qquad F \in L^2(\mathbb{R}^{2d})$$

on the range $\mathbb{H}$ of the short-time Fourier transform [7], where $P\colon L^2(\mathbb{R}^{2d}) \to \mathbb{H}$ is the orthogonal projection onto $\mathbb{H}$. Moreover, such space is a reproducing kernel subspace of $L^2(\mathbb{R}^{2d})$ whose reproducing kernel is proportional to $V_g g$ up to a modulation.

Thus, it is a natural question whether the bounds in [28] can be generalised to the eigenvalue counting function of Toeplitz operators acting on reproducing kernel Hilbert subspaces $\mathbb{H}$ of $L^2(X, \mu)$ with X a metric space and μ a positive, σ-finite Borel measure. Under minimal assumptions on the kernel and the level sets of the distance from the boundary of the concentration domain Ω, we were able to derive the desired estimates, again both for a Gelfand-Shilov-type condition and a polynomial decay of the kernel of $\mathbb{H}$. The test case of Gabor multipliers was studied separately, to exploit its additional structure.

The research for this thesis was conducted at the Applied Harmonic Analysis Cluster of the University of Vienna during a 5-month traineeship funded by an Erasmus+ Scholarship. After completing the thesis, the study has already been expanded to include arbitrary decay assumptions on the kernel, and the work to write an academic paper with the team of the host institution is in progress. Further applications will be also addressed, including graphs.

The dissertation is structured as follows. Chapter 1 collects the preliminary results on Hilbert spaces that are used throughout the thesis; in particular, discrete time-frequency representations are grounded on the theory of frames, whereas the Schatten p-norms and the reproducing kernel Hilbert spaces play an essential role in the investigations of Toeplitz operators. Time-frequency analysis is introduced in Chapter 2, mainly in terms of the short-time Fourier transform and Gabor frames. In Chapter 3, we present the spectral deviation bounds for the concentration operators provided in [28] by Marceca and Romero. In Chapter 4, the test case of Gabor multipliers is tackled. Finally, Chapter 5 deals with the general case of Toeplitz operators acting on reproducing kernel Hilbert spaces, providing threshold-robust spectral bounds and the resulting eigenvalues estimate.

Chapter 1

Preliminaries on Hilbert spaces

We begin with the presentation of the preliminary results concerning Hilbert spaces that are used in the dissertation. Throughout the chapter, $\mathbb{H}$ will denote a Hilbert space.

1.1 Frame theory

The concept of orthonormal basis for a Hilbert space is generalised by the theory of frames, which offers discrete representations of vectors in terms of a fixed collection called frame. The main source for the content of this section is [20, Chapter 5.1].

Definition 1.1.1. A sequence $(e_j)_{j\in J}$ in $\mathbb{H}$ is called a *frame* if there exist constants $A, B > 0$, called *frame bounds*, such that

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq B\|f\|^2 , \qquad f \in \mathbb{H} . \tag{1.1}$$

If $A = B$, a frame $(e_j)_{j\in J}$ is called a *tight frame*; the special case $A = B = 1$ is referred to as a *Parseval frame*.

Example 1.1.2. The union of N orthonormal bases is a tight frame with frame bounds $A = B = N$. The union of an orthonormal basis with N arbitrary unit vectors is a frame with bounds $A = 1$ and $B = N + 1$.

Frames are related to the following operators, whose properties are direct consequences of (1.1).

Definition 1.1.3. Consider a sequence $(e_j)_{j\in J}$ in $\mathbb{H}$.
The *coefficient* (or *analysis*) *operator* C is defined for $f \in \mathbb{H}$ by the sequence

$$Cf = \left(\langle f, e_j\rangle\right)_{j\in J} . \tag{1.2}$$

The *synthesis* (or *reconstruction*) *operator* D is defined for a complex valued finite sequence $c = (c_j)_{j\in J}$ by the vector

$$Dc = \sum_{j\in J} c_j e_j \in \mathbb{H}\,. \tag{1.3}$$

The *frame operator* S is defined for $f \in \mathbb{H}$ by

$$Sf = \sum_{j\in J} \langle f, e_j\rangle\, e_j\,. \tag{1.4}$$

Proposition 1.1.4. *Let $(e_j)_{j\in J}$ be a frame for $\mathbb{H}$, with frame bounds A, B. Then the following statements hold.*

(a) The coefficient operator $C\colon \mathbb{H} \to \ell^2(J)$ is bounded with closed range.

(b) The coefficient operator C and the synthesis operator D are adjoint of each other; that is, $D = C^*$. Consequently, D extends to a bounded operator $D\colon \ell^2(J) \to \mathbb{H}$ and satisfies

$$\Big\| \sum_{j\in J} c_j e_j \Big\| \le B^{1/2} \|c\|_2\,, \qquad c = (c_j)_{j\in J} \in \ell^2(J). \tag{1.5}$$

(c) The frame operator $S\colon \mathbb{H} \to \mathbb{H}$ is positive and invertible, and satisfies

$$S = C^*C = DD^*\,, \tag{1.6}$$

$$A\,\mathrm{I}_{\mathbb{H}} \le S \le B\,\mathrm{I}_{\mathbb{H}} \quad \textit{and} \quad B^{-1}\,\mathrm{I}_{\mathbb{H}} \le S^{-1} \le A^{-1}\,\mathrm{I}_{\mathbb{H}}\,. \tag{1.7}$$

In particular, $(e_j)_{j\in J}$ is a tight frame if and only if $S = A\,\mathrm{I}_{\mathbb{H}}$.

(d) The optimal frame bounds are $B_{opt} = \|S\|_{\mathrm{op}}$ and $A_{opt} = \|S^{-1}\|_{\mathrm{op}}^{-1}$, where $\|\cdot\|_{\mathrm{op}}$ is the operator norm.

Proof. (a) The frame inequality (1.1) implies that C maps in $\ell^2(J)$, it is bounded with $\|C\|_{\mathrm{op}} \le B^{1/2}$ and it is invertible on its range, which is then closed in $\ell^2(J)$.
(b) Let $c = (c_j)_{j\in J}$ be a finite sequence. For every $f \in \mathbb{H}$ we have

$$\langle C^*c, f\rangle = \langle c, Cf\rangle = \sum_{j\in J} c_j \overline{\langle f, e_j\rangle} = \Big\langle \sum_{j\in J} c_j e_j, f \Big\rangle = \langle Dc, f\rangle\,.$$

Then $D = C^* : \ell^2(J) \to \mathbb{H}$ is also bounded with $\|D\|_{\mathrm{op}} \le B^{1/2}$, because C is such.
(c) By definition we have $S = C^*C = DD^*$; consequently, S is self-adjoint and positive. Since

$$\langle Sf, f\rangle = \sum_{j\in J} |\langle f, e_j\rangle|^2\,, \qquad f \in \mathbb{H},$$

inequalities (1.1) yield $A\,\mathrm{I}_{\mathbb{H}} \leq S \leq B\,\mathrm{I}_{\mathbb{H}}$. Then, $S\colon \mathbb{H} \to \mathbb{H}$ is invertible because $A > 0$, and inequalities $A\,S^{-1} \leq SS^{-1} \leq B\,S^{-1}$ follow because S^{-1} is again a positive operator.
(d) The claim is a direct consequence of the fact that, for a positive operator T, its norm is given by $\|T\|_{\mathrm{op}} = \sup\{\langle Tf, f\rangle : \|f\| \leq 1\}$. ☐

The next proposition exploits the convergence properties of the non-orthogonal series defined by the synthesis operator.

Definition 1.1.5. Consider a sequence $(x_j)_{j\in J}$ in $\mathbb{H}$. The series $\sum_{j\in J} x_j$ *converges unconditionally* to a vector $f \in \mathbb{H}$ if for every $\varepsilon > 0$ there exists a finite subset $F_0(\epsilon) \subseteq J$ such that for all finite subsets $F \supseteq F_0(\varepsilon)$

$$\Big\| f - \sum_{j\in F} x_j \Big\| < \epsilon\,. \tag{1.8}$$

Corollary 1.1.6. *Let $(e_j)_{j\in J}$ be a frame for $\mathbb{H}$. Then for every $c \in \ell^2(J)$ the series $Dc = \sum_{j\in J} c_j e_j$ converges unconditionally.*

Proof. Fix $\varepsilon > 0$ and choose $F_0(\varepsilon) \subseteq J$ such that $\sum_{j\notin F} |c_j|^2 < \varepsilon/B^{1/2}$ for every $F \supseteq F_0(\varepsilon)$. Consider the finite sequence $c_F := c \cdot 1_F \in \ell^2(J)$. By definition we have $\sum_{j\in F} c_j e_j = Dc_F$, and therefore

$$\Big\| Dc - \sum_{j\in F} c_j e_j \Big\| = \big\| Dc - Dc_F \big\| = \big\| D(c - c_F) \big\| \leq B^{1/2}\, \|c - c_F\|_2 < \varepsilon\,. \qquad \square$$

We are now interested in a reconstruction formula for a vector in $\mathbb{H}$ from its frame coefficients.

Corollary 1.1.7. *Let $(e_j)_{j\in J}$ be a frame for $\mathbb{H}$, with frame bounds A, B. Then $(S^{-1}e_j)_{j\in J}$ is a frame, called* dual frame, *with frame bounds B^{-1}, A^{-1}. Moreover, every $f \in \mathbb{H}$ has the expansions*

$$f = \sum_{j\in J} \big\langle f, S^{-1}e_j \big\rangle\, e_j \tag{1.9}$$

and

$$f = \sum_{j\in J} \langle f, e_j\rangle\, S^{-1}e_j\,, \tag{1.10}$$

where both sums converge unconditionally in $\mathbb{H}$.

Proof. Proposition 1.1.4(c) implies that

$$B^{-1}\|f\|^2 \leq \big\langle S^{-1}f, f \big\rangle \leq A^{-1}\|f\|^2\,, \qquad f \in \mathbb{H}.$$

The operator S^{-1} is self-adjoint because S is such; thus

$$\sum_{j\in J}\left|\left\langle f, S^{-1}e_j\right\rangle\right|^2 = \sum_{j\in J}\left|\left\langle S^{-1}f, e_j\right\rangle\right|^2 = \left\langle S\left(S^{-1}f\right), S^{-1}f\right\rangle = \left\langle S^{-1}f, f\right\rangle .$$

The combination of these two results shows that the sequence $\left(S^{-1}e_j\right)_{j\in J}$ is a frame with frame bounds B^{-1}, A^{-1}.

Using that $\mathrm{I}_{\mathbb{H}} = S^{-1}S = SS^{-1}$, for $f\in\mathbb{H}$ we obtain

$$f = S\left(S^{-1}f\right) = \sum_{j\in J}\left\langle S^{-1}f, e_j\right\rangle e_j = \sum_{j\in J}\left\langle f, S^{-1}e_j\right\rangle e_j$$

and

$$f = S^{-1}Sf = \sum_{j\in J}\left\langle f, e_j\right\rangle S^{-1}e_j ,$$

where both series converge unconditionally by Corollary 1.1.6. □

Remark 1.1.8. The two series expansions in (1.9) and (1.10) coincide for tight frames, since the frame operator of a tight frame is a multiple of the identity by Proposition 1.1.4(c).

In contrast to orthonormal bases, the coefficients in the frame expansion (1.9) are in general not unique. However, it can be shown that they are canonical in the sense that they minimise the ℓ^2-norm; see [20, Proposition 5.1.4]. If the coefficients in expansion (1.9) are unique, the frame is said to be a *Riesz basis*; see [20, Proposition 5.1.5] for more details.

Frames are used to define discrete time-frequency representations in time-frequency analysis; see Section 2.3.

1.2 Functional calculus and Schatten classes

The main results of the dissertation rely crucially on properties of Hankel operators and their Schatten p-(quasi)norm, with $p>0$. We now recall the spectral theorem for self-adjoint compact operators and its consequences, in order to justify the definition of this family of norms. See, e.g., [37, Chapter 3] as a reference.

Theorem 1.2.1 (Spectral theorem for compact self-adjoint operators)**.** *Consider* $T\colon\mathbb{H}\to\mathbb{H}$ *a compact self-adjoint operator. Then every nonzero element of the spectrum* $\lambda\in\sigma(T)$ *is an eigenvalue of* T *and its eigenspace is finite-dimensional. There also exists a countable orthonormal sequence* $\left(v_j\right)_{j\in J}$ *of eigenvectors, with* $J=\{1,\dots,N\}$ *or* $J=\mathbb{N}$*, and a corresponding sequence of eigenvalues* $\left(\lambda_j\right)_{j\in J}$*, with* $|\lambda_1|\geq|\lambda_2|\geq\ldots>0$*, such that*

$$Tx = \sum_{j\in J}\lambda_j\left\langle x, v_j\right\rangle v_j , \qquad x\in\mathbb{H}. \tag{1.11}$$

In particular, the equality $\sigma(T) \setminus \{0\} = \{\lambda_j \mid j \in J\}$ *holds. If* $J = \mathbb{N}$*, then* $\lambda_j \to 0$ *as* $j \to +\infty$*, and the series in* (1.11) *converges to* T *in the operator norm.*

In this context, it is now possible to define functions of operators, in the following sense. Denote with $\mathcal{K}(\mathbb{H})$ the Banach $*$-algebra of compact operators from $\mathbb{H}$ to itself.

Corollary 1.2.2 (Functional calculus)**.** *Let* $T\colon \mathbb{H} \to \mathbb{H}$ *a compact self-adjoint operator, and let* $\Sigma_T := \sigma(T) \setminus \{0\} = \{\lambda_j \mid j \in J\}$ *with notation as in Theorem 1.2.1. Denote by* $B(\Sigma_T)$ *the Banach* $*$*-algebra of bounded functions* $F\colon \Sigma_T \to \mathbb{C}$ *together with the* sup*-norm. Then, for* $F \in B(\Sigma_T)$ *the assignment*

$$F(T)x = \sum_{j\in J} F(\lambda_j)\langle x, v_j\rangle v_j\,, \qquad x \in \mathbb{H} \tag{1.12}$$

defines a continuous Banach $*$*-algebra homomorphism from* $B(\Sigma_T)$ *to* $\mathcal{K}(\mathbb{H})$*, with* $1_{\Sigma_T}(T) = \mathrm{I}_{\mathbb{H}}$ *and* $\mathrm{I}_{\Sigma_T}(T) = T$*. In particular, if* $F \in B(\Sigma_T)$ *is real valued, then* $F(T)$ *is a compact self-adjoint operator and*

$$\Sigma_{F(T)} = \{F(\lambda_j) \mid j \in J\} \setminus \{0\}\,. \tag{1.13}$$

Proof. Given $F \in B(\Sigma)$ and $x \in \mathbb{H}$, we have

$$\|F(T)x\|^2 = \sum_{j\in J} |F(\lambda_j)|^2\, |\langle x, v_j\rangle|^2 \le \|F\|_\infty^2 \|x\|^2\,.$$

All the other properties follow by direct computation. ☐

The functional calculus can be applied to define new norms for compact operators.

Definition 1.2.3. The *Schatten p-(quasi)norm* of a compact operator $T\colon \mathbb{H} \to \mathbb{H}$ is defined by

$$\|T\|_p = (\mathrm{trace}(|T|^p))^{1/p}\;, \qquad p > 0, \tag{1.14}$$

where $|T| := (T^*T)^{1/2}$ is given by the functional calculus. The operator T belongs to the *Schatten p-class* $\mathcal{J}_p(\mathbb{H})$ if $\|T\|_p < +\infty$.

It can be proved that $\|\cdot\|_p$ is a norm for $p \ge 1$, and is a quasinorm for $0 < p < 1$. In particular, $\mathcal{J}_p(\mathbb{H})$ is a Banach algebra for $p \ge 1$: for $p = 1$, $\mathcal{J}_1(\mathbb{H})$ is known as the space of *trace-class operators*; for $p = 2$, $\mathcal{J}_2(\mathbb{H})$ is the space of *Hilbert-Schmidt operators*. See [36, ch. 1,2] for more details.

In order to better understand the definition of the Schatten p-norms, we introduce the *singular value expansion (SVE)*, an extension of the spectral theorem to compact (non necessarily self-adjoint) operators.

Theorem 1.2.4 (SVE for compact operators)**.** *Let* $\mathbb{H}_1, \mathbb{H}_2$ *be Hilbert spaces, and let* $T\colon \mathbb{H}_1 \to \mathbb{H}_2$ *be a compact operator. Then there exist countable orthonormal sequences* $(v_j)_{j\in J} \subseteq \mathbb{H}_1$ *and* $(u_j)_{j\in J} \subseteq \mathbb{H}_2$*, with* $J = \{1, \dots, N\}$ *or* $J = \mathbb{N}$*, and a sequence of* singular values $(s_j)_{j\in J}$*, with* $s_1 \geq s_2 \geq \cdots > 0$*, such that*

$$Tx = \sum_{j\in J} s_j \langle x, v_j\rangle u_j\,, \qquad x \in \mathbb{H}_1. \tag{1.15}$$

In particular, $Tv_j = s_j u_j$ *for* $j \in J$*. If* $J = \mathbb{N}$*, then* $s_j \to 0$ *as* $j \to +\infty$*, and the series in* (1.15) *converges to* T *in the operator norm.*

Proof. The operator T^*T is compact and self-adjoint. By Theorem 1.2.1,

$$T^*Tx = \sum_{j\in J} \lambda_j \langle x, v_j\rangle v_j\,, \qquad x \in \mathbb{H}_1,$$

with sequences $(v_j)_{j\in J}$ in $\mathbb{H}_1$ and $(\lambda_j)_{j\in J}$ in $\mathbb{C}$ as in the theorem. Note that

$$\lambda_j = \lambda_j \langle v_j, v_j\rangle_{\mathbb{H}_1} = \langle \lambda_j v_j, v_j\rangle_{\mathbb{H}_1} = \langle T^*Tv_j, v_j\rangle_{\mathbb{H}_1} = \langle Tv_j, Tv_j\rangle_{\mathbb{H}_1} \geq 0\,.$$

Then, we define

$$s_j := \sqrt{\lambda_j} \qquad \text{and} \qquad u_j := s_j^{-1} T v_j \tag{1.16}$$

for $j \in J$. The singular values $(s_j)_{j\in J}$ give a non-increasing, infinitesimal sequence of positive real numbers. The sequence $(u_j)_{j\in J}$ is an orthonormal system in $\mathbb{H}_2$, since for $j, k \in J$

$$\begin{aligned}\langle u_j, u_k\rangle_{\mathbb{H}_2} &= \left\langle s_j^{-1}Tv_j, s_k^{-1}Tv_k\right\rangle_{\mathbb{H}_2} = s_j^{-1}s_k^{-1}\langle v_j, T^*Tv_k\rangle_{\mathbb{H}_1}\\ &= s_j^{-1}s_k^{-1}\left\langle v_j, s_k^2 v_k\right\rangle_{\mathbb{H}_1} = s_j^{-1}s_k^{-1}s_k^2\langle v_j, v_k\rangle_{\mathbb{H}_1} = \delta_{j,k}\end{aligned}$$

It is straightforward to show that $\mathcal{N}(T) = \mathcal{N}(T^*T) = \operatorname{span}\{v_j \mid j \in J\}^\perp$. Then, if $P_{\mathcal{N}(T)^\perp}\colon \mathbb{H} \to \mathcal{N}(T)^\perp$ is the projection operator, for $x \in \mathbb{H}_1$ we have

$$Tx = T\left(P_{\mathcal{N}(T)^\perp}x\right) = T\Big(\sum_j \langle x, v_j\rangle_{\mathbb{H}_1} v_j\Big) = \sum_j \langle x, v_j\rangle_{\mathbb{H}_1} Tv_j\,,$$

which is equation (1.15) in the pointwise sense, and in the operator norm if J is finite. If $J = \mathbb{N}$, let $x \in \mathbb{H}_1$ with $\|x\|_{\mathbb{H}_1} = 1$; for every $M \in \mathbb{N}$

$$\begin{aligned}\left\|Tx - \sum_{n=1}^{M} s_j \langle x, v_j\rangle_{\mathbb{H}_1} u_j\right\|_{\mathbb{H}_2}^2 &= \left\|\sum_{n=M+1}^{\infty} s_j \langle x, v_j\rangle_{\mathbb{H}_1} u_j\right\|_{\mathbb{H}_2}^2 = \sum_{n=M+1}^{\infty} s_j^2 \left|\langle x, v_j\rangle_{\mathbb{H}_1}\right|^2\\ &\leq s_{M+1}^2 \sum_{n=M+1}^{\infty} \left|\langle x, v_j\rangle_{\mathbb{H}_1}\right|^2 \leq s_{M+1}^2\|x\|_{\mathbb{H}_1}^2 = s_{M+1}^2\,.\end{aligned}$$

Then, $\left\|T - \sum_{n=1}^{M} s_j \langle \cdot, v_j\rangle_{\mathbb{H}_1} u_j\right\|_{\mathrm{op}} \leq s_{M+1} \to 0$ as $M \to +\infty$, which proves the convergence of (1.15) in the operator norm. □

Remark 1.2.5. By Corollary 1.2.2, it is clear that, for a compact operator $T\colon \mathbb{H} \to \mathbb{H}$, the eigenvalues of $|T| \coloneqq (T^*T)^{1/2}$ are exactly the singular values of T introduced in (1.16). Therefore, by definition, the Schatten p-norms with $p \geq 1$ of the compact operator T are the ℓ^p-norms of the sequence of its singular values $(s_j)_{j\in J}$:

$$\|T\|_p = (\operatorname{trace}(|T|^p))^{1/p} = \Big(\sum_{j\in J} s_j^p\Big)^{1/p}.$$

Consistently, the operator norm $\|T\|_{\mathrm{op}}$ coincides with the ℓ^∞-norm of the singular values – see, e.g., [37, Theorem 3.18]. If we denote $\mathcal{J}_\infty(\mathbb{H}) \coloneqq \mathcal{K}(\mathbb{H})$, the Schatten p-classes $\mathcal{J}_p(\mathbb{H})$ with $p \in [1, +\infty]$ give a classification of compact operators based on the decay of their singular values.

We mention two criteria to compute the norm of trace-class and Hilbert-Schmidt operators on L^2-spaces: they will be used multiple times in the dissertation.

Theorem 1.2.6 (See [36, Theorem 2.11]). *Let (X, μ) be a separable measure space, with μ a positive σ-finite measure, and let $\mathbb{H} \coloneqq L^2(X, \mu)$. Then, for every Hilbert-Schmidt operator $T \in \mathcal{J}_2(\mathbb{H})$ there exists a unique function $K \in L^2(X \times X, \mu \otimes \mu)$ such that for every $f \in \mathbb{H}$ it holds the equality*

$$Tf(x) = \int_X K(x, y) f(y)\, d\mu(y)\,, \qquad x \in X, \tag{1.17}$$

where the integral converges absolutely for a.e. $x \in X$. Conversely, any function $K \in L^2(X \times X, \mu \otimes \mu)$ defines an operator T by (1.17) *which is in $\mathcal{J}_2(\mathbb{H})$ and*

$$\|T\|_2 = \|K\|_2\,. \tag{1.18}$$

Definition 1.2.7. Let X be a set. A function $F\colon X \times X \to \mathbb{C}$ is said to be *positive definite* if for every pair of finite sets $\{c_i\}_{i=1}^n \subseteq \mathbb{C}$ and $\{x_i\}_{i=1}^n \subseteq X$, with $n \in \mathbb{N}$, it holds

$$\sum_{i,j=1}^n c_i \overline{c_j} F(x_i, x_j) \geq 0\,. \tag{1.19}$$

Theorem 1.2.8 (See [36, Theorem 2.12]). *Let X be a locally compact metric space, and μ be a positive Borel measure on X that is finite on compact sets. Let $\mathbb{H} \coloneqq L^2(X, \mu)$. Let $K\colon X \times X \to \mathbb{C}$ be a continuous, positive definite function. Moreover, suppose*

$$\int_X K(x, x)\, d\mu(x) < +\infty\,. \tag{1.20}$$

Then there exists a unique operator $T \in \mathcal{J}_1(\mathbb{H})$ satisfying (1.17)*, and*

$$\|T\|_1 = \int_X K(x, x)\, d\mu(x)\,. \tag{1.21}$$

Finally, through the functional calculus we are also able to give an explicit definition of the so-called *eigenvalue counting function* for a positive, compact, self-adjoint operator $T\colon \mathbb{H} \to \mathbb{H}$. Since $\sigma(T) \subseteq [0, \|T\|_{\mathrm{op}}]$, and 0 is the only possible accumulation point, the eigenvalue counting function counts the eigenvalues of T over a certain threshold $\delta > 0$:

$$\#\{\lambda \in \sigma(T) : \lambda > \delta\} \coloneqq \operatorname{trace}\left[1_{(\delta,+\infty)}(T)\right]. \tag{1.22}$$

The whole dissertation concerns threshold-robust estimates of the eigenvalue counting function for certain positive, compact, self-adjoint operators known as Toeplitz operators: see Sections 3.4.1, 4.3 and 5.1.

1.3 Reproducing kernel Hilbert spaces

The main part of the thesis deals with reproducing kernel Hilbert spaces. Let us recall some of their main properties; see, e.g., [31] as a reference.

Definition 1.3.1. Let X be a set, and let $\mathbb{H} \subseteq X^{\mathbb{C}}$ be a Hilbert space of functions $X \to \mathbb{C}$. The space $\mathbb{H}$ is said to be a *reproducing kernel Hilbert space* (RKHS) if there exists a function $K\colon X \times X \to \mathbb{C}$, called *reproducing kernel*, such that:

1. $K_y \coloneqq K(\cdot, y) \in \mathbb{H}$ for every $y \in X$;
2. for every $f \in \mathbb{H}$, the *reproducing formula* holds:

$$f(y) = \langle f, K_y \rangle, \qquad y \in X. \tag{1.23}$$

Proposition 1.3.2. *Let X be a set and $\mathbb{H} \subseteq X^{\mathbb{C}}$ be a Hilbert space. The following conditions are equivalent:*

(i) the space $\mathbb{H}$ is a RKHS with reproducing kernel $K\colon X \times X \to \mathbb{C}$;

(ii) for every $x \in X$, the evaluation operator $e_x\colon \mathbb{H} \to \mathbb{C}$ is bounded.

Moreover, $\|e_x\|_{op}^2 = \|K_x\|^2 = K(x,x)$ for every $x \in X$.

Proof. By applying Cauchy-Schwartz to the reproducing formula 1.23, it is immediate to see that (i) implies (ii). Conversely, assuming (ii), for every $x \in X$ the Riesz representation theorem yields the existence of a function $K_x \in X$ such that the reproducing formula holds and $\|e_x\|_{\mathrm{op}} = \|K_x\|$. $\square$

The continuity of the evaluation operators implies the pointwise convergence of a sequence that converges in the $\mathbb{H}$-norm.

Lemma 1.3.3. *Let X be a set and $\mathbb{H} \subseteq X^{\mathbb{C}}$ be a RKHS. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{H}$, and $f \in \mathbb{H}$. If $\lim_{n\to+\infty} \|f_n - f\| = 0$, then $\lim_{n\to+\infty} f_n(x) = f(x)$ for every $x \in X$.*

Proof. If $K\colon X\times X\to\mathbb{C}$ is the reproducing kernel, for $x\in X$ the reproducing formula gives $|f(x)-f_n(x)|\le\|f-f_n\|\|K_x\|\to 0$ for $n\to+\infty$. $\square$

Proposition 1.3.4. *Let X be a set and $\mathbb{H}\subseteq X^{\mathbb{C}}$ a RKHS with reproducing kernel $K\colon X\times X\to\mathbb{C}$. The following statements hold:*

(a) the kernel K satisfies

$$K(x,y)=\langle K_y,K_x\rangle=\overline{K(y,x)}\,,\qquad x,y\in X;\tag{1.24}$$

(b) the kernel K is positive definite;

(c) the linear span of $\{K_x\}_{x\in X}$ is dense in $\mathbb{H}$;

(d) if $(e_j)_{j\in J}$ is an orthonormal basis for $\mathbb{H}$, then

$$K(x,y)=\sum_{j\in J}e_j(x)\overline{e_j(y)}\,,\qquad x,y\in X,\tag{1.25}$$

where the series converges pointwise.

Proof. (a) It is a trivial consequence of the reproducing formula.
(b) Given $\{c_i\}_{i=1}^n\subseteq\mathbb{C}$ and $\{x_i\}_{i=1}^n\subseteq X$, with $n\in\mathbb{N}$, then

$$\begin{aligned}\sum_{i,j=1}^n c_i\overline{c_j}K(x_i,x_j)&=\sum_{i,j=1}^n c_i\overline{c_j}\left\langle K_{x_j},K_{x_i}\right\rangle=\left\langle\sum_{j=1}^n\overline{c_j}K_{x_j},\sum_{i=1}^n\overline{c_i}K_{x_i}\right\rangle\\&=\left\|\sum_{i=1}^n\overline{c_i}K_{x_i}\right\|^2\ge 0.\end{aligned}$$

(c) If $f\in\operatorname{span}\{K_x\mid x\in X\}^{\perp}$, then $f=0$ by the reproducing formula.
(d) The reproducing formula gives $\langle K_y,e_j\rangle=\overline{e_j(y)}$ for $y\in X$ and $j\in J$, and therefore $K_y=\sum_{j\in J}\overline{e_j(y)}e_j$; the pointwise convergence follows from the previous lemma. $\square$

Proposition 1.3.5. *Let X be a set and $\mathbb{H}\subseteq X^{\mathbb{C}}$ be a Hilbert space. Let $M\subseteq\mathbb{H}$ be a closed subspace and a RKHS with reproducing kernel $K\colon X\times X\to\mathbb{C}$. Then the orthogonal projection $P_M\colon\mathbb{H}\to M$ acts on a function $f\in\mathbb{H}$ by the formula*

$$P_Mf(y)=\langle f,K_y\rangle\,,\qquad y\in X.\tag{1.26}$$

Proof. Given $f\in\mathbb{H}$, consider the decomposition $f=f_M+f_{M^\perp}$ where $f_M\in M$ and $f_{M^\perp}\in M^\perp$. For every $y\in X$, we have $K_y\in M$, and therefore

$$\langle f,K_y\rangle=\langle f_M+f_{M^\perp},K_y\rangle=\langle f_M,K_y\rangle=f_M(y)=P_Mf(y).\qquad\square$$

In Chapter 5, we will focus on reproducing kernel Hilbert spaces which are subspaces of $L^2(X,\mu)\cap\mathcal{C}(X)$, where X is a metric space and μ is a positive measure.

Chapter 2

Time-frequency analysis

In classical Fourier analysis, a function $f \in L^2(\mathbb{R}^d)$ is described through two complementary representations: the function f itself and its Fourier transform $\hat{f}$. Recall that the Fourier transform of a function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is given by

$$\hat{f}(w) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot w} \, dx \,, \qquad w \in \mathbb{R}^d.$$

The standard interpretation in signal analysis sees $f(x)$ as the temporal behavior $\hat{f}(w)$ as the frequency behavior of a signal.

Time-frequency analysis investigates two-dimensional representations of the form $Vf(x, w)$ that combine the features of $f(x)$ and $\hat{f}(w)$, and whose goal is a simultaneous description of the temporal and spectral behaviors of the signal. The main source for the content of this chapter is [20, Chapters 2–6].

For $x, w \in \mathbb{R}^d$, we will denote with T_x and M_w the translation by x and the modulation by w respectively; for a function $f \colon \mathbb{R}^d \to \mathbb{C}$, they are defined as

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_w f(t) = e^{2\pi i w \cdot t} f(t) \,, \qquad t \in \mathbb{R}^d.$$

Both $T \colon \mathbb{R}^d \to \mathcal{U}(L^2(\mathbb{R}^d))$ and $M \colon \mathbb{R}^d \to \mathcal{U}(L^2(\mathbb{R}^d))$ are unitary representations; that is, both $\{T_x\}_{x \in \mathbb{R}^d}$ and $\{M_w\}_{w \in \mathbb{R}^d}$ are a strongly continuous group of unitary operators on $L^2(\mathbb{R}^d)$. The operators $T_x M_w$ or $M_w T_x$ are called *time-frequency shifts*, and satisfy the canonical commutation relations:

$$T_x M_w = e^{-2\pi i x \cdot w} M_w T_x \,. \tag{2.1}$$

2.1 The short-time Fourier transform

An ideal time-frequency representation aims at combining local information of both the spectra of time and frequency of functions; that is, at providing the occurring frequency spectrum at each instant of time. Nevertheless, a wide collection of mathematical inequalities known as *uncertainty principles* claims that, qualitatively speaking, a function and its Fourier transform cannot be both supported on arbitrarily small sets.

For instance, the following theorem expresses the classical uncertainty principle in dimension $d = 1$, known as Heisenberg-Pauli-Weyl inequality.

Theorem 2.1.1 (See [20, Theorem 2.2.1]). *Let* $f \in L^2(\mathbb{R})$ *and* $a, b \in \mathbb{R}$. *Then*

$$\left(\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}} (w-b)^2 |\hat{f}(w)|^2 \, dw \right)^{1/2} \geq \frac{1}{4\pi} \|f\|_2^2 \, . \tag{2.2}$$

Equality holds if and only if the function f *is a multiple of the time-frequency shift*

$$T_a M_b \varphi_c(x) = e^{2\pi i b(x-a)} \cdot e^{-\pi(x-a)^2/c} \tag{2.3}$$

of the Gaussian function $\varphi_c(x) = e^{-\pi x^2/c}$, $x \in \mathbb{R}$, *for some* $c > 0$.

Quantitative versions of the principle can be formulated in higher dimensions, too. For instance, the next statement is consequence of a more sophisticated inequality by Donoho and Stark [15]; see also [20, Theorem 2.3.1].

Theorem 2.1.2 (See [20, Corollary 2.3.2]). *Let* $f \in L^2(\mathbb{R}^d)$ *with* $\operatorname{supp} f \subseteq T$ *and* $\operatorname{supp} \hat{f} \subseteq \Omega$, *with* $T, \Omega \subseteq \mathbb{R}^d$ *measurable sets. Then* $|T||\Omega| \geq 1$.

Conversely, the following is an important qualitative result.

Theorem 2.1.3 (See [20, Theorem 2.3.3]). *Let* $f \in L^1(\mathbb{R}^d)$ *with* $\operatorname{supp} f \subseteq T$ *and* $\operatorname{supp} \hat{f} \subseteq \Omega$, *with* $T, \Omega \subseteq \mathbb{R}^d$ *measurable sets. If* $|T||\Omega| < +\infty$, *then* $f = 0$.

More details on uncertainty principles can be found in [20, Chapter 2]. These inequalities obstruct the ideal concept of instantaneous frequency at a given time. Indeed, in order to determine the instantaneous frequency spectrum of a signal f at the time x, we need to consider f at least over a short period $[x - \delta, x + \delta]$ for some $\delta > 0$; that is, taking the Fourier transform of $f_{\text{inst}} := f \cdot g_x$, where g_x is some cut-off function with support in $[x - \delta, x + \delta]$. By the uncertainty principle, the support of $\widehat{f_{\text{inst}}}$ cannot be small, and possibly depends even on δ. Thus, for small δ we cannot say that f_{inst} is concentrated on a distinct frequency band. Nevertheless, the idea described above remains the fundamental step towards a simultaneous representation in time and frequency. We now introduce the short-time Fourier transform, which is the main example of time-frequency representation.

Definition 2.1.4. Consider a function $g \in L^2(\mathbb{R}^d)$, called *window function*. The *short-time Fourier transform* (STFT) of a function $f \in L^2(\mathbb{R}^d)$ with respect to g is defined as

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \cdot w} \, dt \, , \qquad (x, w) \in \mathbb{R}^{2d}. \tag{2.4}$$

The following lemma collects some basic properties of the short-time Fourier transform.

Lemma 2.1.5. *Let $f, g \in L^2(\mathbb{R}^d)$. Then $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$, and*

$$\begin{aligned} V_g f(x,w) &= (f \cdot T_x \overline{g})\widehat{\ }(w) && (2.5)\\ &= \langle f, M_w T_x g\rangle && (2.6)\\ &= e^{-\pi i x\cdot w}\int_{\mathbb{R}^d} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}e^{-2\pi i t\cdot w}\,dt\,. && (2.7)\end{aligned}$$

Proof. The identities are simply changes of notation. The uniform continuity follows from identity (2.6) and the continuity of the operator groups $\{T_x\}_{x\in\mathbb{R}^d}$ and $\{M_w\}_{w\in\mathbb{R}^d}$. ☐

Remark 2.1.6. By equality (2.5) we see that, if g is compactly supported with support centered at 0, then $V_g f(x,\cdot)$ is the Fourier transform of a segment of f centered in a neighborhood of x. Indeed, as x varies, the window g slides with it.

Remark 2.1.7. Formula (2.6) can be used to extend the definition of STFT to $\mathcal{S}'(\mathbb{R}^d)$ when the window function is in $\mathcal{S}(\mathbb{R}^d)$. Indeed, given $g \in \mathcal{S}(\mathbb{R}^d)$, the short-time Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined as

$$V_g f(x,w) = \langle f, M_w T_x g\rangle\,, \qquad (x,w)\in\mathbb{R}^{2d}. \tag{2.8}$$

More in general, one can consider the STFT as in (2.8) whenever the bracket $\langle\cdot,\cdot\rangle$ is well defined by some form of duality.

Remark 2.1.8. If we fix the window function $g \in L^2(\mathbb{R}^d)$, then the short-time Fourier transform is a linear form $V_g\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ by identity (2.6). At the same time, the short-time Fourier transform can also be seen as a sesquilinear form $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ given by

$$(f,g) \mapsto V_g f = \mathcal{F}_2\mathcal{T}(f\otimes\overline{g})\,, \tag{2.9}$$

where $\mathcal{T}\colon L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ is the coordinate transformation

$$\mathcal{T}F(x,t) = F(t, t-x)\,, \qquad (x,t)\in\mathbb{R}^{2d},$$

and $\mathcal{F}_2\colon L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ is the partial Fourier transform

$$\mathcal{F}_2 F(x,w) = \int_{\mathbb{R}^d} F(x,t)e^{-2\pi i t\cdot w}\,dt\,, \qquad (x,w)\in\mathbb{R}^{2d}.$$

Notice that both $\mathcal{T}$ and $\mathcal{F}_2$ are unitary operators.

By the factorisation in (2.9), the STFT inherits several properties similar to those possessed by the ordinary Fourier transform. The following theorem on inner products of short-time Fourier transforms corresponds to Parseval's formula.

Theorem 2.1.9 (Orthogonality relations)**.** *Let* $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$*. Then*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)} . \tag{2.10}$$

Proof. We use the factorisation (2.9): the fact that the operators $\mathcal{T}$ and $\mathcal{F}_2$ are unitary yields

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} &= \langle \mathcal{F}_2 \mathcal{T}(f_1 \otimes \overline{g_1}), \mathcal{F}_2 \mathcal{T}(f_2 \otimes \overline{g_2}) \rangle_{L^2(\mathbb{R}^{2d})} \\ &= \langle f_1 \otimes \overline{g_1}, f_2 \otimes \overline{g_2} \rangle_{L^2(\mathbb{R}^{2d})} \\ &= \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)} , \end{aligned}$$

where the last equality follows by the Fubini-Tonelli theorem. □

Corollary 2.1.10. *Let* $f, g \in L^2(\mathbb{R}^d)$*. Then* $V_g f \in L^2(\mathbb{R}^{2d})$ *and*

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2 . \tag{2.11}$$

In particular, if $\|g\|_2 = 1$*, then* $V_g \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ *is a linear isometry.*

We are interested in having an inversion formula for the STFT as a corollary of the orthogonality relations. To get there, for $F \in L^2(\mathbb{R}^{2d})$ consider the weakly defined operator $\Gamma_F \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given by

$$\Gamma_F = \iint_{\mathbb{R}^{2d}} F(x, w) M_w T_x \, dx \, dw . \tag{2.12}$$

It is actually well defined and bounded since $\Gamma_F h$ is a bounded conjugate-linear functional for every $h \in L^2(\mathbb{R}^d)$. Indeed, for every $k \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \big|\langle \Gamma_F h, k \rangle\big| &= \left| \iint_{\mathbb{R}^{2d}} F(x, w) \, \langle M_w T_x h, k \rangle \, dxdw \right| \\ &= \left| \iint_{\mathbb{R}^{2d}} F(x, w) \overline{V_h k(x, w)} \, dxdw \right| \\ &\leq \|F\|_2 \|V_h k\|_2 = \|F\|_2 \|h\|_2 \|k\|_2 . \end{aligned}$$

Corollary 2.1.11 (Inversion formula)**.** *Let* $g, \gamma \in L^2(\mathbb{R}^d)$ *be such that* $\langle \gamma, g \rangle \neq 0$*. Then for every* $f \in L^2(\mathbb{R}^d)$ *it holds*

$$f = \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, w) M_w T_x \gamma \, dwdx . \tag{2.13}$$

where the integral is intended in the weak sense.

Proof. Consider the operator Γ_F as in (2.12) with $F := V_g f$. Then, using the orthogonality relations, we see that for every $k \in L^2(\mathbb{R}^d)$

$$\begin{aligned} \langle \Gamma_F \gamma, k \rangle &= \iint_{\mathbb{R}^{2d}} V_g f(x, w) \overline{\langle k, M_w T_x \gamma \rangle} \, dxdw \\ &= \langle V_g f, V_\gamma k \rangle = \langle f, k \rangle \langle \gamma, g \rangle = \big\langle \langle \gamma, g \rangle f, k \big\rangle . \end{aligned}$$

Therefore, the function $\langle \gamma, g \rangle f$ represents $\Gamma_F \gamma$, which is the assertion. □

Again, a direct computation shows that, for $g \in L^2(\mathbb{R}^d)$, the adjoint operator $V_g^* : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d)$ is given by the weak integral

$$V_g^* F = \int_{\mathbb{R}^{2d}} F(x,w) M_w T_x g \, dx dw \, . \tag{2.14}$$

For $g, \gamma \in L^2(\mathbb{R}^d)$ such that $\langle \gamma, g \rangle \neq 0$, the inversion formula (2.13) thus reads as

$$\mathrm{I} = \frac{1}{\langle \gamma, g \rangle} V_\gamma^* V_g \, . \tag{2.15}$$

In particular, for $\gamma = g$ with $\|g\|_2 = 1$ the equation becomes

$$\mathrm{I} = V_g^* V_g \, . \tag{2.16}$$

Finally, let us define the Wigner distribution – which will be mentioned in Chapter 3 – as an example of quadratic time-frequency representation. Such time-frequency representations were investigated early in quantum mechanics with the goal of finding joint probability distributions for the position and momentum variables (see [20, Lemma 4.3.6]).

Definition 2.1.12. The *cross-Wigner distribution* of two functions $f, g \in L^2(\mathbb{R}^d)$ is defined by

$$W(f,g)(x,w) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t \cdot w} \, dt \, , \qquad (x,w) \in \mathbb{R}^{2d}. \tag{2.17}$$

The *Wigner distribution* of a function $f \in L^2(\mathbb{R}^d)$ is defined by $Wf = W(f,f)$, that is

$$Wf(x,w) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{f\left(x - \frac{t}{2}\right)} e^{-2\pi i t \cdot w} \, dt \, , \qquad (x,w) \in \mathbb{R}^{2d}. \tag{2.18}$$

Remark 2.1.13. From equation (2.7), it is straightforward to see that for every $f, g \in L^2(\mathbb{R}^d)$ the following equality holds:

$$W(f,g)(x,w) = 2^d e^{4\pi i x \cdot w} V_{\check{g}} f(2x, 2w) \, , \qquad (x,w) \in \mathbb{R}^{2d}. \tag{2.19}$$

Therefore, many properties can be translated easily from the STFT to the Wigner distribution (see [20, Proposition 4.3.2]). We only mention that Wf is real-valued for every $f \in L^2(\mathbb{R}^d)$, a fact that will give sense to equation (3.5) further in the dissertation.

The reader is referred to [20, Chapters 3, 4] for a wider discussion on the short-time Fourier transform and quadratic time-frequency representations.

2.2 Decay properties of the STFT

In this section we introduce two standard classes of functions according to the decay of their short-time Fourier transform. In Chapters 3 and 5, we will consider window functions belonging to these spaces.

Definition 2.2.1. Let $r, s \geq 0$. The *Gelfand-Shilov space* $\mathcal{S}_r^s(\mathbb{R}^d)$ is the space of functions $g \in \mathcal{C}^\infty(\mathbb{R}^d)$ for which there exist constants $A, B, C > 0$ such that for every multi-indexes $\alpha, \beta \in \mathbb{N}_0^d$

$$\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta g(x)| \leq C A^{|\alpha|} B^{|\beta|} |\alpha|!^r |\beta|!^s. \tag{2.20}$$

Remark 2.2.2. The Gelfand-Shilov spaces are subspaces of $\mathcal{S}(\mathbb{R}^d)$.

Lemma 2.2.3 (See [19, Chapter IV.8])**.** *Given* $r, s \geq 0$*, the space* $\mathcal{S}_r^s(\mathbb{R}^d)$ *is nonempty if and only if either* $r + s > 1$*, or* $r + s = 1$ *and* $rs > 0$*.*

The definition of Gelfand-Shilov spaces was originally formulated in greater generality in [19]. The usefulness of these spaces in time-frequency analysis is due to their characterisations in terms of the Fourier transform.

Theorem 2.2.4 (See [6, Theorem 2.3, Corollary 2.5])**.** *Let* $r, s > 0$ *such that* $r + s \geq 1$*, and let* $g\colon \mathbb{R}^d \to \mathbb{C}$*. The following conditions are equivalent:*

(i) $g \in \mathcal{S}_r^s(\mathbb{R}^d)$*;*

(ii) there exist $A, B, C > 0$ *such that for every* $\alpha, \beta \in \mathbb{N}_0^d$

$$\sup_{x\in\mathbb{R}^d} |x^\alpha g(x)| \leq C A^{|\alpha|} |\alpha|!^r \qquad \text{and} \qquad \sup_{x\in\mathbb{R}^d} |\partial^\beta g(x)| \leq C B^{|\beta|} |\beta|!^s\,;$$

(iii) there exist $A, B, C > 0$ *such that for every* $\alpha, \beta \in \mathbb{N}_0^d$

$$\sup_{x\in\mathbb{R}^d} |x^\alpha g(x)| \leq C A^{|\alpha|} |\alpha|!^r \qquad \text{and} \qquad \sup_{\xi\in\mathbb{R}^d} |\xi^\beta \hat{g}(\xi)| \leq C B^{|\beta|} |\beta|!^s\,;$$

(iv) there exist $a, b > 0$ *such that*

$$\sup_{x\in\mathbb{R}^d} e^{a|x|^{1/r}} |g(x)| < +\infty \qquad \text{and} \qquad \sup_{\xi\in\mathbb{R}^d} e^{b|\xi|^{1/s}} |\hat{g}(\xi)| < +\infty\,.$$

Since they involve decay properties of both the function and its Fourier transform, Gelfand-Shilov spaces can be also characterised in terms of the STFT. We state this characterisation in the specific case $r = s$, which will be considered in Chapters 3 and 5.

Theorem 2.2.5 (See [21, Corollary 3.11]). *Let* $r > 0$ *and* $g \in \mathcal{S}_r^r(\mathbb{R}^d) \setminus \{0\}$*. For* $f \in (\mathcal{S}_r^r(\mathbb{R}^d))'$*, the following conditions are equivalent:*

(i) $f \in \mathcal{S}_r^r(\mathbb{R}^d)$;

(ii) $V_g f \in \mathcal{S}_r^r(\mathbb{R}^{2d})$.

Remark 2.2.6. In the previous theorem, we are implicitely using the embedding of $\mathcal{S}_r^r(\mathbb{R}^d)$ in $(\mathcal{S}_r^r(\mathbb{R}^d))'$. As in the standard case of Schwartz functions and tempered distributions, it is given by the assignment

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(x) g(x)\, dx\,, \qquad f, g \in \mathcal{S}_r^r(\mathbb{R}^d).$$

Combining Theorem 2.2.4 and 2.2.5, one sees that Gelfand-Shilov spaces classify the functions whose short-time Fourier transform has an exponential decay. In particular, let us write explicitly the case – that we will use later in the dissertation – in which we consider the short-time Fourier transform of the window function g itself.

Corollary 2.2.7. *Let* $r \geq \frac{1}{2}$*. If* $g \in \mathcal{S}_r^r(\mathbb{R}^d)$*, then* $V_g g \in \mathcal{S}_r^r(\mathbb{R}^{2d})$*, and in particular there exist* $A, C > 0$ *such that for every* $n \in \mathbb{N}_0$

$$|V_g g(z)| \leq C A^n n!^r (1 + |z|)^{-n}\,, \qquad z \in \mathbb{R}^{2d}. \tag{2.21}$$

Moreover, also $V_{\hat{g}} g \in \mathcal{S}_r^r(\mathbb{R}^{2d})$*, and in particular there exist* $a, D > 0$ *such that*

$$|V_{\hat{g}} g(z)| \leq C e^{-a|z|^{1/r}}\,, \qquad z \in \mathbb{R}^{2d}. \tag{2.22}$$

Remark 2.2.8. In view of Lemma 2.2.3, the conditions $r, s > 0$, $r + s \geq 1$ in Theorem 2.2.4, and therefore the condition $r \geq 1/2$ in the previous corollary, are not restrictive. Otherwise, one gets $V_g f = 0$ (resp. $V_g g = 0$) which implies $f = 0$ or $g = 0$ (resp. $g = 0$) by Corollary 2.1.10.

We now turn to the second family of spaces that we will be interested in.

Definition 2.2.9. Let $g \in \mathcal{S}(\mathbb{R}^d)$ be a nonzero function. Let $p, q \in [1, +\infty]$ and $s \in \mathbb{R}$. The *modulation space* $M_s^{p,q}(\mathbb{R}^d)$ is the space of the tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{M_s^{p,q}} := \Big\| w \mapsto \Big[\|x \mapsto (1 + |(x,w)|)^s V_g f(x,w)\|_{L^p(\mathbb{R}^d)} \Big] \Big\|_{L^q(\mathbb{R}^d)} < +\infty\,. \tag{2.23}$$

Modulation spaces were introduced by Feichtinger in [16], and they can be defined with a more general weight function in place of the polynomial one that we use here; see [20, Chapter 11] for a more complete discussion. The weight describes the decay property of the short-time Fourier transform of the functions in the space. Indeed, if $p, q \in [1, +\infty)$ we can be more explicit in writing

$$\|f\|_{M_s^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (1 + |(x,w)|)^{ps} |V_g f(x,w)|^p \, dx \right)^{q/p} dw \right)^{1/q}, \tag{2.24}$$

whereas if $p = \infty$ or $q = \infty$ the integral is replaced by the essential supremum.

It is important to mention that the definition of modulation spaces is actually independent of the choice of the window function, as a consequence of the following change of window formula.

Lemma 2.2.10 (See [20, Proposition 11.3.3]). *Let $g, \gamma, g_1 \in \mathcal{S}(\mathbb{R}^d)$ such that $\langle g, \gamma \rangle \neq 0$. Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then*

$$|V_{g_1} f(z)| \leq \frac{1}{|\langle g, \gamma \rangle|} \big(|V_g f| * |V_{g_1} \gamma| \big)(z), \qquad z \in \mathbb{R}^{2d}. \tag{2.25}$$

Proposition 2.2.11 (See [20, Proposition 11.3.2(c)]). *Given $p, q \in [1, +\infty]$ and $s \in \mathbb{R}$, the definition of $M_s^{p,q}(\mathbb{R}^d)$ does not depend on the choice of the window function. In particular, a function $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $M_s^{p,q}(\mathbb{R}^d)$ if there exists some nonzero $g \in \mathcal{S}(\mathbb{R}^d)$ for which* (2.23) *holds..*

The Gelfand-Shilov-type condition (2.21) and the polynomial decay condition (2.24) with $p = q = 1$ on the short-time Fourier transform will be considered later in the investigation of the spectral deviation bounds for Toeplitz operators.

2.3 Gabor frames

The inversion formula (2.13) shows that a function in $L^2(\mathbb{R}^d)$ can be expressed as a continuous superposition of time-frequency shifts with the short-time Fourier transform as weight function. However, since $L^2(\mathbb{R}^d)$ is a separable Hilbert space, a series expansion with respect to a countable subset of time-frequency shifts should suffice to represent every $f \in L^2(\mathbb{R}^d)$. In order to obtain such a discrete time-frequency representation, we turn to the theory of frames.

Consider a window function $g \in L^2(\mathbb{R}^d)$, $g \neq 0$, and two constants $\alpha, \beta > 0$: given $f \in L^2(\mathbb{R}^d)$, for $k, n \in \mathbb{Z}^d$ we observe that the samples

$$V_g f(\alpha k, \beta n) = \langle f, M_{\beta n} T_{\alpha k} g \rangle = e^{-2\pi i \alpha\beta k \cdot n} \langle f, T_{\alpha k} M_{\beta n} g \rangle \tag{2.26}$$

of the STFT on the lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ are just inner products of f with the functions from the sequence $(T_{\alpha k} M_{\beta n} g)_{(k,n) \in \mathbb{Z}^{2d}}$, which are time-frequency shift of the window g. If this collection is a frame, then f is uniquely determined by the samples of $V_g f$ by formula (1.9). The next definition is motivated by this observation.

Definition 2.3.1. Consider a window function $g \in L^2(\mathbb{R}^d)$, $g \neq 0$, and lattice parameters $\alpha, \beta > 0$. The sequence of time-frequency shifts

$$\mathcal{G}(g, \alpha, \beta) = (T_{\alpha k} M_{\beta n} g)_{(k,n) \in \mathbb{Z}^{2d}} \tag{2.27}$$

is called a *Gabor system.* If $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$, it is called a *Gabor frame*, and its frame operator – the *Gabor frame operator* – has the form

$$\begin{align} Sf &= \sum_{(k,n) \in \mathbb{Z}^{2d}} \langle f, T_{\alpha k} M_{\beta n} g \rangle \, T_{\alpha k} M_{\beta n} g \tag{2.28} \\ &= \sum_{(k,n) \in \mathbb{Z}^{2d}} V_g f(\alpha k, \beta n) M_{\beta n} T_{\alpha k} g \, . \tag{2.29} \end{align}$$

Let us investigate the structure of the dual frame of a Gabor frame. Unless stated otherwise, in the rest of this section $g \in L^2(\mathbb{R}^d)$, $g \neq 0$ will be the window function and $\alpha, \beta > 0$ will be the lattice parameters.

Proposition 2.3.2. *If the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$, then there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that the dual frame of $\mathcal{G}(g, \alpha, \beta)$ is $\mathcal{G}(\gamma, \alpha, \beta)$. Consequently, every $f \in L^2(\mathbb{R}^d)$ can be written in the expansions*

$$\begin{align} f &= \sum_{(k,n) \in \mathbb{Z}^{2d}} \langle f, T_{\alpha k} M_{\beta n} g \rangle \, T_{\alpha k} M_{\beta n} \gamma \tag{2.30} \\ &= \sum_{(k,n) \in \mathbb{Z}^{2d}} \langle f, T_{\alpha k} M_{\beta n} \gamma \rangle \, T_{\alpha k} M_{\beta n} g \, , \tag{2.31} \end{align}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. Moreover,

$$\begin{align} A \|f\|_2^2 &\leq \sum_{(k,n) \in \mathbb{Z}^{2d}} |V_g f(\alpha k, \beta n)|^2 \leq B \|f\|_2^2 \, , \tag{2.32} \\ B^{-1} \|f\|_2^2 &\leq \sum_{(k,n) \in \mathbb{Z}^{2d}} |\langle f, T_{\alpha k} M_{\beta n} \gamma \rangle|^2 \leq A^{-1} \|f\|_2^2 \, . \tag{2.33} \end{align}$$

In particular, one can take $\gamma = S^{-1} g$.

Proof. First, we show that the Gabor frame operator S commutes with time-frequency shifts $T_{\alpha r} M_{\beta s}$, with $(r, s) \in \mathbb{Z}^{2d}$. Given $f \in L^2(\mathbb{R}^d)$,

$$\begin{align} (T_{\alpha r} M_{\beta s})^{-1} \, S \, (T_{\alpha r} M_{\beta s}) f &= \sum_{(k,n) \in \mathbb{Z}^{2d}} \langle T_{\alpha r} M_{\beta s} f, T_{\alpha k} M_{\beta n} g \rangle \, (T_{\alpha r} M_{\beta s})^{-1} \, T_{\alpha k} M_{\beta n} g \\ &= \sum_{(k,n) \in \mathbb{Z}^{2d}} \left\langle f, T_{\alpha (k-r)} M_{\beta (n-s) g} \right\rangle T_{\alpha (k-r)} M_{\beta (n-s)} g = Sf \, , \end{align}$$

where we used that $(T_{\alpha r}M_{\beta s})^{-1}\, T_{\alpha k}M_{\beta n} = e^{-2\pi i \alpha\beta(k-r)\cdot s}T_{\alpha(k-r)}M_{\beta(n-s)}$ by the commutation relations (2.1). Therefore, S^{-1} also commutes with $T_{\alpha r}M_{\beta s}$, and the dual frame consists of the functions

$$S^{-1}\left(T_{\alpha k}M_{\beta n}g\right) = T_{\alpha k}M_{\beta n}S^{-1}g$$

Thus we may take $\gamma = S^{-1}g$ as the dual window. The other assertions follow by Corollary 1.1.7 and equation (2.26). ☐

Remark 2.3.3. Proposition 2.3.2 provides a discrete time-frequency representation of signals. In fact, if $\mathcal{G}(g,\alpha,\beta)$ is a frame and $\gamma \in L^2(\mathbb{R}^d)$ is a dual window, combining equation (2.26) with equation (2.30) we get that every function $f \in L^2(\mathbb{R}^d)$ can be written as

$$f = \sum_{(k,n)\in\mathbb{Z}^{2d}} V_g f(\alpha k, \beta n) M_{\beta n} T_{\alpha k}\gamma\,.$$

This is an explicit reconstruction of f from the samples of its STFT, and it can be seen as a discrete version of the inversion formula (2.13).

Remark 2.3.4. If we denote with C_g and C_γ the coefficient operators related to the functions g and γ respectively, then equations (2.30) and (2.31) become

$$\mathrm{I} = C_\gamma^* C_g = C_g^* C_\gamma\,. \tag{2.34}$$

As already mentioned in Remark 1.1.8, if $\mathcal{G}(g,\alpha,\beta)$ is a tight frame, the two expansions coincide since the frame operator is a multiple of the identity and the canonical dual window $\gamma = S^{-1}g$ is a multiple of g.

In particular, if $\mathcal{G}(g,\alpha,\beta)$ is a Parseval frame, we have

$$\mathrm{I} = S = C_g^* C_g\,. \tag{2.35}$$

Compare equations (2.34) and (2.35) with equations (2.15) and (2.16).

Questions related to the existence of Gabor frames can be found in [20, Chapter 6]. For example, the following result provides an important necessary condition for a Gabor system to be a frame.

Theorem 2.3.5 (See [20, Theorem 7.5.1])**.** *If the Gabor system $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$, then $\alpha\beta \le 1$.*

At the threshold value $\alpha\beta = 1$, a special situation occurs.

Theorem 2.3.6 (See [20, Corollary 7.5.2])**.** *For a Gabor system $\mathcal{G}(g,\alpha,\beta)$, the following statements hold:*

(a) the system $\mathcal{G}(g,\alpha,\beta)$ is a Riesz basis for $L^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(g,\alpha,\beta)$ is a frame and $\alpha\beta = 1$;

(b) the system $\mathcal{G}(g, \alpha, \beta)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(g, \alpha, \beta)$ is a tight frame, $\|g\|_2 = 1$ and $\alpha\beta = 1$.

These results motivate the following common terminology in signal analysis:

- *oversampling* for $\alpha\beta < 1$;
- *critical sampling* for $\alpha\beta = 1$;
- *undersampling* for $\alpha\beta > 1$.

However, the condition $\alpha\beta \leq 1$ does not guarantee that $\mathcal{G}(g, \alpha, \beta)$ is a frame. For example, if we consider $g = 1_{[0,1]}$ and $a = 2$ in dimension $d = 1$, then the Gabor system $\mathcal{G}(1_{[0,1]}, 2, \beta)$ is never a frame for $L^2(\mathbb{R}^d)$ for any $\beta > 0$, because the functions with support in $\cup_{k\in\mathbb{Z}}[2k+1, 2k+2)$ are not in the span of $\mathcal{G}(1_{[0,1]}, 2, \beta)$.

Sufficient conditions for Gabor frames can be formulated with additional assumptions on the window function. For instance, a result by Walnut [38] claims that, if the window function g is chosen in the so-called *Wiener space* $W(\mathbb{R}^d)$, which is the space of functions in $L^\infty(\mathbb{R}^d)$ such that

$$\|g\|_W \coloneqq \sum_{n\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d} |g(x+n)| < +\infty$$

then there exist appropriate choices of the parameters $\alpha, \beta > 0$ such that $\mathcal{G}(g, \alpha, \beta)$ is a frame; see [20, Definition 6.1.1] and [20, Theorem 6.5.1].

Lastly, let us mention the case of the normalised Gaussian window function $\varphi_d(x) = 2^{d/4}e^{-\pi|x|^2}$, $x \in \mathbb{R}^d$. Its inspection is of special interest in time-frequency analysis: Gaussian functions minimise the uncertainty principle (Theorem 2.1.1) and, more in general, they are linked to optimal resolution of signals in the time-frequency plane. In dimension $d = 1$, the precise range of the lattice parameters that give a Gabor frame when φ_1 is the window function is completely known. The following theorem was conjectured by Daubechies and Grossmann [10], and then proved independently by Lyubarski [27] and Seip and Wallsten [34, 35] using methods from complex analysis.

Theorem 2.3.7 (See [20, Theorem 7.5.3],[24])**.** *The Gabor system $\mathcal{G}(\varphi_1, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 1$.*

Gabor frames will be used in Chapter 4 to introduce the Gabor multipliers.

Chapter 3

Concentration operators for the STFT

In this chapter, we present the content of the article *Spectral deviation of concentration operators for the short-time Fourier transform* by F. Marceca and J. L. Romero [28]. It gives the background which the original content of this dissertation is based on.

3.1 Defining concentration operators

Operators that localise both in time and in frequency are of interest for applications in optics and signal analysis, in which one can often observe signals (time-dependent signals or optical data) only within a certain frequency window and during a limited time (or space) interval. Initially, the localisation operators which treated independently the two sets of variables were used. Considering a frequency window $[-W, W]$ and a time interval $[-T, T]$ in dimension $d = 1$, the effective observation of a signal f can be described as

$$L_{T,W} f = Q_T P_W f$$

where Q_T and P_W are projection operators on the relevant intervals in, respectively, time and frequency; that is,

$$\begin{aligned} Q_T f &= f \cdot 1_{[-T,T]}\,, \\ P_W f(t) &= \int_{-\infty}^{+\infty} f(t')\,\frac{\sin(2\pi W(t-t'))}{\pi(t-t')}\,dt'\,, \qquad t \in \mathbb{R}. \end{aligned}$$

On the contrary, the so called *time-frequency concentration operators* were introduced in [8] to treat the joint time-frequency variable $z = (x, w) \in \mathbb{R}^{2d}$ as a single entity. Consider the inversion formula given in (2.13): by taking $\gamma = g$

with $\|g\|_2 = 1$, for a function $f \in L^2(\mathbb{R}^d)$ we get

$$f(t) = \int_{\mathbb{R}^{2d}} V_g f(x,w)\, g(t-x) e^{2\pi i t\cdot w}\, dx\, dw, \qquad t \in \mathbb{R}^d, \tag{3.1}$$

where the integral converges in quadratic mean by Corollary 2.1.11. The concentration operator for the short-time Fourier transform $L_\Omega \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by restricting integration in (3.1) to a compact time-frequency domain $\Omega \subseteq \mathbb{R}^{2d}$; that is, for $f \in L^2(\mathbb{R}^d)$,

$$L_\Omega f(t) = \int_{\Omega} V_g f(x,w)\, g(t-x) e^{2\pi i t\cdot w}\, dx\, dw, \qquad t \in \mathbb{R}^d. \tag{3.2}$$

In terms of the adjoint short-time Fourier transform (2.14), the operator L_Ω can be written as

$$L_\Omega f = V_g^* \left[1_\Omega \cdot V_g f \right], \qquad f \in L^2(\mathbb{R}^d). \tag{3.3}$$

Equation (3.3) should be compared to equation (2.16). The time-frequency concentration operator L_Ω is a combination of three consecutive steps: analysis (V_g), processing (multiplication by 1_Ω) and synthesis (V_g^*). From a heuristic point of view, L_Ω represents the projection onto the space of functions whose short-time Fourier transform is essentially localised on Ω. To understand the extent to which this intuition is correct, a classic procedure is studying the profile of the spectrum $\sigma(L_\Omega)$. In fact, straightforward computations show that L_Ω is positive semi-definite and contractive; moreover, it is compact and self-adjoint. Therefore, all the non-zero elements in $\sigma(L_\Omega)$ are eigenvalues, and $\sigma(L_\Omega) \subseteq [0,1]$. The goal is understanding the location of the eigenvalues in the interval $[0,1]$ by studying the eigenvalue counting function defined in (1.22). Heuristically, they are expected to concentrate near 1 and 0.

3.2 The eigenvalue counting function

A standard approach to studying the eigenvalue counting function of the concentration operator L_Ω consists in considering dilations of a fixed compact set $\Omega \subseteq \mathbb{R}^{2d}$ by a parameter $R > 0$ and searching for an asymptotic in the limit as $R \to \infty$.

A two-term asymptotic expansion for the eigenvalue counting function of a time-frequency concentration operator with Schwartz window g and under increasing dilation of a compact set Ω with C^2-boundary was derived in [30]. Given a threshold $\delta \in (0,1)$, it reads

$$\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \lambda > \delta\} = |R\cdot\Omega| + A_1(g, \partial\Omega, \delta)\cdot R^{2d-1} + o_{\delta,\Omega,g}(R^{2d-1}) \tag{3.4}$$

as $R \to +\infty$; the coefficient $A_1(g, \partial\Omega, \delta)$ is given by

$$A_1(g, \partial\Omega, \delta) := C_d \cdot \int_{\partial\Omega} \inf \Big\{ \lambda \geq 0 \colon \int_{\{z \cdot n_u > \lambda\}} Wg(z)\, dz < \delta \Big\}\, d\mathcal{H}^{2d-1}(u), \quad (3.5)$$

where Wg is the Wigner distribution of g, n_u is the outer unit normal of $\partial\Omega$ at u, and C_d is a constant depending only on the dimension d. In particular, the first coefficient of the asymptotic is related to the measure of Ω, whereas the second coefficient is related to $\partial\Omega$ and a time-frequency representation of g.

A similar pattern can be found also with weaker assumptions on g and Ω. For the first term of the asymptotic, in [17] – which refines the earlier investigations in [33] – the following asymptotic is provided:

$$\frac{\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \lambda > \delta\}}{|R \cdot \Omega|} \longrightarrow 1\,, \qquad R \to +\infty.$$

On the other hand, the second term of the asymptotic is related to the number of intermediate eigenvalues in the so-called *plunge region*, that is

$$\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \delta < \lambda < 1 - \delta\}\,.$$

Indeed, this essentially encodes the error of the one-term approximation

$$\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \lambda > \delta\} \approx |R \cdot \Omega|\,,$$

as it is the difference of two such error terms. In [13], the authors obtained the following bounds with suitably general windows g and domains Ω:

$$0 < c_{g,\delta,\Omega} \leq \frac{\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \delta < \lambda < 1 - \delta\}}{\mathcal{H}^{2d-1}\big(\partial[R \cdot \Omega]\big)} \leq C_{g,\delta,\Omega}\,,$$

for all sufficiently large R and sufficiently small $\delta > 0$ – depending on Ω and g.

Turning back to equation (3.4), the leading error term $A_1(g, \partial\Omega, \delta)$ depends explicitly on δ, g and Ω, but the error bound $o_{\delta,\Omega,g}(R^{2d-1})$ is not uniform on these parameters. Therefore, the asymptotic expansion (3.4) is *non-robust with respect to δ*: this means that it is only valid when δ is considered fixed with respect to R. This way, applications where δ is allowed to vary with R are precluded. In contrast, many methods regarding applications require *threshold-robust* formulas. For example, Landau's method to study the discrepancy of sampling and interpolating sets [25] relies on threshold-robust formulas: see, e.g., [2, Proposition 1.3] for an application requiring the choice of δ proportional to a negative power of $|\Omega|$. This motivates the interest in *threshold-robust upper bounds* for the error

$$\big|\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \lambda > \delta\} - |R \cdot \Omega|\big|\,.$$

Estimates of this kind are possible even for a fixed (non-dilated) domain. For example, if $\mathcal{H}^{2d-1}(\partial\Omega) < +\infty$, comparing the first two moments of L_Ω as done, e.g., in [1, Proposition 3.3], for $\delta \in (0,1)$ one gets

$$\big|\#\{\lambda \in \sigma(L_\Omega) : \lambda > \delta\} - |\Omega|\big| \leq C_g \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot \max\left\{\tfrac{1}{\delta}, \tfrac{1}{1-\delta}\right\}, \tag{3.6}$$

where

$$C_g := \int_{\mathbb{R}^{2d}} |z||V_g g(z)|^2 \, dz .$$

Nevertheless, the estimate in (3.6) is not sharp in the dependence on δ – see Section 3.3.2. In [28], Marceca and Romero provide a better threshold-robust spectral deviation bound for time-frequency concentration operators (3.2), which is valid for a (non-dilated) compact set Ω and fully explicit on the threshold δ, as in (3.6). Moreover, their estimates are proved to be almost sharp in δ, up to $\log\log$ factors.

3.3 Threshold-robust spectral bounds

We now present the estimates provided in [28] for the spectral deviation of concentration operators for the short-time Fourier transform. In the article, the authors consider the following regularity condition on the compact set $\Omega \subseteq \mathbb{R}^{2d}$ related to the concentration operator.

Definition 3.3.1. A compact set $\Omega \subseteq \mathbb{R}^{2d}$ has *regular boundary at scale* $\eta > 0$ if there exists a constant $\kappa > 0$ such that

$$\mathcal{H}^{2d-1}\big(\partial\Omega \cap B_r(z)\big) \geq \kappa \cdot r^{2d-1}, \qquad 0 < r \leq \eta, \quad z \in \partial\Omega. \tag{3.7}$$

In this case, the largest possible constant κ is denoted

$$\kappa_{\partial\Omega,\eta} := \inf_{0<r\leq\eta} \inf_{z\in\partial\Omega} \frac{1}{r^{2d-1}} \cdot \mathcal{H}^{2d-1}\big(\partial\Omega \cap B_r(z)\big) . \tag{3.8}$$

This condition means that $\partial\Omega$ satisfies the lower estimate in the definition of *Ahlfors regularity* – see [12, Definition I.1.13]. Aside from Lipschitz domains, which satisfy (3.7), this boundary regularity allows for ridges; for example, sets of the form $[0,1]^n \times E \subseteq \mathbb{R}^{n+2}$, where $\partial E \subseteq \mathbb{R}^2$ is compact and connected, are included.

3.3.1 Spectral deviation

The following two theorems are the main results of [28]. They describe threshold-robust spectral bounds for time-frequency concentration operators according to two different assumptions on the decay assumption for the STFT $V_g g$ of the window g; that are, a Gelfand-Shilov-type decay [28, v5, Theorem 1.1] and a polynomial decay [28, v5, Theorem 1.4] respectively.

Theorem 3.3.2 (Threshold-robust spectral bounds, Gelfand-Shilov-type condition)**.** *Let* $g \in L^2(\mathbb{R}^d)$ *satisfy* $\|g\|_2 = 1$ *and the following Gelfand-Shilov-type condition with parameter* $\beta \geq 1/2$*: there exist* $C_g, A > 0$ *such that for every* $n \in \mathbb{N}_0$

$$\big|V_g g(z)\big| \leq C_g A^n n!^\beta (1+|z|)^{-n}, \qquad z \in \mathbb{R}^{2d}. \tag{3.9}$$

Let $\Omega \subseteq \mathbb{R}^{2d}$ *be a compact set with regular boundary at scale* $\eta > 0$*, and consider the concentration operator* L_Ω *defined in* (3.2)*.*

For $\delta \in (0,1)$*, set* $\tau := \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$*. Then*

$$\begin{aligned} &\big|\#\{\lambda \in \sigma(L_\Omega) : \lambda > \delta\} - |\Omega|\big| \\ &\qquad \leq C_g' \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot (\log\tau)^\beta \left(1 + \frac{(\log\tau)^{(2d-1)\beta}}{\eta^{2d-1}}\right) \frac{\log\big[\log(\tau)+1\big]}{\kappa_{\partial\Omega,\eta}}, \end{aligned} \tag{3.10}$$

where $C_g' := C_g^{1/2} \cdot A^{3d+2} \cdot C_d^\beta$*, and* C_d *depends only on* d*.*

Remark 3.3.3. Condition (3.9) is satisfied whenever g belongs to the Gelfand-Shilov class $\mathcal{S}_\beta^\beta$ with $\beta \geq 1/2$, as seen in Corollary 2.2.7. As mentioned in Remark 2.2.8, the condition $\beta \geq 1/2$ is not restrictive.

Theorem 3.3.4 (Threshold-robust spectral bounds, polynomial condition)**.** *Let* $g \in L^2(\mathbb{R}^d)$ *satisfy* $\|g\|_2 = 1$ *and for some* $s \geq 1$

$$C_g := \int_{\mathbb{R}^{2d}} (1+|z|)^s |V_g g(z)|^2\, dz < +\infty. \tag{3.11}$$

Let $\Omega \subseteq \mathbb{R}^{2d}$ *be a compact set with regular boundary at scale* $\eta \in (0,1]$*, and consider the concentration operator* L_Ω *defined in* (3.2)*.*

For $\delta \in (0,1)$*, set* $\tau := \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$*. Then*

$$\big|\#\{\lambda \in \sigma(L_\Omega) : \lambda > \delta\} - |\Omega|\big| \leq C_g' \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot \tau^{\frac{2d}{2d+s-1}} \cdot \left(\frac{\log(C_g\tau)}{\kappa_{\partial\Omega,\eta} \cdot \eta^{2d-1}}\right)^{\frac{s-1}{2d+s-1}}, \tag{3.12}$$

where $C_g' := C_d \cdot C_g^{\frac{2d}{2d+s-1}}$*, and* C_d *depends only on* d*.*

Remark 3.3.5. Condition (3.11) is satisfied whenever g belongs to the modulation space $M_s^{1,1}(\mathbb{R}^d)$ with $s \geq 1$. Indeed, let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ a nonzero window function such that

$$\|g\|_{M_s^{1,1}} = \int_{\mathbb{R}^{2d}} (1+|z|)^s |V_\varphi g(z)|\, dz < +\infty\,.$$

The change of window formula in Lemma 2.2.10 yields

$$|V_g g| \leq |V_\varphi g| * |V_g \varphi|.$$

Since, for every $z \in \mathbb{R}^{2d}$, $|V_g g(z)| \leq \|g\|_2^2 = 1$ and $|V_g \varphi(z)| = |V_\varphi g(-z)|$ by (2.6), we conclude that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} (1+|z|)^s |V_g g(z)|^2 \, dz &\leq \int_{\mathbb{R}^{2d}} (1+|z|)^s |V_g g(z)| \, dz \\ &\leq \int_{\mathbb{R}^{2d}} (1+|z|)^s \int_{\mathbb{R}^{2d}} |V_\varphi g(z')||V_\varphi g(z'-z)| \, dz' \, dz \\ &\leq \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} (1+|z'|)^s |V_\varphi g(z')|(1+|z'-z|)^s |V_\varphi g(z'-z)| \, dz \, dz' \\ &= \|g\|^2_{M^{1,1}_s} < +\infty \, .\end{aligned}$$

The proofs in [28] rely on specific properties of the STFT and its relation to Toeplitz operators, as shown further in the chapter.

3.3.2 The dilation regime

The most effective way to appreciate Theorem 3.3.2 is specialising it to the dilation regime, that is dilating the compact set Ω by an increasing factor $R > 0$.

First, note that, by Definition 3.3.1, if Ω has regular boundary at scale η with parameter $\kappa_{\partial\Omega,\eta}$, then $R \cdot \Omega$ has regular boundary at scale $R \cdot \eta$ with parameter $\kappa_{\partial\Omega,\eta}$. Therefore, the following corollary [28, v5, Corollary 1.3] can be derived from Theorem 3.3.2.

Corollary 3.3.6 (Threshold-robust spectral bounds, the dilation regime)**.** *Let $g \in L^2(\mathbb{R}^d)$ and $\Omega \subseteq \mathbb{R}^{2d}$ satisfy the conditions of Theorem 3.3.2. Then, with the same notation, for all $R > 0$ it holds*

$$\begin{aligned}&\big|\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \lambda > \delta\} - |R \cdot \Omega|\big| \\ &\quad \leq C'_g \cdot \mathcal{H}^{2d-1}\big(\partial[R\cdot\Omega]\big) \cdot (\log \tau)^\beta \left(1 + \frac{(\log \tau)^{(2d-1)\beta}}{(R\cdot\eta)^{2d-1}}\right) \frac{\log\big[\log(\tau)+1\big]}{\kappa_{\partial\Omega,\eta}} \, . \qquad (3.13)\end{aligned}$$

In order to compare Corollary 3.3.6 with the spectral deviation estimate that follows from the asymptotic (3.4), suppose that g belongs to the Gelfand-Shilov space $\mathcal{S}^\beta_\beta(\mathbb{R}^d)$, with $\beta \geq 1/2$, and Ω has C^2-boundary. Then, by equation (2.22) in Corollary 2.2.7 and by equation (2.19), there exist constants $c, C' > 0$ such that

$$|Wg(z)| = |V_{\check{g}} g(2z)| \leq C' e^{-c|2z|^{1/\beta}} \leq C' e^{-c|z|^{1/\beta}} \, , \qquad z \in \mathbb{R}^{2d};$$

see also [5]. Inspection of (3.5) for $\delta < 1/2$ (see below) gives

$$|A_1(g, \partial\Omega, \delta)| \leq C_g \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot [\log(1/\delta)]^\beta \, . \qquad (3.14)$$

The asymptotic expansion (3.4) now shows that there exist $R_0 = R_0(\Omega, \delta, g) > 0$ such that

$$\big|\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \lambda > \delta\} - |R\cdot\Omega|\big| \leq C_{g,d,\beta} \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot R^{2d-1} \cdot [\log(1/\delta)]^{\beta}$$

for

$$R \geq R_0(\Omega, g, \delta)\,.$$

Since no information on the dependence of R_0 on δ is available, applications of the previous equation where R is taken as a function of δ (or vice-versa) are not possible. In contrast, still with $\delta < 1/2$, estimates (3.13) provides the threshold-robust bound

$$\begin{aligned}\big|\#\{\lambda \in \sigma(L_{R\cdot\Omega}) : \lambda > \delta\} &- |R\cdot\Omega|\big| \\ &\leq C'_g \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot R^{2d-1} \cdot [\log(1/\delta)]^{\beta} \cdot \log\big[\log(1/\delta) + 1\big],\end{aligned}$$

for

$$R \geq \frac{[\log(1/\delta)]^{\beta}}{\eta}\,.$$

Let us now prove (3.14) for $\delta < 1/2$. Fix $u \in \partial\Omega$. Denote

$$C'_\beta := C' \int_{\{z_1 > 0\}} e^{-c2^{(1/2\beta)-1}|z|^{1/\beta}}\,dz \qquad \text{and} \qquad \xi_\beta := \left(\frac{\delta}{C'_\beta}\right)^{\frac{1}{c2^{(1/2\beta)-1}}}.$$

For C' sufficiently large, it holds $\xi_\beta \leq 1$ for every $\beta \geq 1/2$. Consider

$$\lambda_0 := \log\left(\frac{1}{\xi_\beta}\right)^{\beta} > 0\,.$$

For every $z = (z_1, z') \in \mathbb{R}^{2d}$ with $z_1 > 0$ we have

$$|z + \lambda_0 e_1|^2 = |z|^2 + \lambda_0^2 + 2\langle z, \lambda_0 e_1\rangle = |z|^2 + \lambda_0^2 + 2z_1\lambda_0 > |z|^2 + \lambda_0^2\,,$$

where e_1 is the first vector of the canonical orthonormal basis. There follows

$$|z + \lambda_0 e_1|^{1/\beta} > (|z|^2 + \lambda_0^2)^{1/2\beta} \geq 2^{(1/2\beta)-1}(|z|^{1/\beta} + \lambda_0^{1/\beta})\,,$$

where we applied the power mean inequality $(\frac{1}{n}\sum_{i=1}^{n} a_i^{k_1})^{\frac{1}{k_1}} \geq (\frac{1}{n}\sum_{i=1}^{n} a_i^{k_2})^{\frac{1}{k_2}}$ for $k_1 \geq k_2$, $a_1, \ldots, a_n \geq 0$. Therefore,

$$\begin{aligned}\int_{\{z\cdot n_u > \lambda_0\}} Wg(z)\,dz &\leq \int_{\{z\cdot n_u > \lambda_0\}} |Wg(z)|\,dz \leq C' \int_{\{z\cdot n_u > \lambda_0\}} e^{-c|z|^{1/\beta}}\,dz \\ &= C' \int_{\{z_1 > \lambda_0\}} e^{-c|z|^{1/\beta}}\,dz = C' \int_{\{z_1 > 0\}} e^{-c|z+\lambda_0 e_1|^{1/\beta}}\,dz \\ &< C' \int_{\{z_1 > 0\}} e^{-c2^{(1/2\beta)-1}(|z|^{1/\beta}+\lambda_0^{1/\beta})}\,dz \\ &= C' e^{-c2^{(1/2\beta)-1}\lambda_0^{1/\beta}} \int_{\{z_1 > 0\}} e^{-c2^{(1/2\beta)-1}|z|^{1/\beta}}\,dz \\ &\leq C'_\beta \xi_\beta^{c2^{(1/2\beta)-1}} = \delta\,.\end{aligned}$$

For C' sufficiently large, it holds $\log C'_\beta \geq \log 2$ and we obtain

$$\begin{aligned}\inf\left\{\lambda \geq 0 : \int_{\{z\cdot n_u > \lambda\}} Wg(z)\,dz < \delta\right\} &\leq \lambda_0 \\ &= \left(\frac{1}{c2^{(1/2\beta)-1}}\right)^\beta \left[\log C'_\beta + \log(1/\delta)\right]^\beta \\ &\lesssim \left(\frac{2}{c}\right)^\beta \left[\log C'_\beta \frac{\log(1/\delta)}{\log 2} + \log(1/\delta)\frac{\log C'_\beta}{\log 2}\right]^\beta \\ &= \left(\frac{4\log C'_\beta}{c\log 2}\right)^\beta [\log(1/\delta)]^\beta\end{aligned}$$

which yields (3.14).

3.3.3 Eigenvalues estimate

The concentration operator is compact, self-adjoint, positive semi-definite and contractive. Therefore, its spectrum is contained in the interval $[0,1]$ and we can consider its eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ in decreasing order. More precisely, we have $\{\lambda_k : k \in \mathbb{N}\} \setminus \{0\} = \sigma(T_\Omega) \setminus \{0\}$, as sets with multiplicities, where

$$\lambda_k = \inf\{\|L_\Omega - S\| : \ S \in \mathcal{L}(L^2(\mathbb{R}^d)),\, \dim(R(S)) < k\}, \qquad k \in \mathbb{N}\,;$$

see, e.g., [14, Lemma 4.3].

Spectral deviation bounds can be applied to deduce estimates for the individual eigenvalues. Set $A_\Omega = \lceil |\Omega| \rceil$. Heuristically, eigenvalues are expected to concentrate near 1 and 0: given the inequalities (3.10) and (3.12), we expect A_Ω to be the index $k \in \mathbb{N}$ where the eigenvalue λ_k crosses the threshold $1/2$. In fact, if we choose as threshold δ the eigenvalue λ_k for some $k \in \mathbb{N}$, the right hand side of those inequalities is minimised when λ_k is approximately $1/2$ – that is when $\tau \geq 2$ is approximately 2 – and in that case the left hand side suggests that k is approximately $|\Omega|$. Precisely, Theorem 3.3.2 yields the following estimates [28, v5, Corollary 1.7].

Corollary 3.3.7 (Eigenvalues estimate, Gelfand-Shilov-type condition)**.** *Suppose $g \in L^2(\mathbb{R}^d)$ and $\Omega \subseteq \mathbb{R}^{2d}$ satisfy the conditions of Theorem 3.3.2 with $\eta \in (0,1]$. Using the theorem's notation, write*

$$\gamma := \frac{2C'_g}{\kappa_{\partial\Omega,\eta}\cdot\eta^{2d-1}} \cdot \mathcal{H}^{2d-1}(\partial\Omega).$$

Then the following statements hold.

(i) For every $k = A_\Omega + \gamma h \in \mathbb{N}$ with $h \geq 1$,

$$\lambda_k \leq e^{-\left(\frac{h}{e(1+\log h)}\right)^{1/2d\beta}}. \tag{3.15}$$

(ii) For every $k = A_\Omega - \gamma h \in \mathbb{N}$ *with* $h \geq 1$*,*

$$\lambda_k \geq 1 - e^{-\left(\frac{h}{e(1+\log h)}\right)^{1/2d\beta}}. \tag{3.16}$$

Similarly, Theorem 3.3.4 gives the following result [28, v5, Corollary 1.8].

Corollary 3.3.8 (Eigenvalues estimate, polynomial decay)**.** *Suppose* $g \in L^2(\mathbb{R}^d)$ *and* $\Omega \subseteq \mathbb{R}^{2d}$ *satisfy the conditions of Theorem 3.3.4. Using the theorem's notation, write*

$$\gamma \coloneqq C_g' \cdot \left(\kappa_{\partial\Omega,\eta} \cdot \eta^{2d-1}\right)^{-\frac{s-1}{2d+s-1}} \cdot \mathcal{H}^{2d-1}(\partial\Omega).$$

Then the following statements hold.

(i) For every $k = A_\Omega + \gamma h \in \mathbb{N}$ *with* $h \geq 1$*,*

$$\lambda_k \leq e^{\left(\frac{2d+s-1}{2d}\right)^2} \cdot h^{-\frac{2d+s-1}{2d}} \cdot (1 + \log(C_g h))^{\frac{s-1}{2d}}. \tag{3.17}$$

(ii) For every $k = A_\Omega - \gamma h \in \mathbb{N}$ *with* $h \geq 1$*,*

$$\lambda_k \geq 1 - e^{\left(\frac{2d+s-1}{2d}\right)^2} \cdot h^{-\frac{2d+s-1}{2d}} \cdot (1 + \log(C_g h))^{\frac{s-1}{2d}}. \tag{3.18}$$

The sharpness of bounds in Corollaries 3.3.6, 3.3.7 and 3.3.8 is tested in [28, Chapter 4.3] using the Gaussian as window function and the disk as compact set. In fact, it is known that in this case the concentration operator is diagonal in the Hermite basis, and the spectrum can be computed explicitly [8]. With the examples provided, Marceca and Romero show that the dependence of (3.10) and (3.12) in $\tau = \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$ is the best possible up to $\log\log$ factors.

3.4 The main tools in the proofs

3.4.1 Toeplitz operators for the concentration operators

As seen in Corollary 2.1.10, for a normalised window function $g \in L^2(\mathbb{R}^d)$, $\|g\|_2 = 1$, the short-time Fourier transform defines an isometry $V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$. Its range is the closed subspace

$$\mathbb{H} \coloneqq \left\{V_g f \colon f \in L^2(\mathbb{R}^d)\right\} \subseteq L^2(\mathbb{R}^{2d}),$$

which is a reproducing kernel Hilbert space. Indeed, for $f \in L^2(\mathbb{R}^d)$, its STFT $V_g f$ is a continuous function by Lemma 2.1.5, and we can apply the inversion formula (2.13) to write

$$\begin{aligned}
V_g f(x,w) &= \langle f, M_w T_x g\rangle \\
&= \int_{\mathbb{R}^{2d}} V_g f(x',w')\langle M_{w'}T_{x'}g, M_w T_x g\rangle \, dx'\, dw' \\
&= \int_{\mathbb{R}^{2d}} V_g f(x',w')\langle g, M_{w-w'}T_{x-x'}g\rangle e^{2\pi i x'\cdot(w'-w)} \, dx'\, dw' \\
&= \int_{\mathbb{R}^{2d}} V_g f(x',w') \left[V_g g((x,w)-(x',w'))e^{2\pi i x'\cdot(w'-w)}\right] dx'\, dw' \,,
\end{aligned} \tag{3.19}$$

with $(x,w) \in \mathbb{R}^{2d}$. Then the reproducing kernel is

$$K(z,z') = V_g g(z-z')e^{2\pi i(w'-w)\cdot x'}\,, \qquad z=(x,w), z'=(x',w') \in \mathbb{R}^{2d}. \tag{3.20}$$

By Proposition 1.3.5, the orthogonal projection $P : L^2(\mathbb{R}^{2d}) \to \mathbb{H}$ is the integral operator

$$PF(z) = \int_{\mathbb{R}^{2d}} K(z,z')F(z')\, dz'\,, \qquad F \in L^2(\mathbb{R}^{2d}).$$

Recalling formula (2.14) for the adjoint operator V_g^*, computations analogous to (3.19) show that

$$P = V_g V_g^*\,. \tag{3.21}$$

Given a compact domain $\Omega \subseteq \mathbb{R}^{2d}$, we define the *Gabor-Toeplitz operator* $T_\Omega : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ by

$$T_\Omega F = P\big(1_\Omega \cdot PF\big)\,, \qquad F \in L^2(\mathbb{R}^{2d}).$$

Writing the operator T_Ω with respect to the decomposition $L^2(\mathbb{R}^{2d}) = \mathbb{H} \bigoplus \mathbb{H}^\perp$, by equation (3.21) one has

$$T_\Omega = \begin{bmatrix} V_g L_\Omega V_g^* & 0 \\ 0 & 0 \end{bmatrix}\,; \tag{3.22}$$

see, e.g., [13]. This shows that the time-frequency concentration operator is unitarily equivalent to the contraction of a Toeplitz operator on the range of the short-time Fourier transform. Thus, the spectrum of T_Ω and L_Ω coincide apart from the multiplicity of the eigenvalue $\lambda = 0$, and for every $\delta \in (0,1)$ we have

$$\{\lambda \in \sigma(L_\Omega) : \lambda > \delta\} = \{\lambda \in \sigma(T_\Omega) : \lambda > \delta\}$$

Moreover, applying Theorems 1.2.6 and 1.2.8, a direct calculation shows what follows.

Lemma 3.4.1 (See [1, Lemma 2.1]). *The operator* T_Ω *is trace-class, and so is* L_Ω*, and* $0 \le T_\Omega \le I$*. Moreover,*

$$\operatorname{trace}(L_\Omega) = \operatorname{trace}(T_\Omega) = |\Omega| \,, \tag{3.23}$$

$$\operatorname{trace}(L_\Omega^2) = \operatorname{trace}(T_\Omega^2) = \int_\Omega \int_\Omega |V_g g(z - z')|^2 \, dz \, dz' \,. \tag{3.24}$$

The proofs in [28] rely on these specific characteristic of time-frequency concentration operators, namely that they are unitarily equivalent to Toeplitz operators on the range of the short-time Fourier transform, and that such range is a reproducing kernel subspace of $L^2(\mathbb{R}^{2d})$. More specifically, the proofs are based on the analysis of the *Gabor-Hankel operator* $H_\Omega : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$, defined as

$$H_\Omega F = (I - P)\big(1_\Omega \cdot PF\big) \,, \qquad F \in L^2(\mathbb{R}^{2d}) \,,$$

which is related to T_Ω by

$$H_\Omega^* H_\Omega = T_\Omega - T_\Omega^2 \,. \tag{3.25}$$

In what follows, we will omit the dependence on Ω and denote T_Ω and H_Ω by T and H respectively.

3.4.2 The Schatten p-norm of the Hankel operator

The core of the theorems in [28] is estimating the Schatten p-norms of the Hankel operator H, with a value of p that is optimised as a function of the spectral parameter δ. In fact, the next lemma shows how the spectral deviation of the eigenvalue counting function is related to $\|H\|_p$ for $p \in (0, 2]$. The proof is included for the sake of completeness.

Lemma 3.4.2 (See [28, v5, Lemma 4.1]). *Let* $g \in L^2(\mathbb{R}^d)$ *with* $\|g\|_{L^2(\mathbb{R}^d)} = 1$*. Let* $\Omega \subseteq \mathbb{R}^{2d}$ *be compact, and let* H *be the corresponding Gabor-Hankel operator. Then, for every* $\delta \in (0, 1)$ *and for every* $p \in (0, 2]$,

$$\big|\#\{\lambda \in \sigma(L_\Omega) : \lambda > \delta\} - |\Omega|\big| \le (\delta(1 - \delta))^{-p/2} \cdot \|H\|_p^p \,. \tag{3.26}$$

Proof. Define the function $G\colon [0, 1] \to [0, 1]$ by

$$G(t) = \begin{cases} -t & \text{if } 0 \le t \le \delta \\ 1 - t & \text{if } \delta < t \le 1 \end{cases} \,.$$

Since $\delta < 1$ and $1 - \delta < 1$, for $t \le \delta$ we have

$$|G(t)| = t \le \frac{t}{\delta} \le \frac{t - t^2}{\delta - \delta^2} \le 1 \,,$$

whereas for $\delta < t \le 1$

$$|G(t)| = 1 - t \le \frac{1-t}{1-\delta} \le \frac{t-t^2}{\delta-\delta^2} \le 1\,.$$

Since $p/2 \le 1$, we get

$$|G(t)| \le \left(\frac{t-t^2}{\delta-\delta^2}\right)^{p/2}, \qquad t \in [0,1].$$

The Gabor-Toeplitz operator T is related to L_Ω by (3.22), and to H by (3.25). Since $0 \le T \le I$, and by equations (1.22) and (3.23), we apply the functional calculus to conclude

$$\begin{aligned} \big|\#\{\lambda \in \sigma(L_\Omega) : \lambda > \delta\} - |\Omega|\big| &= |\operatorname{trace}(1_{(\delta,+\infty)}(T)) - \operatorname{trace}(T)| \\ &= |\operatorname{trace}(G(T))| \le \operatorname{trace}(|G|(T)) \\ &\le (\delta-\delta^2)^{-p/2} \cdot \operatorname{trace}\left[(T-T^2)^{p/2}\right] \\ &= (\delta(1-\delta))^{-p/2} \cdot \|H\|_p^p\,. \end{aligned} \qquad \square$$

Then, the Schatten p-norms of the Hankel operator are bounded as follows. An auxiliary parameter $\alpha > 0$ is introduced, and will allow for more flexibility in the optimisation of the final bounds.

Proposition 3.4.3 (See [28, v5, Proposition 3.1]). *Let $\Omega \subseteq \mathbb{R}^{2d}$ be a compact set with regular boundary at scale $\eta > 0$, and let H be the corresponding Gabor-Hankel operator. Let $p \in (0,2]$ and $\alpha > 0$. Then*

$$\|H\|_p^p \le C_d \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot \frac{\left\|(1+|\cdot|\eta^{-1})^{(2d-1)(2-p)/2p}(1+|\cdot|)^{(1+\alpha)(2-p)/2p+1/2}V_g g\right\|_2^p}{(\kappa_{\partial\Omega,\eta}\cdot\alpha)^{1-p/2}}, \tag{3.27}$$

where C_d is a constant that only depends on d.

In particular, if $\eta \in (0,1]$,

$$\|H\|_p^p \le C_d \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot \frac{\left\|(1+|\cdot|)^{(2d+\alpha)(2-p)/2p+1/2}V_g g\right\|_2^p}{(\kappa_{\partial\Omega,\eta}\cdot\eta^{2d-1}\cdot\alpha)^{1-p/2}}\,. \tag{3.28}$$

We do not include the proof of the proposition. However, its layout will be followed in Chapter 4 to extend the results to Gabor multipliers, and this will give a clear insight into it. Generally speaking, the proof strongly relies on the fact that the range of the short-time Fourier transform $\mathbb{H}$ is a reproducing kernel Hilbert space, and makes use of the next three results.

First, this elementary lemma is applied to $A = H^*H$, which is a positive operator on the Hilbert space $X = \mathbb{H}$, and $q = p/2$, with $p \in (0,2]$, in the investigation of the Schatten p-norm of H.

Lemma 3.4.4 (See [28, v5, Lemma 2.1], [39, Proposition 6.3.3]). *Let A be a positive operator on a Hilbert space X, and let $q \in (0,1]$. Then*

$$\langle A^q x, x\rangle \leq \langle Ax, x\rangle^q , \qquad x \in X, \|x\| = 1. \tag{3.29}$$

Further in the proof, integrals of the reproducing kernel of $\mathbb{H}$ (3.20) over the concentration domain Ω are involved. In particular, in order to compute

$$\int_\Omega \int_{\Omega^c} (1+|z'-z|\eta^{-1})^{(2d-1)(2-p)/p}(1+|z'-z|)^{(1+\alpha)(2-p)/p}|V_g g(z'-z)|^2 \, dz' dz$$

with η, p and α as in the statement of Proposition 3.4.3, the next lemma is applied to the function

$$\varphi(z) = (1+|z|\eta^{-1})^{(2d-1)(2-p)/p}(1+|z|)^{(1+\alpha)(2-p)/p}|V_g g(z)|^2 , \qquad z \in \mathbb{R}^{2d},$$

since

$$\int_\Omega \int_{\Omega^c} \varphi(z'-z)\, dz'\, dz \leq \left\| (\textstyle\int \varphi) \cdot 1_\Omega - 1_\Omega * \varphi \right\|_1 .$$

The lemma separates explicitly the geometrical contribution of Ω and the integral decay properties of φ, i.e. of $V_g g$ in our setting.

Lemma 3.4.5 (See [28, v5, Lemma 2.2]). *Let $\Omega \subseteq \mathbb{R}^{2d}$ be a compact set and let $\varphi \in L^1(\mathbb{R}^{2d})$. Then, for $E \in \{\Omega, \Omega^c\}$ it holds*

$$\left\| 1_E * \varphi - \big(\textstyle\int \varphi\big) \cdot 1_E \right\|_1 \leq \mathcal{H}^{2d-1}(\partial\Omega) \cdot \int_{\mathbb{R}^{2d}} |z||\varphi(z)|\, dz . \tag{3.30}$$

Lastly, the following bound for the Haudorff measure of the level sets of the distance from $\partial\Omega$ is a key technical tool in computations.

Proposition 3.4.6 (See [28, v5, Proposition 2.3], [4, Theorems 5, 6]). *Let $\Omega \subseteq \mathbb{R}^{2d}$ be a compact set with regular boundary at scale $\eta > 0$. Then*

$$\mathcal{H}^{2d-1}\big(\{z \in \mathbb{R}^{2d} \colon\ \mathrm{d}(z, \partial\Omega) = r\}\big) \leq \frac{C_d}{\kappa_{\partial\Omega,\eta}} \cdot \mathcal{H}^{2d-1}(\partial\Omega) \cdot \Big(1 + \frac{r}{\eta}\Big)^{2d-1} \tag{3.31}$$

for almost every $r > 0$, where C_d is a constant that depends only on d.

In addition, $|\nabla\, \mathrm{d}(z, \partial\Omega)| = 1$ for almost every $z \in \mathbb{R}^{2d}$.

Finally, given Proposition 3.4.3, the last step consists in estimating the L^2-norms in inequalities (3.27) and (3.28) using the decay assumptions on the STFT of the window function g, namely (3.9) and (3.11). Theorems 3.3.2 and 3.3.4 follow by the combination of these estimates, Lemma 3.4.2 and Proposition 3.4.3, where the values of p and α is chosen in function of δ to get the results (3.10) and (3.12) respectively. See [28] for complete details. However, in dealing with Gabor multipliers in the next chapter the structure of the proof will be preserved and made clear.

Chapter 4

Gabor multipliers

As seen in the previous chapter, the spectral deviation bound for concentration operators proved in [28] is essentially due to the fact that the range of the short-time Fourier transform is a reproducing kernel Hilbert subspace of $L^2(\mathbb{R}^{2d})$. Thus, we are interested in applying the same layout of the proofs presented in [28] to other operators that share the same structure with the concentration operator.

This chapter deals with Gabor multipliers. After some preliminary work on the geometry of the space, we are able to prove the exact same bounds as the ones valid for the concentration operators.

In what follows, we will denote with $\Lambda_{\alpha,\beta}$ the lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d \subseteq \mathbb{R}^{2d}$ for some parameters $\alpha, \beta > 0$. The time-frequency shift of a function $g \in L^2(\mathbb{R}^d)$ will be written as

$$\pi(\lambda)g = T_{\alpha k} M_{\beta n} g\,, \qquad \lambda = (\alpha k, \beta n) \in \Lambda_{\alpha,\beta}.$$

In particular, for $\alpha = \beta = \frac{1}{L}$ with $L \geq 1$, the lattice will be denoted Λ_L. Notice that the condition $L \geq 1$ is not restrictive, due to Theorem 2.3.5.

4.1 Defining Gabor multipliers

In great generality, Gabor multipliers can be defined as follows. Fix two functions $g_1, g_2 \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda \subseteq \mathbb{R}^{2d}$, and let $m = (m_\lambda)_{\lambda\in\Lambda}$ be a complex-valued sequence on Λ. Then the *Gabor multiplier* associated to the triple (g_1, g_2, Λ) with symbol m is given by

$$M_m(f) = M_{g_1,g_2,\Lambda,m}(f) := \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g_1\rangle \pi(\lambda)g_2$$

with $f\colon \mathbb{R}^d \to \mathbb{C}$ a function. Depending on the properties of the *analysis window* g_1, the *synthesis window* g_2 and the multiplier sequence m, the operator $M_{g_1,g_2,\Lambda,m}$ is bounded between various spaces; see, e.g., [18, Theorem 5.4.1], and in general [18] for a wider dissertation on Gabor multipliers.

We will focus our attention on the following specific setting. Consider a function $g \in L^2(\mathbb{R}^d)$ and parameters $\alpha, \beta > 0$ such that $(\pi(\lambda)g)_{\lambda\in\Lambda_{\alpha,\beta}} = \mathcal{G}(g,\alpha,\beta)$ is a Gabor frame, and let $\gamma = S^{-1}g$ be its dual window as in Theorem 2.3.2. Set $g_1 = g$ and $g_2 = \gamma$, and let $m = 1_\Omega$ with $\Omega \subseteq \Lambda_{\alpha,\beta}$ compact (that is, finite). Then, the Gabor multiplier $M_\Omega = M_{g,\gamma,\Lambda_{\alpha,\beta},1_\Omega} \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ reads as

$$M_\Omega f = \sum_{\lambda\in\Omega} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma\,, \qquad f \in L^2(\mathbb{R}^d),$$

or, equivalently,

$$M_\Omega = C_\gamma^*[1_\Omega \cdot C_g]\,,$$

where $C_g, C_\gamma \colon L^2(\mathbb{R}^d) \to \ell^2(\Lambda_{\alpha,\beta})$ are the coefficient operators associated to the frames $(\pi(\lambda)g)_{\lambda\in\Lambda_{\alpha,\beta}}$ and $(\pi(\lambda)\gamma)_{\lambda\in\Lambda_{\alpha,\beta}}$ respectively.

By Proposition 1.1.4, the range of the operator C_g is the closed subspace

$$\mathbb{H} \coloneqq \{C_g f \colon f \in L^2(\mathbb{R}^d)\} \subseteq \ell^2(\Lambda_{\alpha,\beta}) \tag{4.1}$$

which is a reproducing kernel Hilbert space. In fact, for $f \in L^2(\mathbb{R}^d)$, the inversion formula (2.30) gives

$$\begin{aligned} C_g f(\lambda) = \langle f, \pi(\lambda)g\rangle &= \sum_{\lambda'\in\Lambda_{\alpha,\beta}} \langle f, \pi(\lambda')g\rangle \langle \pi(\lambda')\gamma, \pi(\lambda)g\rangle \\ &= \sum_{\lambda'\in\Lambda_{\alpha,\beta}} C_g f(\lambda') \langle \pi(\lambda')\gamma, \pi(\lambda)g\rangle \end{aligned}$$

for $\lambda \in \Lambda_{\alpha,\beta}$, and the reproducing kernel is

$$K(\lambda,\lambda') = \langle \pi(\lambda')\gamma, \pi(\lambda)g\rangle = \langle g, \pi(\lambda-\lambda')g\rangle = V_g\gamma(\lambda-\lambda')e^{2\pi i(w'-w)\cdot x'}$$

for $\lambda = (x,w), \lambda' = (x',w') \in \Lambda_{\alpha,\beta}$.

In what follows, we will only focus on the case $\Lambda = \Lambda_L$ for some $L \geq 1$ and $(\pi(\lambda)g)_{\lambda\in\Lambda_L}$ being a Parseval frame with $\|g\|_2 = 1$: these choices lead to a complete superposition of the structures of the concentration operator and the Gabor multiplier. Indeed, in this case we have $\gamma = g$. The Gabor multiplier reads

$$M_\Omega f = \sum_{\lambda\in\Omega} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g\,, \qquad f \in L^2(\mathbb{R}^d), \tag{4.2}$$

that is,

$$M_\Omega = C_g^*[1_\Omega \cdot C_g]\,, \tag{4.3}$$

whereas the reproducing kernel of $\mathbb{H}$ becomes

$$K(\lambda,\lambda') = V_g g(\lambda-\lambda')e^{2\pi i(w'-w)\cdot x'}\,, \qquad \lambda = (x,w), \lambda' = (x',w') \in \Lambda_L. \tag{4.4}$$

Compare equations (4.2), (4.3) and (4.4) with equations (3.2), (3.3) and (3.20).

Recall that, by Proposition 1.3.4(a), we get

$$K(\lambda, \lambda') = \sum_{\mu \in \Lambda_L} K(\lambda, \mu) K(\mu, \lambda'), \qquad \lambda, \lambda' \in \Lambda_L. \tag{4.5}$$

By Proposition 1.3.5, the orthogonal projection $P : \ell^2(\Lambda_L) \to \mathbb{H}$ is the operator

$$Pv(\lambda) = \sum_{\lambda' \in \Lambda_L} v(\lambda') K(\lambda, \lambda'), \qquad v \in \ell^2(\Lambda_L), \tag{4.6}$$

and easy computations show that

$$P = C_g C_g^*. \tag{4.7}$$

Notice that, by Cauchy-Schwartz, it holds $\|V_g g\|_{\ell^\infty(\Lambda_L)} \leq \|g\|^2_{\ell^2(\Lambda_L)} = 1$, while for the Parseval frame condition with $\|g\|_2 = 1$ – see (2.32) – we get

$$\|V_g g\|^2_{\ell^2(\Lambda_L)} = \sum_{\lambda \in \Lambda_L} |\langle g, \pi(\lambda) g\rangle|^2 = \|g\|_2^2 = 1.$$

Consequently, by Hölder's inequality,

$$\|V_g g\|_{\ell^1(\Lambda_L)} \geq 1. \tag{4.8}$$

4.2 Preliminaries on the lattice Λ_L

In order to adapt the results in [28] to Gabor multipliers, we will need the following results related to the structure of the lattice Λ_L.

4.2.1 Perimeter and level sets

Motivated by the presence of the Hausdorff measure in the estimates for the concentration operator, we look for a definition of boundary for a discrete set. In fact, the topological boundary for a discrete set is empty and therefore useless in this context.

Definition 4.2.1. The *discrete boundary* of a set $E \subseteq \Lambda_L$ is given by

$$\partial E := \left\{ \lambda \in E \mid \min\{|\lambda - \mu| \colon \mu \in \Lambda_L \setminus E\} = L^{-1} \right\}. \tag{4.9}$$

Remark 4.2.2. Equivalently,

$$\partial E = \{\lambda \in E \mid \mathrm{d_{gr}}(\lambda, \Lambda_L \setminus E) = 1\},$$

where $\mathrm{d_{gr}}$ is the graph-distance – which is defined as the number of edges of a geodesic (i.e. shortest) path connecting two vertices – of the square grid graph whose set of vertices is the lattice Λ_L.

Remark 4.2.3. Notice that $\frac{L^{-1}}{\sqrt{2d}}\,\mathrm{d_{gr}}(\lambda,\rho) \leq |\lambda-\rho|$ for $\lambda,\rho\in\Lambda_L$. Therefore, for a set $E\subseteq\Lambda_L$ we have

$$\frac{L^{-1}}{\sqrt{2d}}\,\mathrm{d_{gr}}(\lambda,\partial E) \leq \mathrm{d}(\lambda,\partial E), \qquad \lambda\in\Lambda_L\,, \tag{4.10}$$

where d is the Euclidean distance.

In the study of concentration operators L_Ω for the STFT, the boundary of the compact set $\Omega\subseteq\mathbb{R}^{2d}$ becomes relevant in terms of its $(2d-1)$-Hausdorff measure $\mathcal{H}^{2d-1}(\partial\Omega)$. In this discrete setting, given Definition 4.2.1, the Hausdorff measure is naturally replaced with the counting measure, and the following lemma provides an estimate for the measure of the level sets of the distance function from the boundary of a compact set. Compare it to Proposition 3.4.6.

Lemma 4.2.4. *Let $\Omega\subseteq\Lambda_L$ be a compact set. Then*

$$\#\{\lambda\in\Lambda_L \mid \mathrm{d_{gr}}(\lambda,\partial\Omega)=n\} \leq C_d\cdot\#\partial\Omega\cdot(1+n)^{2d-1}\,, \qquad n\in\mathbb{N}_0, \tag{4.11}$$

where C_d is a constant that depends on d only.

Proof. If we fix $\mu\in\Lambda_L$ and $n\in\mathbb{N}$, then

$$\begin{aligned}\#\{\lambda\in\Lambda_L \mid \mathrm{d_{gr}}(\lambda,\mu)=n\} &\leq \#\big(\Lambda_L\cap\partial(L^{-1}[-n,n]^{2d})\big)\\ &= (2n+1)^{2d}-(2n-1)^{2d}\\ &= \sum_{k=0}^{2d}\binom{2d}{k}(2n)^k - \sum_{k=0}^{2d}\binom{2d}{k}(2n)^k(-1)^{2d-k}\\ &= 2\sum_{h=1}^{d}\binom{2d}{2h-1}(2n)^{2h-1} \leq C_d\cdot n^{2d-1}\,,\end{aligned}$$

with $C_d\geq 1$. Therefore $\#\{\lambda\in\Lambda_L \mid \mathrm{d_{gr}}(\lambda,\mu)=n\}\leq C_d\cdot(1+n)^{2d-1}$ for every $n\in\mathbb{N}_0$. Multiplying by $\#\partial\Omega$, the proof is concluded. ☐

We will also need the following integral property of the lattice Λ_L.

Lemma 4.2.5. *Consider* $0<\alpha\leq 1$. *Then:*

$$\sum_{\lambda\in\Lambda_L}\frac{1}{(1+L\cdot|\lambda|)^{2d+\alpha}} \leq C_d\cdot\frac{1}{\alpha}\,, \tag{4.12}$$

where C_d is a constant that depends on d only.

Proof. For every $n\in\mathbb{N}$, $1\leq n\leq 2d$, consider $\Lambda_L^n := L^{-1}\mathbb{Z}^n$. We will prove by induction that for every $n\in\mathbb{N}$, $1\leq n\leq 2d$,

$$\sum_{\lambda\in\Lambda_L^n}\frac{1}{(1+L\cdot|\lambda|)^{2d+\alpha}} \lesssim D_n\cdot\frac{1}{\alpha}, \qquad D_n := \prod_{k=1}^{n}(2^k+k), \tag{4.13}$$

which gives the assertion for $n = 2d$.

For $n = 1$, we use the integral bound and the condition $\alpha \le 1$ to get

$$\sum_{\lambda \in \Lambda_L^1} \frac{1}{(1 + L \cdot |\lambda|)^{2d+\alpha}} \le 1 + 2 \int_0^{+\infty} \frac{1}{(1+x)^{2d+\alpha}}\, dx$$
$$= 1 + 2 \frac{1}{2d + \alpha - 1} \le 3 \cdot \frac{1}{\alpha} = D_1 \cdot \frac{1}{\alpha}.$$

Given $n \in \mathbb{N}$, $1 < n \le 2d$, a change of variables leads to

$$\sum_{\lambda \in \Lambda_L^n} \frac{1}{(1 + L \cdot |\lambda|)^{2d+\alpha}} \le 2^n \sum_{\lambda \in \Lambda_L^n \cap (\mathbb{R}_{>0})^n} \frac{1}{(1 + L \cdot |\lambda|)^{2d+\alpha}} + n \sum_{\lambda \in \Lambda_L^{n-1}} \frac{1}{(1 + L \cdot |\lambda|)^{2d+\alpha}}\,. \tag{4.14}$$

For the first sum in the right side of (4.14), we use the integral bound again:

$$\sum_{\lambda \in \Lambda_L^n \cap (\mathbb{R}_{>0})^n} \frac{1}{(1 + L \cdot |\lambda|)^{2d+\alpha}} \le \int_{(\mathbb{R}_{\ge 0})^n} \frac{1}{(1 + |z|)^{2d+\alpha}}\, dz \lesssim \frac{1}{\alpha}\,.$$

By induction in (4.14) we get

$$\sum_{\lambda \in \Lambda_L^n} \frac{1}{(1 + L \cdot |\lambda|)^{2d+\alpha}} \lesssim 2^n \cdot \frac{1}{\alpha} + n \cdot D_{n-1} \cdot \frac{1}{\alpha} \le D_n \cdot \frac{1}{\alpha}$$

that is (4.13). ☐

4.2.2 Discrete convolution

The proof of Proposition 3.4.3 involves convolution in Lemma 3.4.5. The lemma can be converted to the discrete setting of Λ_L, to which the notion of convolution still applies.

Lemma 4.2.6. *Let $\Omega \subseteq \Lambda_L$ be a compact set, and let $\varphi \in \ell^1(\Lambda_L)$ be a non-negative sequence. Then, for $E \in \{\Omega, \Omega^c\}$ it holds*

$$\Big\| 1_E * \varphi - \big(\Sigma\varphi\big) \cdot 1_E \Big\|_{\ell^1(\Lambda_L)} \le C_d \cdot \#\partial\Omega \cdot L \sum_{\lambda \in \Lambda_L} |\lambda| \varphi(\lambda)\,, \tag{4.15}$$

where C_d is a constant depending only on d.

Proof. Let $Q_1 := [0,1)^{2d}$ and $Q_2 := (-1,1)^{2d}$. Let $(e_n)_{n=1}^{2d}$ be the canonical orthonormal basis of $\mathbb{R}^{2d}$ and let $V := \{\sum_{n=1}^{2d} \delta_n e_n \mid \delta_n \in \{0,1\}\}$ be the set of vertices of $\overline{Q_1}$.

Let us initially prove the inequality for $E = \Omega$. Consider $F := \Omega + L^{-1} Q_1$, whose closure $\overline{F}$ is compact in $\mathbb{R}^{2d}$. As in the proof of [29, Theorem 1.3], we note that ∂F is an almost disjoint union of faces of cubes (the intersection of any two faces has zero $\mathcal{H}^{2d-1}$-measure). Moreover, each of those faces is contained in $k+\partial(L^{-1}Q_1)$ for exactly one $k\in\partial\Omega$. In particular,

$$\partial F \subseteq \bigcup_{k\in\partial\Omega} k+\partial(L^{-1}Q_1)\,.$$

Thus,

$$\mathcal{H}^{2d-1}(\partial F) \leq 2(2d)\cdot L^{-(2d-1)}\cdot \#\partial\Omega\,. \tag{4.16}$$

Since $\mathbb{R}^{2d} = \cup_{\lambda\in\Lambda_L}(\lambda+L^{-1}Q_1)$, for every $z\in\mathbb{R}^{2d}$ there exists a unique $\lambda_z\in\Lambda_L$ such that $z\in\lambda_z+L^{-1}Q_1$; then, define the function

$$\tilde{\varphi}(z) := \max\left\{\varphi(\mu) \mid \mu\in(\lambda_z+L^{-1}V)\setminus\{0\}\right\}\,,\qquad z\in\mathbb{R}^{2d}.$$

By definition $\tilde{\varphi}$ is a non-negative function, and $\tilde{\varphi}\in L^1(\mathbb{R}^{2d})$ since

$$\begin{aligned}
\int_{\mathbb{R}^{2d}}\tilde{\varphi}(z)\,dz &= \sum_{\lambda\in\Lambda_L}\int_{\lambda+L^{-1}Q_1}\tilde{\varphi}(z)\,dz\\
&\leq \sum_{\lambda\in\Lambda_L}\int_{\lambda+L^{-1}Q_1}\Big(\sum_{\mu\in(\lambda+L^{-1}V)\setminus\{0\}}\varphi(\mu)\Big)\,dz\\
&\leq \sum_{\lambda\in\Lambda_L}\int_{\lambda+L^{-1}Q_1}\Big(\sum_{\mu\in\lambda+L^{-1}V}\varphi(\mu)\Big)\,dz\\
&= \sum_{\lambda\in\Lambda_L}\int_{\lambda+L^{-1}Q_1}\Big(\sum_{e\in V}\varphi(\lambda+L^{-1}e)\Big)\,dz\\
&= L^{-2d}\sum_{\lambda\in\Lambda_L}\sum_{e\in V}\varphi(\lambda+L^{-1}e) = L^{-2d}\sum_{e\in V}\sum_{\lambda\in\Lambda_L}\varphi(\lambda+L^{-1}e)\\
&= L^{-2d}\sum_{e\in V}\sum_{\lambda\in\Lambda_L}\varphi(\lambda) = 2^{2d}L^{-2d}\|\varphi\|_{\ell^1(\Lambda_L)} < +\infty\,,
\end{aligned}$$

Thus, by Lemma 3.4.5 we get

$$\Big\|1_F*\tilde{\varphi}-\big(\textstyle\int\tilde{\varphi}\big)\cdot 1_F\Big\|_{L^1(\mathbb{R}^{2d})} \leq \mathcal{H}^{2d-1}(\partial F)\cdot\int_{\mathbb{R}^{2d}}|z|\tilde{\varphi}(z)\,dz\,. \tag{4.17}$$

Let us consider separately the two sides of inequality (4.17). Since $\tilde{\varphi}$ is non-negative, a direct computation of the left hand side shows that

$$\begin{aligned}
\Big\|1_F*\tilde{\varphi}-\big(\textstyle\int\tilde{\varphi}\big)\cdot 1_F\Big\|_{L^1(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^{2d}}\left|\int_F\tilde{\varphi}(z-z')\,dz' - 1_F(z)\int_{\mathbb{R}^{2d}}\tilde{\varphi}(z-z')\,dz'\right|dz\\
&= \int_F\left|\int_{F^c}\tilde{\varphi}(z-z')\,dz'\right|dz + \int_{F^c}\left|\int_F\tilde{\varphi}(z-z')\,dz'\right|dz\\
&= \int_{F^c}\int_F\tilde{\varphi}(z-z')\,dz\,dz' + \int_F\int_{F^c}\tilde{\varphi}(z-z')\,dz\,dz'.
\end{aligned} \tag{4.18}$$

By definition $F = \cup_{\lambda\in\Omega}(\lambda + L^{-1}Q_1)$, and then $F^c = \cup_{\lambda\in\Omega^c}(\lambda + L^{-1}Q_1)$. For the first integral in (4.18),

$$\begin{aligned}\int_{F^c}\int_F \tilde{\varphi}(z-z')\,dz\,dz' &= \sum_{\lambda\in\Omega^c}\sum_{\mu\in\Omega}\int_{\lambda+L^{-1}Q_1}\int_{\mu+L^{-1}Q_1}\tilde{\varphi}(z-z')\,dz\,dz'\\ &= \sum_{\lambda\in\Omega^c}\sum_{\mu\in\Omega}\int_{L^{-1}Q_1}\int_{L^{-1}Q_1}\tilde{\varphi}((\lambda-\mu)+(z-z'))\,dz\,dz'.\end{aligned}$$

For $z, z' \in L^{-1}Q_1$ we have that $z - z' \in L^{-1}Q_2$. Moreover, for $\lambda \in \Omega^c$ and $\mu \in \Omega$ we have $\lambda - \mu \neq 0$; therefore, by definition $\tilde{\varphi}((\lambda-\mu)+(z-z')) \geq \varphi(\lambda-\mu)$. Then

$$\int_{F^c}\int_F \tilde{\varphi}(z-z')\,dz\,dz' \geq L^{-2(2d)}\sum_{\lambda\in\Omega^c}\sum_{\mu\in\Omega}\varphi(\lambda-\mu). \tag{4.19}$$

Similarly, we get to the same conclusion for the second integral in (4.18):

$$\int_{F}\int_{F^c} \tilde{\varphi}(z-z')\,dz\,dz' \geq L^{-2(2d)}\sum_{\lambda\in\Omega}\sum_{\mu\in\Omega^c}\varphi(\lambda-\mu). \tag{4.20}$$

Combining the two estimates (4.19) and (4.20) with (4.18), we get

$$\begin{aligned}\left\|1_F * \tilde{\varphi} - \big(\textstyle\int\tilde{\varphi}\big)\cdot 1_F\right\|_{L^1(\mathbb{R}^{2d})} &\geq L^{-2(2d)}\left[\sum_{\lambda\in\Omega^c}\sum_{\mu\in\Omega}\varphi(\lambda-\mu) + \sum_{\lambda\in\Omega}\sum_{\mu\in\Omega^c}\varphi(\lambda-\mu)\right]\\ &= L^{-2(2d)}\left\|1_\Omega * \varphi - \big(\Sigma\varphi\big)\cdot 1_\Omega\right\|_{\ell^1(\Lambda_L)}.\end{aligned} \tag{4.21}$$

where the equality follows by computations analogous to (4.18) and the fact that φ is non-negative.

Now, consider the integral in the right side of (4.17):

$$\begin{aligned}\int_{\mathbb{R}^{2d}}|z|\tilde{\varphi}(z)\,dz &= \sum_{\lambda\in\Lambda_L}\int_{\lambda+L^{-1}Q_1}|z|\tilde{\varphi}(z)\,dz\\ &\leq \sum_{\lambda\in\Lambda_L}\int_{\lambda+L^{-1}Q_1}|z|\Big(\sum_{\mu\in(\lambda+L^{-1}V)\setminus\{0\}}\varphi(\mu)\Big)\,dz\\ &= \sum_{\lambda\in\Lambda_L}\sum_{\mu\in(\lambda+L^{-1}V)\setminus\{0\}}\varphi(\mu)\int_{L^{-1}Q_1}|\lambda+z|\,dz\,.\end{aligned} \tag{4.22}$$

Fix $\lambda \in \Lambda_L$ and $\mu \in (\lambda + L^{-1}V) \setminus \{0\}$; that is, $\mu \neq 0$ and $\mu = \lambda + L^{-1}e$ for some $e \in V$. In particular, $|\mu| \geq L^{-1}$. Fix $z \in L^{-1}Q_1$. If $\lambda \notin L^{-1}Q_2$, then

$$\begin{aligned}|\lambda+z| &\leq |\lambda+L^{-1}e| + |z-L^{-1}e| = |\mu| + |z-L^{-1}e|\\ &\leq |\mu| + L^{-1}\sqrt{2d} \leq (1+\sqrt{2d})|\mu| \leq 2\sqrt{2d}|\mu|\end{aligned}$$

because $z - L^{-1}e \in L^{-1}\overline{Q_2}$. Conversely, if $\lambda \in L^{-1}Q_2$, then

$$|\lambda + z| \leq |\lambda| + |z| \leq 2\sqrt{2d}L^{-1} \leq 2\sqrt{2d}|\mu|.$$

Applying this bound in (4.22), we get

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} |z|\tilde{\varphi}(z)\,dz &\leq 2\sqrt{2d}L^{-2d} \sum_{\lambda\in\Lambda_L} \sum_{\mu\in(\lambda+L^{-1}V)\setminus\{0\}} |\mu|\varphi(\mu) \\
&= 2\sqrt{2d}L^{-2d} \sum_{\lambda\in\Lambda_L} \sum_{\mu\in(\lambda+L^{-1}V)} |\mu|\varphi(\mu) \\
&= 2\sqrt{2d}L^{-2d} \sum_{\lambda\in\Lambda_L} \sum_{e\in V} |\lambda + L^{-1}e|\,\varphi(\lambda + L^{-1}e) \\
&= 2\sqrt{2d}L^{-2d} \sum_{e\in V} \sum_{\lambda\in\Lambda_L} |\lambda + L^{-1}e|\,\varphi(\lambda + L^{-1}e) \\
&= 2\sqrt{2d}L^{-2d} \sum_{e\in V} \sum_{\lambda\in\Lambda_L} |\lambda|\varphi(\lambda) \\
&= 2^{2d+1}\sqrt{2d}L^{-2d} \sum_{\lambda\in\Lambda_L} |\lambda|\varphi(\lambda)\,. \qquad (4.23)
\end{aligned}$$

Combining inequality (4.17) with (4.21), (4.23) and (4.16) we get:

$$\begin{aligned}
\Big\|1_\Omega * \varphi - \big(\Sigma\varphi\big)\cdot 1_\Omega\Big\|_{\ell^1(\Lambda_L)} &\leq L^{2(2d)}\Big\|1_F * \tilde{\varphi} - \big(\textstyle\int\tilde{\varphi}\big)\cdot 1_F\Big\|_{L^1(\mathbb{R}^{2d})} \\
&\leq L^{2(2d)} \cdot \mathcal{H}^{2d-1}(\partial F)\cdot \int_{\mathbb{R}^{2d}} |z|\tilde{\varphi}(z)\,dz \\
&\leq 2^{2d+2}(2d)^{3/2}\cdot \#\partial\Omega \cdot L \sum_{\lambda\in\Lambda_L} |\lambda|\varphi(\lambda)\,, \qquad (4.24)
\end{aligned}$$

which is the assertion for $E = \Omega$.

The case $E = \Omega^c$ can be proved either in the same way, and noticing that $\#\partial(\Omega^c) \leq 2^{2d}\#\partial\Omega$ in the right hand side of the final inequality, or directly from (4.24), using that $1_\Omega * \varphi - \big(\Sigma\varphi\big)\cdot 1_\Omega = \big(\Sigma\varphi\big)\cdot 1_{\Omega^c} - 1_{\Omega^c} * \varphi$. □

4.3 Toeplitz operators for Gabor multipliers

As seen in Section 4.1, there is a complete superposition of the definition of the concentration operator from the short-time Fourier transform and the definition of the Gabor multiplier (4.2) from the coefficient operator. Therefore, in the setting of equations (4.2)–(4.4), we are led to consider the Gabor-Toeplitz operator $T_\Omega : \ell^2(\Lambda_L) \to \ell^2(\Lambda_L)$ defined by

$$T_\Omega v = P\big(1_\Omega \cdot Pv\big)\,, \qquad v \in \ell^2(\Lambda_L).$$

With respect to the decomposition $\ell^2(\Lambda_L) = \mathbb{H} \bigoplus \mathbb{H}^\perp$, where we recall that $\mathbb{H}$ is the range of the coefficient operator (4.1), by (4.3) and (4.7) we have

$$T_\Omega = \begin{bmatrix} C_g M_\Omega C_g^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, M_Ω and T_Ω have the same non-zero eigenvalues.

Lemma 4.3.1. *The operator* T_Ω *is trace-class, and so is* M_Ω*, and* $0 \le T_\Omega \le I$*. Moreover,*

$$\operatorname{trace}(M_\Omega) = \operatorname{trace}(T_\Omega) = \#\Omega\,, \tag{4.25}$$

$$\operatorname{trace}(M_\Omega^2) = \operatorname{trace}(T_\Omega^2) = \sum_{\lambda\in\Omega}\sum_{\mu\in\Omega} |V_g g(\lambda-\mu)|^2\,. \tag{4.26}$$

Proof. The proof works as in [1, Lemma 2.1]. For $v \in \mathbb{H}$, we compute

$$\langle T_\Omega v, v\rangle = \langle P(\mathbf{1}_\Omega \cdot v), v\rangle = \langle \mathbf{1}_\Omega \cdot v, v\rangle = \sum_{\lambda\in\Omega} |v(\lambda)|^2\,.$$

Therefore, we get $0 \le T_\Omega \le I$; in particular, T_Ω is bounded and positive semi-definite. Let us introduce the function $\Theta(\lambda) := |V_g g(\lambda)|^2$ for $\lambda \in \Lambda_L$. Notice that $\Theta(\lambda-\rho) = |K(\lambda,\rho)|^2$ for $\lambda,\rho \in \Lambda_L$, and

$$\|\Theta\|_{\ell^1(\Lambda_L)} = \sum_{\lambda\in\Lambda_L} |V_g g(\lambda)|^2 = \sum_{\lambda\in\Lambda_L} |\langle g, \pi(\lambda) g\rangle|^2 = \|g\|_2^2 = 1 \tag{4.27}$$

because $(\pi(\lambda)g)_{\lambda\in\Lambda_L}$ is a Parseval frame.

We are now interested in the integral kernel of T_Ω, which is related to its spectral properties. For $v \in \ell^2(\Lambda_L)$,

$$\begin{aligned} T_\Omega v(\lambda) = P(\mathbf{1}_\Omega \cdot Pv)(\lambda) &= \sum_{\mu\in\Lambda_L} (\mathbf{1}_\Omega \cdot Pv)(\mu) K(\lambda,\mu) \\ &= \sum_{\mu\in\Lambda_L}\sum_{\rho\in\Lambda_L} \mathbf{1}_\Omega(\mu) v(\rho) K(\mu,\rho) K(\lambda,\mu) \\ &= \sum_{\rho\in\Lambda_L} v(\rho) \sum_{\mu\in\Lambda_L} \mathbf{1}_\Omega(\mu) K(\lambda,\mu) K(\mu,\rho) \end{aligned}$$

where we use equation (4.6) twice. Thus, the operator T_Ω has integral kernel

$$K_{T_\Omega}(\lambda,\rho) = \sum_{\mu\in\Lambda_L} \mathbf{1}_\Omega(\mu) K(\lambda,\mu) K(\mu,\rho).$$

By (4.27), we compute

$$\begin{aligned} \sum_{\lambda\in\Lambda_L} K_{T_\Omega}(\lambda,\lambda) &= \sum_{\lambda\in\Lambda_L}\sum_{\mu\in\Lambda_L} \mathbf{1}_\Omega(\mu) |K(\lambda,\mu)|^2 \\ &= \sum_{\mu\in\Omega}\sum_{\lambda\in\Lambda_L} \Theta(\lambda-\mu) = \#\Omega \cdot \|\Theta\|_{\ell^1(\Lambda_L)} = \#\Omega\,. \end{aligned}$$

By Theorem 1.2.8, we have that T_Ω and M_Ω are trace-class and (4.25) holds. Moreover, by Theorem 1.2.6 we get

$$\begin{aligned}
\operatorname{trace}(M_\Omega^2) = \operatorname{trace}(T_\Omega^2) = \|K_{T_\Omega}\|^2_{\ell^2(\Lambda_L \times \Lambda_L)} &= \sum_{\lambda \in \Lambda_L} \sum_{\mu \in \Lambda_L} |K_{T_\Omega}(\lambda, \mu)|^2 \\
&= \sum_{\lambda \in \Lambda_L} \sum_{\mu \in \Lambda_L} K_{T_\Omega}(\lambda, \mu) K_{T_\Omega}(\mu, \lambda) \\
&= \sum_{\rho \in \Lambda_L} \sum_{\eta \in \Lambda_L} 1_\Omega(\rho) 1_\Omega(\eta) \sum_{\lambda \in \Lambda_L} \sum_{\mu \in \Lambda_L} K(\lambda, \rho) K(\rho, \mu) K(\mu, \eta) K(\eta, \lambda) \\
&= \sum_{\rho \in \Lambda_L} \sum_{\eta \in \Lambda_L} 1_\Omega(\rho) 1_\Omega(\eta) K(\rho, \eta) K(\eta, \rho) \\
&= \sum_{\rho \in \Lambda_L} \sum_{\eta \in \Lambda_L} 1_\Omega(\rho) 1_\Omega(\eta) \Theta(\rho - \eta)\,,
\end{aligned}$$

which is equation (4.26). □

Lastly, we define the Gabor-Hankel operator $H_\Omega : \ell^2(\Lambda_L) \to \ell^2(\Lambda_L)$ by

$$H_\Omega v = (I - P) 1_\Omega \cdot P v, \qquad v \in \ell^2(\Lambda_L).$$

The Toeplitz operator and the Hankel operator are related by

$$H_\Omega^* H_\Omega = T_\Omega - T_\Omega^2. \tag{4.28}$$

In what follows, we will omit the dependence on Ω and denote T_Ω and H_Ω by T and H respectively.

4.4 Hankel operators estimates

We are now ready to reproduce in the contest of Gabor multipliers the estimates provided in [28] for the concentration operators. The layout of the proofs is preserved, even though some differences will occur when in [28] specific properties of $\mathbb{R}^{2d}$ are used. In particular, results in Section 4.2 replace Lemma 3.4.5 and Proposition 3.4.6.

Proposition 4.4.1. *Let $\Omega \subseteq \Lambda_L$ be a compact set, and let H be the corresponding Gabor-Hankel operator. Let $p \in (0, 2]$ and $\alpha \in (0, 1]$. Then*

$$\|H\|_p^p \le C_d \cdot \#\partial\Omega \cdot \frac{\|V_g g\|^p_{\ell^1(\Lambda_L)} \cdot \|(1 + L|\cdot|)^{[(2d-1)+(1+\alpha)](2-p)/2p+1/2} V_g g\|^p_{\ell^2(\Lambda_L)}}{\alpha^{1-p/2}}\,, \tag{4.29}$$

where C_d is a constant that depends only on d.

Proof. We argue as in the proof of Proposition 3.4.3; see [28, v2, Proposition 3.1].

Step 1. Put $A = (H^*H)^{p/2}$. Since $A|_{\mathbb{H}^\perp} = 0$, for an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathbb{H}$ we have

$$\begin{aligned}\|H\|_p^p &= \operatorname{trace}(A) = \sum_{n\in\mathbb{N}} \langle Ae_n, e_n\rangle = \sum_{n\in\mathbb{N}}\sum_{\lambda\in\Lambda_L} Ae_n(\lambda)\overline{e_n(\lambda)} \\ &= \sum_{n\in\mathbb{N}}\sum_{\lambda\in\Lambda_L} PAe_n(\lambda)\overline{e_n(\lambda)} = \sum_{n\in\mathbb{N}}\sum_{\lambda\in\Lambda_L} \langle Ae_n, K_\lambda\rangle\overline{e_n(\lambda)} \\ &= \sum_{\lambda\in\Lambda_L} \big\langle A\sum_{n\in\mathbb{N}} e_n\overline{e_n(\lambda)}, K_\lambda\big\rangle = \sum_{\lambda\in\Lambda_L}\langle AK_\lambda, K_\lambda\rangle = \sum_{\lambda\in\Lambda_L}\left\langle (H^*H)^{p/2}K_\lambda, K_\lambda\right\rangle,\end{aligned}$$

where we applied Proposition 1.3.4(d). Since $p/2 \le 1$, by Lemma 3.4.4 we have $\langle (H^*H)^{p/2}K_\lambda, K_\lambda\rangle \le \langle (H^*H)K_\lambda, K_\lambda\rangle^{p/2}$, and consequently

$$\|H\|_p^p \le \sum_{\lambda\in\Lambda_L}\langle H^*HK_\lambda, K_\lambda\rangle^{p/2} = \sum_{\lambda\in\Lambda_L}\|HK_\lambda\|^p_{\ell^2(\Lambda_L)}\,. \tag{4.30}$$

Let us write the function HK_λ explicitly: for $\mu \in \Lambda_L$,

$$\begin{aligned}HK_\lambda(\mu) &= (I-P)\big[\mathbf{1}_\Omega \cdot PK_\lambda\big](\mu) = (I-P)\big[\mathbf{1}_\Omega\cdot K_\lambda\big](\mu) \\ &= \mathbf{1}_\Omega(\mu)K(\mu,\lambda) - \sum_{\rho\in\Lambda_L}\mathbf{1}_\Omega(\rho)K(\rho,\lambda)K(\mu,\rho) \\ &= \sum_{\rho\in\Lambda_L}\big(\mathbf{1}_\Omega(\mu) - \mathbf{1}_\Omega(\rho)\big)K(\mu,\rho)K(\rho,\lambda) \\ &= \sum_{\rho\in\Lambda_L}\big(\mathbf{1}_\Omega(\mu)\mathbf{1}_{\Omega^c}(\rho) - \mathbf{1}_{\Omega^c}(\mu)\mathbf{1}_\Omega(\rho)\big)K(\mu,\rho)K(\rho,\lambda)\,.\end{aligned}$$

By the triangle inequality,

$$\|HK_\lambda\|_{\ell^2(\Lambda_L)} \le \Big(\sum_{\mu\in\Omega}\Big|\sum_{\rho\in\Omega^c}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac12} + \Big(\sum_{\mu\in\Omega^c}\Big|\sum_{\rho\in\Omega}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac12}.$$

We combine the last estimate with (4.30) and use the inequality $(a+b)^p \le 2(a^p+b^p)$ for $a, b \ge 0$, $0 < p \le 2$, to split the sums in four parts:

$$\begin{aligned}\|H\|_p^p &\lesssim \sum_{\lambda\in\Lambda_L}\Big(\sum_{\mu\in\Omega}\Big|\sum_{\rho\in\Omega^c}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac p2} + \sum_{\lambda\in\Lambda_L}\Big(\sum_{\mu\in\Omega^c}\Big|\sum_{\rho\in\Omega}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac p2} \\ &= \sum_{\lambda\in\Omega}\Big(\sum_{\mu\in\Omega}\Big|\sum_{\rho\in\Omega^c}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac p2} + \sum_{\lambda\in\Omega^c}\Big(\sum_{\mu\in\Omega}\Big|\sum_{\rho\in\Omega^c}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac p2} \\ &\quad+ \sum_{\lambda\in\Omega}\Big(\sum_{\mu\in\Omega^c}\Big|\sum_{\rho\in\Omega}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac p2} + \sum_{\lambda\in\Omega^c}\Big(\sum_{\mu\in\Omega^c}\Big|\sum_{\rho\in\Omega}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{\frac p2}.\end{aligned}$$

Hence,

$$\|H\|_p^p \lesssim \max\{I_j(E)\colon\ j \in \{1,2\}, E \in \{\Omega,\Omega^c\}\}\,, \tag{4.31}$$

where

$$\begin{aligned} I_1(E) &\coloneqq \sum_{\lambda\in E}\Big(\sum_{\mu\in E}\Big|\sum_{\rho\in E^c} K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{p/2},\\ I_2(E) &\coloneqq \sum_{\lambda\in E}\Big(\sum_{\mu\in E^c}\Big|\sum_{\rho\in E} K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{p/2}. \end{aligned}$$

Step 2. Let E denote either Ω or Ω^c, and introduce the quantity

$$I_3(E) \coloneqq \sum_{\lambda\in E}\Big(\sum_{\rho\in E^c}|V_g g(\rho-\lambda)|^2\Big)^{p/2}.$$

Let us show that

$$I_1(E) \le \|V_g g\|_{\ell^1(\Lambda_L)}^p I_3(E) \qquad\text{and}\qquad I_2(E) \le 4\|V_g g\|_{\ell^1(\Lambda_L)}^p I_3(E)\,. \tag{4.32}$$

For I_1, we consider the function

$$G_\lambda(\rho) \coloneqq K(\rho,\lambda)1_{E^c}(\rho)\,, \qquad \lambda,\rho\in\Lambda_L;$$

recall (4.4) and use Young's inequality for the convolution to get

$$\begin{aligned} \sum_{\mu\in\Lambda_L}\Big|\sum_{\rho\in E^c}K(\mu,\rho)K(\rho,\lambda)\Big|^2 &\le \sum_{\mu\in\Lambda_L}\Big(\sum_{\rho\in\Lambda_L}\big|V_g g(\mu-\rho)\big|\big|G_\lambda(\rho)\big|\Big)^2\\ &= \||V_g g| * |G_\lambda|\|_{\ell^2(\Lambda_L)}^2 \le \|V_g g\|_{\ell^1(\Lambda_L)}^2\|G_\lambda\|_{\ell^2(\Lambda_L)}^2\\ &= \|V_g g\|_{\ell^1(\Lambda_L)}^2\sum_{\rho\in E^c}|V_g g(\rho-\lambda)|^2\,, \end{aligned} \tag{4.33}$$

which yields $I_1(E) \le \|V_g g\|_{\ell^1(\Lambda_L)}^p I_3(E)$. Regarding I_2, by formula (4.5) we get

$$\sum_{\rho\in E}K(\mu,\rho)K(\rho,\lambda) = K(\mu,\lambda) - \sum_{\rho\in E^c}K(\mu,\rho)K(\rho,\lambda)\,.$$

By the triangle inequality and applying again (4.33),

$$\begin{aligned} &\Big(\sum_{\mu\in E^c}\Big|\sum_{\rho\in E}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{1/2}\\ &\qquad\le \Big(\sum_{\mu\in E^c}|K(\mu,\lambda)|^2\Big)^{1/2} + \Big(\sum_{\mu\in\Lambda_L}\Big|\sum_{\rho\in E^c}K(\mu,\rho)K(\rho,\lambda)\Big|^2\Big)^{1/2}\\ &\qquad\le \Big(\sum_{\mu\in E^c}|V_g g(\mu-\lambda)|^2\Big)^{1/2} + \|V_g g\|_{\ell^1(\Lambda_L)}\Big(\sum_{\rho\in E^c}|V_g g(\rho-\lambda)|^2\Big)^{1/2}\\ &\qquad\le 2\|V_g g\|_{\ell^1(\Lambda_L)}\Big(\sum_{\rho\in E^c}|V_g g(\rho-\lambda)|^2\Big)^{1/2}. \end{aligned}$$

where we used (4.8). Then, $I_2(E) \leq 2^p \|V_g g\|^p_{\ell^1(\Lambda_L)} I_3(E) \leq 4\|V_g g\|^p_{\ell^1(\Lambda_L)} I_3(E)$.

Step 3. By (4.31) and (4.32),

$$\|H\|_p^p \lesssim \|V_g g\|^p_{\ell^1(\Lambda_L)} \max\left\{I_3(E)\colon E \in \{\Omega, \Omega^c\}\right\}. \tag{4.34}$$

We now bound $I_3(E)$ for either $E = \Omega$ or $E = \Omega^c$. Consider the parameter $\alpha > 0$ as in the statement of the proposition, and define the function

$$h(\lambda) := (1 + L \cdot \mathrm{d}(\lambda, \partial\Omega))^{(2d-1)+(1+\alpha)}, \qquad \lambda \in \Lambda_L,$$

where d is the Euclidean distance. By Hölder's inequality,

$$\begin{aligned} I_3(E) &= \sum_{\lambda\in E} \frac{h(\lambda)^{1-p/2}}{h(\lambda)^{1-p/2}} \Big(\sum_{\rho\in E^c} |V_g g(\rho-\lambda)|^2\Big)^{p/2} \\ &\leq \Big(\sum_{\lambda\in E} \frac{1}{h(\lambda)}\Big)^{1-p/2} \Big(\sum_{\lambda\in E}\sum_{\rho\in E^c} h(\lambda)^{(2-p)/p} |V_g g(\rho-\lambda)|^2\Big)^{p/2}. \end{aligned} \tag{4.35}$$

We study the two integrals in (4.35) separately. For the first one, we apply inequality (4.10), Lemma 4.2.4 and the condition $\alpha \leq 1$:

$$\begin{aligned} \sum_{\lambda\in E} \frac{1}{h(\lambda)} &\leq \sum_{\lambda\in\Lambda_L} \frac{1}{(1 + \frac{1}{\sqrt{2d}}\,\mathrm{d}_{\mathrm{gr}}(\lambda, \partial\Omega))^{(2d-1)+(1+\alpha)}} \\ &\leq C_d \cdot \sum_{\lambda\in\Lambda_L} \frac{1}{(1 + \mathrm{d}_{\mathrm{gr}}(\lambda, \partial\Omega))^{(2d-1)+(1+\alpha)}} \\ &= C_d \cdot \sum_{n=0}^{+\infty} \sum_{\substack{\lambda\in\Lambda_L \\ \mathrm{d}_{\mathrm{gr}}(\lambda,\partial\Omega)=n}} \frac{1}{(1 + \mathrm{d}_{\mathrm{gr}}(\lambda, \partial\Omega))^{(2d-1)+(1+\alpha)}} \\ &= C_d \cdot \sum_{n=0}^{+\infty} \frac{1}{(1+n)^{(2d-1)+(1+\alpha)}} \cdot \#\{\lambda \in \Lambda_L \mid \mathrm{d}_{\mathrm{gr}}(\lambda, \partial\Omega) = n\} \\ &\leq C_d \cdot \#\partial\Omega \cdot \sum_{n=0}^{+\infty} \frac{1}{(1+n)^{1+\alpha}} \leq C_d \cdot \#\partial\Omega \cdot \Big(1 + \int_0^{+\infty} \frac{1}{(1+x)^{1+\alpha}}\,dx\Big) \\ &= C_d \cdot \#\partial\Omega \cdot \Big(1 + \frac{1}{\alpha}\Big) \lesssim C_d \cdot \#\partial\Omega \cdot \frac{1}{\alpha}. \end{aligned} \tag{4.36}$$

Next, we bound the second integral in (4.35). For $\lambda \in E, \rho \in E^c$,

$$h(\lambda) = (1 + L \cdot \mathrm{d}(\lambda, \partial\Omega))^{(2d-1)+(1+\alpha)} \leq (1 + L \cdot |\lambda - \rho|)^{(2d-1)+(1+\alpha)}.$$

So we get

$$\begin{aligned} &\sum_{\lambda\in E}\sum_{\rho\in E^c} h(\lambda)^{(2-p)/p} |V_g g(\rho-\lambda)|^2 \\ &\qquad\qquad \leq \sum_{\lambda\in E}\sum_{\rho\in E^c} (1 + L \cdot |\rho-\lambda|)^{[(2d-1)+(1+\alpha)](2-p)/p} |V_g g(\rho-\lambda)|^2. \end{aligned}$$

Considering the function

$$\varphi(\lambda) := (1 + L \cdot |\lambda|)^{[(2d-1)+(1+\alpha)](2-p)/p} |V_g g(\lambda)|^2 , \qquad \lambda \in \Lambda_L,$$

Lemma 4.2.6 gives

$$\begin{aligned}\sum_{\lambda \in E} \sum_{\rho \in E^c} h(\lambda)^{(2-p)/p} |V_g g(\rho - \lambda)|^2 &\leq \sum_{\rho \in E^c} \sum_{\lambda \in E} \varphi(\rho - \lambda) \\ &\leq \big\| (\Sigma\varphi) \cdot 1_{E^c} - 1_{E^c} * \varphi \big\|_{\ell^1(\Lambda_L)} \\ &\leq C_d \cdot \#\partial\Omega \cdot L \sum_{\lambda \in \Lambda_L} |\lambda| \varphi(\lambda) . \end{aligned} \tag{4.37}$$

Combining (4.35), (4.36) and (4.37) we obtain

$$\begin{aligned} I_3 &\leq C_d \cdot \frac{\#\partial\Omega}{\alpha^{1-p/2}} \cdot \|(L| \cdot |)\varphi\|_{\ell^1(\Lambda_L)}^{p/2} \\ &\leq C_d \cdot \frac{\#\partial\Omega}{\alpha^{1-p/2}} \cdot \big\|(1 + L| \cdot |)^{[(2d-1)+(1+\alpha)](2-p)/p+1} |V_g g|^2 \big\|_{\ell^1(\Lambda_L)}^{p/2} , \end{aligned}$$

which, together with (4.34), yields the assertion. □

4.5 Spectral deviation

Finally, we can apply the result from the previous section to find threshold-robust spectral bounds for Gabor multipliers. As in [28], both the cases of a Gelfand-Shilov-type condition and a polynomial decay on the kernel are taken into account.

First, we relate the eigenvalue counting function to the Schatten p-norm of the Hankel operator.

Lemma 4.5.1. *Let $\Omega \subseteq \Lambda_L$ be a compact set, and let H be the corresponding Gabor-Hankel operator. Then, for every $\delta \in (0, 1)$ and every $0 < p \leq 2$,*

$$\big|\#\{\lambda \in \sigma(M_\Omega) : \lambda > \delta\} - \#\Omega\big| \leq (\delta(1 - \delta))^{-p/2} \cdot \|H\|_p^p. \tag{4.38}$$

Proof. The proof works as in Lemma 3.4.2: equation (4.28) and formula (4.25) are used in place of (3.25) and (3.23). □

We are in a position to state and prove the spectral deviation bounds with the two different decay assumption.

Theorem 4.5.2 (Threshold-robust spectral bounds, Gelfand-Shilov-type condition). *Let $\Lambda_L = L^{-1}\mathbb{Z}^{2d}$. Let $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$ be such that:*

(i) $(\pi(\lambda)g)_{\lambda \in \Lambda_L}$ is a Parseval frame;

(ii) the following Gelfand-Shilov-type condition is satisfied for some $\beta \geq 1/2$*: there exist* $C_g, A > 0$ *such that for every* $n \in \mathbb{N}_0$

$$\big|V_g g(\mu)\big| \leq C_g A^n n!^{\beta}(1 + L \cdot |\mu|)^{-n}, \qquad \mu \in \Lambda_L. \tag{4.39}$$

Let $\Omega \subseteq \Lambda_L$ *be a compact set, and consider the Gabor multiplier* M_Ω *as in* (4.2)*. For* $\delta \in (0,1)$*, set* $\tau := \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$*. Then*

$$\big|\#\{\lambda \in \sigma(M_\Omega) : \lambda > \delta\} - \#\Omega\big| \leq C_g' \cdot \#\partial\Omega \cdot (\log \tau)^{2d\beta} \log\big[\log(\tau) + 1\big], \tag{4.40}$$

where $C_g' := C_g \cdot A^{3d+3} \cdot C_d^{\beta}$*, and* C_d *depends only on* d*.*

Proof. The proof works as in [28, v2, Theorem 1.1]. Let $0 < p \leq 1/2$ and $0 < \alpha \leq 1$. By Lemma 4.5.1,

$$\begin{aligned}
|\#\{\lambda \in \sigma(M_\Omega) :\ \lambda > \delta\} - \#\Omega| &\leq (\delta(1-\delta))^{-p/2}\|H\|_p^p \\
&\leq 2^{p/2}\max\{\delta^{-1}, (1-\delta)^{-1}\}^{p/2}\|H\|_p^p \\
&\leq 2\tau^{p/2}\|H\|_p^p\,.
\end{aligned} \tag{4.41}$$

We use Proposition 4.4.1 to bound $\|H\|_p^p$. First note that by Lemma 4.2.5 we get

$$\begin{aligned}
&\big\|(1 + L|\cdot|)^{[(2d-1)+(1+\alpha)](2-p)/2p+1/2}V_g g\big\|_{\ell^2(\Lambda_L)}^p \\
&\qquad = \left(\sum_{\lambda \in \Lambda_L} \frac{\big((1 + L|\lambda|)^{(2d+\alpha)/p+1/2}|V_g g(\lambda)|\big)^2}{(1 + L|\lambda|)^{2d+\alpha}}\right)^{p/2} \\
&\qquad \leq \big\|(1 + L|\cdot|)^{-(2d+\alpha)}\big\|_{\ell^1(\Lambda_L)}^{p/2} \cdot \big\|(1 + L|\cdot|)^{(2d+\alpha)/p+1/2}V_g g\big\|_{\ell^\infty(\Lambda_L)}^p \\
&\qquad \leq C_d \cdot \frac{1}{\alpha^{p/2}} \cdot \big\|(1 + L|\cdot|)^{(2d+\alpha)/p+1/2}V_g g\big\|_{\ell^\infty(\Lambda_L)}^p\,.
\end{aligned} \tag{4.42}$$

By (4.39), we get

$$\begin{aligned}
\big\|(1 + L|\cdot|)^{(2d+\alpha)/p+1/2}V_g g\big\|_{\ell^\infty(\Lambda_L)}^p &\leq C_g^p A^{p\lceil(2d+\alpha)/p+1/2\rceil}(\lceil(2d+\alpha)/p+1/2\rceil)!^{p\beta} \\
&\leq C_g^p A^{2d+\alpha+3p/2}\left(\frac{2d+\alpha+3p/2}{p}\right)^{\beta(2d+\alpha+3p/2)} \\
&\leq C_g^{1/2} A^{2d+2}\,(2d+2)^{\beta(2d+2)}\,(p^{-p})^{3\beta/2}p^{-\beta(2d+\alpha)} \\
&\leq C_g^{1/2} A^{2d+2}\,(2d+2)^{\beta(2d+2)}\,e^{3\beta/2}p^{-\beta(2d+\alpha)} \\
&= C_g^{1/2} A^{2d+2} C_d^{\beta} p^{-\beta(2d+\alpha)}\,,
\end{aligned} \tag{4.43}$$

where we used the elementary bound $p^{-p} \leq e$. Joining (4.42) and (4.43) we obtain

$$\big\|(1 + L|\cdot|)^{[(2d-1)+(1+\alpha)](2-p)/2p+1/2}V_g g\big\|_{\ell^2(\Lambda_L)}^p \leq C_g^{1/2} A^{2d+2} C_d^{\beta}\frac{1}{\alpha^{p/2}p^{\beta(2d+\alpha)}}\,. \tag{4.44}$$

On the other hand, by Lemma 4.2.5 we find

$$\begin{aligned}\|V_g g\|_{\ell^1(\Lambda_L)}^p &= \left(\sum_{\lambda\in\Lambda_L} \frac{(1+L|\lambda|)^{2d+1}|V_g g(\lambda)|}{(1+L|\lambda|)^{2d+1}}\right)^p \leq C_d \left\|(1+L|\cdot|)^{2d+1} V_g g\right\|_{\ell^\infty(\Lambda_L)}^p \\ &\leq C_d C_g^p A^{p(2d+1)} (2d+1)!^{\beta p} \leq C_d C_g^{1/2} A^{d+1} (2d+1)^{\beta(d+1)}\,.\end{aligned}$$

Combining this with (4.44) and Proposition 4.4.1, we get

$$\begin{aligned}\|H\|_p^p &\leq C_d \cdot \#\partial\Omega \cdot \frac{\|V_g g\|_{\ell^1(\Lambda_L)}^p \cdot \|(1+L|\cdot|)^{[(2d-1)+(1+\alpha)](2-p)/2p+1/2} V_g g\|_{\ell^2(\Lambda_L)}^p}{\alpha^{1-p/2}} \\ &\leq \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta \cdot \frac{1}{\alpha p^{\beta(2d+\alpha)}}\,. \qquad (4.45)\end{aligned}$$

We now choose $\alpha = 1/(\log(1/p+1))$ and assume for the moment that this choice is indeed compatible with the assumption $\alpha \leq 1$, to obtain

$$\|H\|_p^p \lesssim \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta e^\beta \cdot \frac{\log(1/p+1)}{p^{2d\beta}}\,.$$

This together with (4.41) gives

$$\left|\#\{\lambda\in\sigma(M_\Omega):\ \lambda>\delta\} - \#\Omega\right| \lesssim \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta \cdot \tau^{p/2} \frac{\log(1/p+1)}{p^{2d\beta}}\,.$$

Taking $p = 1/\log\tau$ yields (4.40), provided that this choice indeed leads to $p \leq 1/2$ and $\alpha \leq 1$.

To complete the proof, we first observe that, if $\tau \geq e^2$, then, indeed,

$$p = \frac{1}{\log\tau} \leq \frac{1}{2}, \quad \text{and} \quad \alpha = \frac{1}{\log\big((1/p)+1\big)} \leq \frac{1}{\log(2+1)} \leq 1,$$

are valid choices for p and α. On the other hand, if $\tau \leq e^2$ we choose $p = 1/2$ and $\alpha = 1$ in (4.45) to get

$$\|H\|_p^p \lesssim \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta \cdot 2^{\beta(2d+1)} \lesssim \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta\,. \qquad (4.46)$$

Note that $\tau \geq 2$, as either $\delta \leq 1/2$ or $1-\delta \leq 1/2$. Combining this observation, the assumption $\tau \leq e^2$, (4.41) and (4.46), we conclude

$$\begin{aligned}\left|\#\{\lambda\in\sigma(M_\Omega):\ \lambda>\delta\} - \#\Omega\right| &\lesssim \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta \cdot \tau^{1/4} \\ &\leq \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta \cdot \frac{\sqrt{e}(\log\tau)^{2d\beta} \log\big[\log(\tau)+1\big]}{(\log 2)^{2d\beta} \log\big[\log(2)+1\big]} \\ &\leq \#\partial\Omega \cdot C_g A^{3d+3} C_d^\beta \cdot (\log\tau)^{2d\beta} \log\big[\log(\tau)+1\big].\end{aligned}$$

Hence, we have proved (4.40) for all possible values of τ. □

Theorem 4.5.3 (Threshold-robust spectral bounds, polynomial decay)**.** *Consider* $\Lambda_L = L^{-1}\mathbb{Z}^{2d}$. *Let* $g \in L^2(\mathbb{R}^d)$ *with* $\|g\|_2 = 1$ *be such that:*

(i) $(\pi(\lambda)g)_{\lambda\in\Lambda_L}$ *is a Parseval frame;*

(ii) for some $s \geq 1$ *it holds*

$$C_g := \|V_g g\|^2_{\ell^1(\Lambda_L)} \cdot \sum_{\lambda\in\Lambda_L} (1 + L \cdot |\lambda|)^s |V_g g(\lambda)|^2 < +\infty. \tag{4.47}$$

Let $\Omega \subseteq \Lambda$ *be a compact set, and consider the Gabor multiplier* M_Ω *as in* (4.2). *For* $\delta \in (0, 1)$, *set* $\tau := \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$. *Then*

$$\left|\#\{\lambda \in \sigma(M_\Omega) : \lambda > \delta\} - \#\Omega\right| \leq C'_g \cdot \#\partial\Omega \cdot \tau^{\frac{2d}{2d+s-1}} \cdot (\log(C_g\tau))^{\frac{s-1}{2d+s-1}}\,, \tag{4.48}$$

where $C'_g := C_g^{\frac{2d}{2d+s-1}} \cdot C_d$, *and* C_d *depends only on* d*.*

Proof. Fix $\alpha > 0$ and set

$$p = 2\frac{2d+\alpha}{2d+\alpha+s-1}, \quad \text{so that} \quad s = 1 + \frac{(2d+\alpha)(2-p)}{p}\,.$$

Observe that $p \leq 2$ because $s - 1 \geq 0$. Applying Proposition 4.4.1 we obtain

$$\begin{aligned}\|H\|_p^p &\leq C_d \cdot \#\partial\Omega \cdot \frac{\|V_g g\|^p_{\ell^1(\Lambda_L)} \cdot \left\|(1 + L|\cdot|)^{(2d+\alpha)(2-p)/2p+1/2} V_g g\right\|^p_{\ell^2(\Lambda_L)}}{\alpha^{1-p/2}} \\ &\leq C_d \cdot \#\partial\Omega \cdot \frac{C_g^{p/2}}{\alpha^{1-p/2}}\,.\end{aligned}$$

As in the proof of Theorem 4.5.2, by Lemma 4.5.1, we derive (4.41). In combination with the previous estimate this leads to the bound

$$\left|\#\{\lambda \in \sigma(M_\Omega) : \lambda > \delta\} - \#\Omega\right| \leq C_d \cdot \#\partial\Omega \cdot \frac{(C_g\tau)^{p/2}}{\alpha^{1-p/2}}\,. \tag{4.49}$$

We now choose α so that the above expression is small. Note that $C_g\tau \geq 2$ since $C_g \geq \|V_g g\|^2_{\ell^1(\Lambda_L)}\|V_g g\|^2_{\ell^2(\Lambda_L)} \geq 1$ by (4.8) and $\max\{\delta^{-1}, (1-\delta)^{-1}\} \geq 2$. Therefore, we can choose

$$\alpha = \frac{\log 2}{\log(C_g\tau)} \leq 1\,.$$

Let us now estimate (4.49) for this choice of α. First note that

$$\begin{aligned}\frac{p}{2} &= \frac{2d+\alpha}{2d+\alpha+s-1} = \frac{2d}{2d+s-1+\alpha} + \frac{\alpha}{2d+2s-1+\alpha} \\ &\leq \frac{2d}{2d+s-1} + \frac{\log 2}{\log(C_g\tau)}\,,\end{aligned}$$

and, consequently,

$$(C_g\tau)^{p/2} \leq 2(C_g\tau)^{2d/(2d+s-1)}\,. \tag{4.50}$$

In addition, we have,

$$1-\frac{p}{2}=\frac{s-1}{2d+s-1+\alpha}\leq\frac{s-1}{2d+s-1}\leq 1\,.$$

Hence:

$$\left(\frac{\log(C_g\tau)}{\log 2}\right)^{1-p/2}\leq\left(\frac{\log(C_g\tau)}{\log 2}\right)^{(s-1)/(2d+s-1)}\leq\frac{(\log(C_g\tau))^{(s-1)/(2d+s-1)}}{\log 2}\,. \tag{4.51}$$

Finally, (4.48) follows by applying the estimates (4.50) and (4.51) to (4.49). □

Theorems 4.5.2 and 4.5.3 can be used to deduce asymptotic estimates for the individual eigenvalues as in [28]. We will omit this conclusion for the moment. In fact, in the next chapter we will be able to obtain similar results for a more general class of operators, including Gabor multipliers, and the eigenvalue estimates will be derived from those results.

Chapter 5

Toeplitz operators acting on RKHS

The original contribution of this dissertation is the extension of the estimates provided in [28] to Toeplitz operators acting on reproducing kernel Hilbert space. The proofs of Theorems 3.3.2 and 3.3.4 rely on the fact that the concentration operator and its associated Toeplitz operator share the same non-zero eigenvalues: then, the Toeplitz operator is investigated in place of the concentration operator. This more abstract point of view, for which we only need the projector of an L^2-space over a reproducing kernel Hilbert subspace, is the one considered in this chapter.

5.1 Defining Toeplitz operators on RKHS

From now onwards, we will consider the following setting. Let X be a locally compact metric space, and let μ be a positive, σ-finite, Borel measure on X that is finite on compact sets. Let $\mathbb{H} \subseteq L^2(X,\mu) \cap \mathcal{C}(X)$ be a reproducing kernel Hilbert space with reproducing kernel $K\colon X \times X \to \mathbb{C}$; in particular, $\mathbb{H}$ is a subspace of $L^2(X,\mu)$ whose elements have a continuous representative. This gives sense to $\mathbb{H}$ as a set of functions. Recall that Proposition 1.3.4(a) gives the formula

$$K(x,y) = \int_X K(x,z)K(z,y)\,d\mu(z)\,, \qquad x,y \in X, \tag{5.1}$$

and that by Proposition 1.3.5 the projection $P\colon L^2(X,\mu) \to \mathbb{H}$ is given by

$$Pf(x) = \int_X f(y)K(x,y)\,d\mu(y)\,, \qquad f \in L^2(X,\mu). \tag{5.2}$$

Let $\Omega \subseteq X$ be a compact set. As done in the case of concentration operators and Gabor multipliers, consider the Toeplitz operator $T_\Omega\colon L^2(X,\mu) \to L^2(X,\mu)$, defined as

$$T_\Omega f = P(1_\Omega \cdot Pf)\,, \qquad f \in L^2(X,\mu),$$

and the Hankel operator $H_\Omega\colon L^2(X,\mu) \to L^2(X,\mu)$, defined as

$$H_\Omega f = (I-P)1_\Omega \cdot Pf\,, \qquad f \in L^2(X,\mu).$$

They are related by

$$H_\Omega^* H_\Omega = T_\Omega - T_\Omega^2\,, \tag{5.3}$$

and, as before, we may drop the dependence on Ω in the notation.

In order to have the properties that the Toeplitz operator enjoys in the case of the concentration operator for the STFT and of the Gabor multiplier for a Parseval frame, we will make some assumptions on the kernel and its decay. First, we assume the following condition on its L^2-norm, which replaces the fact that, in the setting of the concentration operator, $\|V_g g\|_2 = \|g\|_2 = 1$ and that $\|V_g g\|_2$ is invariant under translations.

[C1] $\|K_y\|_2 = 1$ for every $y \in X$.

Applying the Cauchy-Schwarz inequality to formula (5.1), by Condition **[C1]** we get $\|K_y\|_\infty \le 1$ for every $y \in X$. Therefore, by Hölder's inequality

$$\|K_y\|_1 \ge 1\,, \qquad y \in X. \tag{5.4}$$

We can now compute the trace of the Toeplitz operator; compare Lemma 5.1.1 with Lemma 3.4.1.

Lemma 5.1.1. *Under assumption* **[C1]**, *the operator* T_Ω *is trace-class and*

$$\operatorname{trace}(T_\Omega) = \mu(\Omega)\,, \tag{5.5}$$

$$\operatorname{trace}(T_\Omega^2) = \int_\Omega \int_\Omega |K(x,y)|^2 \, d\mu(x)\, d\mu(y)\,. \tag{5.6}$$

Proof. The proof works exactly as in Lemma 3.4.1. For $f \in \mathbb{H}$, one has

$$\langle T_\Omega f, f\rangle = \langle P(\mathbf{1}_\Omega \cdot f), f\rangle = \langle \mathbf{1}_\Omega \cdot f, f\rangle = \int_\Omega |f(x)|^2 \, d\mu(x).$$

Therefore, $0 \le T_\Omega \le I$ and T_Ω is bounded and positive semi-definite.

For $f \in L^2(X,\mu)$, it holds

$$\begin{aligned} T_\Omega f(x) = P(\mathbf{1}_\Omega \cdot Pf)(x) &= \int_X \mathbf{1}_\Omega(y) Pf(y) K(x,y)\, d\mu(y) \\ &= \int_X \int_X \mathbf{1}_\Omega(y) f(z) K(y,z) K(x,y)\, d\mu(z)\, d\mu(y) \\ &= \int_X f(z) \left[\int_X \mathbf{1}_\Omega(y) K(x,y) K(y,z)\, d\mu(y) \right] d\mu(z)\,, \end{aligned}$$

where we used the expression (5.2) for the projector. So the operator T_Ω has integral kernel

$$K_{T_\Omega}(x,z) = \int_X \mathbf{1}_\Omega(y) K(x,y) K(y,z)\, d\mu(y)\,.$$

Then, by Condition **[C1]**

$$\begin{aligned}\int_X K_{T_\Omega}(x,x)\, d\mu(x) &= \int_X \int_X \mathbf{1}_\Omega(y) |K(x,y)|^2\, d\mu(y)\, d\mu(x)\\ &= \int_\Omega \int_X |K(x,y)|^2\, d\mu(x)\, d\mu(y) = \int_\Omega \|K_y\|_2^2\, d\mu(y) = \mu(\Omega)\,.\end{aligned}$$

By Theorem 1.2.8, we have that T_Ω is trace-class and $\mathrm{trace}(T_\Omega) = \mu(\Omega)$. Moreover, by Theorem 1.2.6 we get

$$\begin{aligned}\mathrm{trace}(T_\Omega^2) &= \|K_{T_\Omega}\|^2_{L^2(X\times X)}\\ &= \int_X \int_X |K_{T_\Omega}(x,y)|^2\, d\mu(x)\, d\mu(y)\\ &= \int_X \int_X K_{T_\Omega}(x,y) K_{T_\Omega}(y,x)\, d\mu(x)\, d\mu(y)\\ &= \int_\Omega \int_\Omega \left[\int_X \int_X K(x,z)K(z,y)K(y,w)K(w,x)\, d\mu(y)\, d\mu(x)\right] d\mu(w)\, d\mu(z)\\ &= \int_\Omega \int_\Omega K(z,w)K(w,z)\, d\mu(w)\, d\mu(z)\\ &= \int_\Omega \int_\Omega |K(z,w)|^2\, d\mu(w)\, d\mu(z)\,,\end{aligned}$$

which concludes the proof. ☐

Remark 5.1.2. The only reason for which we ask the metric space X to be locally compact and the measure μ to be finite on compact sets is to be able to apply Theorem 1.2.8.

5.2 Hankel operators estimates

The estimate for the Schatten p-norms of the Hankel operator in Proposition 3.4.3 aims at separating the geometric contribution of the compact set Ω from the integral decay of the kernel. In order to imitate it, from now on we make the following extra assumptions.

[C2] There exist $C_K > 0$ such that $\|K_y\|_1 \leq C_K$ for every $y \in X$.

[C3] There exist $C_\Omega, \gamma > 0$ such that

$$\mu(\{x \in X \mid n \leq d(x, \partial\Omega) < n+1\}) \leq C_\Omega \cdot (1+n)^\gamma\,, \qquad n \in \mathbb{N}_0;$$

without loss of generality, we can assume $\gamma \geq 1$.

Condition [**C2**] replaces the fact that the L^1-norm of $V_g g$ is invariant under translation, while Condition [**C3**] replaces the property proved in Proposition 3.4.6 for compact sets with regular boundary as in Definition 3.3.1.

Remark 5.2.1. By equation (5.4), we have $C_K \geq 1$ in Condition [**C2**].

The next proposition should be compared to Proposition 3.4.3: the bound for the Schatten p-norms of the Hankel operator stops before assuming any extra decay condition on the kernel, and it is provided in two different versions.

Proposition 5.2.2. *Let $\Omega \subseteq X$ be a compact set, and let H be the corresponding Hankel operator. Let $p \in (0,2]$ and $\alpha \in (0,1]$. Assume* [**C2**] *and* [**C3**]. *Then*

$$\begin{align} \|H\|_p^p &\lesssim C_K^p \Big(\frac{C_\Omega}{\alpha}\Big)^{1-\frac{p}{2}} \Big(\int_\Omega \int_{\Omega^c} (1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z',z)|^2 \, d\mu(z') \, d\mu(z) \Big)^{\frac{p}{2}} \tag{5.7} \\ &\lesssim C_K^{3p/2} \Big(\frac{C_\Omega}{\alpha}\Big)^{1-\frac{p}{2}} \Big(\int_\Omega \sup_{z' \in \Omega^c} \Big((1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z',z)| \Big) \, d\mu(z) \Big)^{\frac{p}{2}}. \tag{5.8} \end{align}$$

Proof. The proof is similar to [28, v2, Proposition 3.1].

Step 1. Exactly as in [28, v2, Proposition 3.1] and in Theorem 4.4.1, we find that

$$\|H\|_p^p \lesssim \max\left\{ I_j(E) \colon \ j \in \{1,2\}, E \in \{\Omega, \Omega^c\} \right\}, \tag{5.9}$$

where

$$\begin{aligned} I_1(E) &:= \int_E \Big(\int_E \Big| \int_{E^c} K(w,z')K(z',z) \, d\mu(z') \Big|^2 d\mu(w) \Big)^{p/2} d\mu(z) \,, \\ I_2(E) &:= \int_E \Big(\int_{E^c} \Big| \int_{E} K(w,z')K(z',z) \, d\mu(z') \Big|^2 d\mu(w) \Big)^{p/2} d\mu(z) \,. \end{aligned}$$

In fact, the proof is based exclusively on the fact that $\mathbb{H} \subseteq L^2(X,\mu)$ is a RKHS.

Step 2. Let E denote either Ω or Ω^c, and introduce the quantity

$$I_3(E) := \int_E \Big(\int_{E^c} |K(z',z)|^2 \, d\mu(z') \Big)^{p/2} d\mu(z) \,.$$

Let us show that

$$I_1(E) \leq C_K^p I_3(E) \qquad \text{and} \qquad I_2(E) \leq 4 C_K^p I_3(E). \tag{5.10}$$

For I_1, for every fixed $z \in X$ consider the function

$$G_z(z') := K(z',z) 1_{E^c}(z') \,, \qquad z' \in X,$$

and write

$$\int_X \Big| \int_{E^c} K(w,z')K(z',z)\,d\mu(z')\Big|^2\,d\mu(w) = \int_X \Big| \int_X K(w,z')G_z(z')\,d\mu(z')\Big|^2\,d\mu(w)\,.$$

We argue as in the proof of Young's inequality to achieve a similar bound. For every $w \in X$, by Hölder's inequality

$$\begin{aligned}\Big| \int_X K(w,z')G_z(z')\,d\mu(z')\Big| &\le \int_X |K(w,z')||G_z(z')|\,d\mu(z')\\ &= \int_X \Big(|K(w,z')|^{1/2}|G_z(z')|\Big)|K(w,z')|^{1/2}\,d\mu(z')\\ &\le \Big(\int_X |K(w,z')||G_z(z')|^2\,d\mu(z')\Big)^{1/2}\Big(\int_X |K(w,z')|\,d\mu(z')\Big)^{1/2}\\ &= \Big(\int_X |K(w,z')||G_z(z')|^2\,d\mu(z')\Big)^{1/2}\|K_w\|_1^{1/2}\,.\end{aligned}$$

Therefore, using Condition **[C2]** we get

$$\begin{aligned}\int_X \Big| \int_{E^c} K(w,z')K(z',z)\,d\mu(z')\Big|^2\,d\mu(w)&\\ &\le \int_X \Big(\int_X |K(w,z')||G_z(z')|^2\,d\mu(z')\Big)\|K_w\|_1\,d\mu(w)\\ &\le C_K \int_X |G_z(z')|^2\|K_{z'}\|_1\,d\mu(z')\\ &\le C_K^2 \int_{E^c} |K(z',z)|^2\,d\mu(z')\,. \qquad (5.11)\end{aligned}$$

This shows that $I_1(E) \le C_K^p \cdot I_3(E)$. Regarding I_2, we observe that, by (5.1),

$$\int_E K(w,z')K(z',z)\,d\mu(z') = K(w,z) - \int_{E^c} K(w,z')K(z',z)\,d\mu(z').$$

By the triangle inequality, and applying again (5.11),

$$\begin{aligned}&\Big(\int_{E^c}\Big| \int_E K(w,z')K(z',z)\,d\mu(z')\Big|^2\,d\mu(w)\Big)^{\frac12}\\ &\quad\le \Big(\int_{E^c} |K(w,z)|^2\,d\mu(w)\Big)^{\frac12} + \Big(\int_X \Big| \int_{E^c} K(w,z')K(z',z)\,d\mu(z')\Big|^2 d\mu(w)\Big)^{\frac12}\\ &\quad\le \Big(\int_{E^c} |K(w,z)|^2\,d\mu(w)\Big)^{\frac12} + C_K\Big(\int_{E^c} |K(z',z)|^2\,d\mu(z')\Big)^{\frac12}\\ &\quad\le 2C_K\Big(\int_{E^c} |K(z',z)|^2\,d\mu(z')\Big)^{\frac12},\end{aligned}$$

where we used $C_K \ge 1$ by Remark 5.2.1. Then $I_2(E) \le 2^p C_K^p I_3(E) \le 4C_K^p I_3(E)$.

Step 3. By (5.9) and (5.10),

$$\|H\|_p^p \lesssim C_K^p \cdot \max\left\{I_3(E)\colon E \in \{\Omega, \Omega^c\}\right\}. \tag{5.12}$$

We now bound I_3 for either $E = \Omega$ or $E = \Omega^c$. Consider the parameter $\alpha \in (0,1]$ as in the statement of the proposition and the exponent $\gamma \geq 1$ as in **[C3]**, and define the function

$$h(z) = (1 + d(z, \partial\Omega))^{\alpha+\gamma+1}, \qquad z \in X.$$

By Hölder's inequality,

$$\begin{aligned} I_3 &= \int_E \frac{h(z)^{1-p/2}}{h(z)^{1-p/2}} \Big(\int_{E^c} |K(z', z)|^2 \, d\mu(z') \Big)^{p/2} d\mu(z) \\ &\leq \Big(\int_E \frac{1}{h(z)} \, d\mu(z) \Big)^{1-p/2} \Big(\int_E \int_{E^c} h(z)^{(2-p)/p} |K(z', z)|^2 \, d\mu(z') \, d\mu(z) \Big)^{p/2}. \end{aligned} \tag{5.13}$$

We study the two integrals in (5.13) separately. For the first one, we use Condition **[C3]** and the assumption $\alpha \leq 1$ to obtain

$$\begin{aligned} \int_X \frac{1}{h(z)} \, d\mu(z) &= \sum_{n \in \mathbb{N}} \int_{\{z \in X | n \leq d(z, \partial\Omega) < n+1\}} \frac{1}{h(z)} \, d\mu(z) \\ &\leq \sum_{n \in \mathbb{N}} \frac{1}{(1+n)^{\alpha+\gamma+1}} \mu(\{z \in X \mid n \leq d(z, \partial\Omega) < n+1\}) \\ &\leq C_\Omega \sum_{n \in \mathbb{N}} \frac{1}{(1+n)^{\alpha+1}} \leq C_\Omega \Big(1 + \int_0^{+\infty} \frac{1}{(1+x)^{\alpha+1}} dx \Big) \\ &= C_\Omega \left(1 + \frac{1}{\alpha} \right) \lesssim \frac{C_\Omega}{\alpha}. \end{aligned} \tag{5.14}$$

Next we bound the second integral in (5.13). For $z \in E$ and $z' \in E^c$,

$$h(z) = (1 + d(z, \partial\Omega))^{\alpha+\gamma+1} \leq (1 + d(z, z'))^{\alpha+\gamma+1}.$$

By Condition **[C2]** we get

$$\begin{aligned} &\int_E \int_{E^c} h(z)^{\frac{2-p}{p}} |K(z', z)|^2 \, d\mu(z') \, d\mu(z) \\ &\quad \leq \int_E \int_{E^c} (1 + d(z, z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z', z)|^2 \, d\mu(z') \, d\mu(z) \\ &\quad = \int_\Omega \int_{\Omega^c} (1 + d(z, z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z', z)|^2 \, d\mu(z') \, d\mu(z) \\ &\quad \leq \int_\Omega \sup_{z' \in \Omega^c} \Big((1 + d(z, z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z', z)| \Big) \left[\int_{\Omega^c} |K(z', z)| d\mu(z') \right] d\mu(z) \\ &\quad \leq C_K \int_\Omega \sup_{z' \in \Omega^c} \Big((1 + d(z, z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z', z)| \Big) \, d\mu(z), \end{aligned}$$

Combining this estimate with (5.13) and (5.14), we obtain

$$\begin{aligned} I_3 &\lesssim \Big(\frac{C_\Omega}{\alpha}\Big)^{1-\frac{p}{2}} \bigg(\int_\Omega \int_{\Omega^c} (1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z',z)|^2 \, d\mu(z')\, d\mu(z)\bigg)^{\frac{p}{2}} \\ &\leq C_K^{p/2} \Big(\frac{C_\Omega}{\alpha}\Big)^{1-\frac{p}{2}} \bigg(\int_\Omega \sup_{z'\in\Omega^c} \Big((1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z',z)|\Big)\, d\mu(z)\bigg)^{\frac{p}{2}}, \end{aligned}$$

which, together with (5.12), yields (5.7) and (5.8). □

Remark 5.2.3. The additional assumption $\alpha \leq 1$ is coherent with the fact that in Theorems 5.3.2 and 5.3.6 this extra assumption is also used.

5.3 Spectral deviation

We now want to apply Proposition 5.2.2 to deduce estimates on the eigenvalue counting function. First, we relate the eigenvalue counting function to the Schatten p-norms of the Hankel operator, as in Lemma 3.4.2.

Lemma 5.3.1. *Let $\Omega \subseteq X$ be compact, and let H be the corresponding Hankel operator. Assume* **[C1]**. *Then, for every $\delta \in (0,1)$ and every $0 < p \leq 2$,*

$$\big|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\big| \leq (\delta(1-\delta))^{-p/2} \cdot \|H\|_p^p. \tag{5.15}$$

Proof. The proof works as in Lemma 3.4.2: equation (5.3) and formula (5.5) are used in place of (3.25) and (3.23). □

In what follows, we will combine Lemma 5.3.1 with estimates (5.7) or (5.8) to get a spectral deviation bound in the case of a Gelfand-Shilov-type decay or a polynomial decay of the kernel, respectively.

5.3.1 Gelfand-Shilov-type condition

The main theorem reads as follows.

Theorem 5.3.2 (Threshold-robust spectral bounds, Gelfand-Shilov-type condition – I)**.** *Let (X,d) be a locally compact metric space and μ be a positive, σ-finite, Borel measure on X that is finite on compact sets. Let $\mathbb{H} \subseteq L^2(X,\mu)$ be a RKHS whose reproducing kernel $K_y(x) = K(x,y)$ satisfies the following conditions:*

- **[C1]** $\|K_y\|_2 = 1$ *for every $y \in X$;*
- **[C2]** *there exists $C_K > 0$ such that $\|K_y\|_1 \leq C_K$ for every $y \in X$;*
- **[GS]** *a Gelfand-Shilov-type condition with parameter $\beta > 0$: there exist constants $D_K, A \geq 1$ such that for every $m \in \mathbb{N}_0$*

$$|K(z,z')| \leq D_K A^m m!^\beta (1+d(z,z'))^{-m}, \qquad z,z' \in X. \tag{5.16}$$

Let $\Omega \subseteq X$ be a compact set such that

[C3] *there exist $C_\Omega > 0$, $\gamma \geq 1$ such that*

$$\mu(\{x \in X \mid n \leq d(x, \partial\Omega) < n+1\}) \leq C_\Omega (1+n)^\gamma\,, \qquad n \in \mathbb{N}_0. \quad (5.17)$$

For $\delta \in (0,1)$, set $\tau := \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$. Then

$$\big|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\big| \leq C'_K \cdot C_\Omega \cdot (\log\tau)^{\beta(\gamma+1)} \log\big[\log(\tau)+1\big], \quad (5.18)$$

where $C'_K := C_K \cdot C_\gamma^\beta \cdot A^{\gamma+3} \cdot D_K^{1/4}$, and C_γ depends only on γ.

Proof. Let $0 < p \leq 1/2$ and $0 < \alpha \leq 1$. By Lemma 5.3.1,

$$\begin{aligned} |\#\{\lambda \in \sigma(T_\Omega):\ \lambda > \delta\} - \mu(\Omega)| &\leq (\delta(1-\delta))^{-p/2}\|H\|_p^p \\ &\leq 2^{p/2}\max\{\delta^{-1}, (1-\delta)^{-1}\}^{p/2}\|H\|_p^p \\ &\leq 2\tau^{p/2}\|H\|_p^p\,. \qquad (5.19) \end{aligned}$$

We use Proposition 5.2.2 to bound $\|H\|_p^p$. In particular, we continue the estimates in inequality (5.8) by using **[GS]** with $m_0 := \lceil(\alpha+\gamma+1)2/p\rceil$:

$$\begin{aligned} &\int_\Omega \sup_{z' \in \Omega^c}\Big((1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p}}|K(z',z)|\Big)\,d\mu(z) \\ &\leq \int_\Omega \sup_{z' \in \Omega^c}\Big((1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} D_K A^{m_0} m_0!^\beta (1+d(z,z'))^{-(\alpha+\gamma+1)\frac{2}{p}}\Big)\,d\mu(z) \\ &= D_K A^{m_0} m_0!^\beta \sum_{n \in \mathbb{N}} \int_{\{z \in \Omega | n \leq d(z,\partial\Omega) < n+1\}} \sup_{z' \in \Omega^c}\big((1+d(z,z'))^{-(\alpha+\gamma+1)}\big)\,d\mu(z) \\ &\leq D_K A^{m_0} m_0!^\beta \sum_{n \in \mathbb{N}} (1+n)^{-(\alpha+\gamma+1)} \mu(\{z \in \Omega \mid n \leq d(z,\partial\Omega) < n+1\}) \\ &\leq C_\Omega D_K A^{m_0} m_0!^\beta \sum_{n \in \mathbb{N}} (1+n)^{-(\alpha+1)} \\ &\lesssim D_K A^{m_0} m_0!^\beta \frac{C_\Omega}{\alpha}\,, \qquad (5.20) \end{aligned}$$

where we used that, given $z \in \{z \in \Omega \mid n \leq d(z,\partial\Omega) < n+1\}$, for every $z' \in \Omega^c$ we have $d(z,z') \geq d(z,\partial\Omega) \geq n$. Combining (5.20) with inequality (5.8), we get

$$\begin{aligned} \|H\|_p^p &\lesssim C_K^{3p/2} \cdot \frac{C_\Omega}{\alpha}\Big(D_K A^{m_0} m_0!^\beta\Big)^{p/2} \\ &\leq C_K^{3p/2} \cdot \frac{C_\Omega}{\alpha} \cdot D_K^{p/2} A^{\alpha+\gamma+1+p/2}\Big(\frac{2(\alpha+\gamma+1)+p}{p}\Big)^{\beta(\alpha+\gamma+1+p/2)} \\ &\leq C_K^{3p/2} \cdot \frac{C_\Omega}{\alpha} \cdot D_K^{p/2} A^{\gamma+3}(2(\gamma+2)+1)^{\beta(\gamma+3)} p^{-\beta(\alpha+\gamma+1)} e^{-\beta/2} \\ &\leq C_K^{3p/2} \cdot \frac{C_\Omega}{\alpha} \cdot D_K^{p/2} A^{\gamma+3} C_\gamma^\beta \frac{1}{p^{\beta(\alpha+\gamma+1)}}\,, \qquad (5.21) \end{aligned}$$

where C_γ is a constant depending only on γ.

We now choose $\alpha = 1/(\log(1/p+1))$ and assume for the moment that this choice is indeed compatible with the assumption $\alpha \leq 1$, to obtain

$$\|H\|_p^p \lesssim C_K^{3p/2} C_\gamma^\beta C_\Omega D_K^{p/2} A^{\gamma+3} \cdot \frac{\log(1/p+1)}{p^{\beta(\gamma+1)}}.$$

Together with (5.19), this gives

$$\big|\#\{\lambda \in \sigma(T_\Omega) : \ \lambda > \delta\} - \mu(\Omega)\big| \lesssim C_K^{3p/2} C_\gamma^\beta C_\Omega D_K^{p/2} A^{\gamma+3} \cdot \frac{\tau^{p/2}\log(1/p+1)}{p^{\beta(\gamma+1)}}.$$

Taking $p = 1/\log\tau$ yields (5.18), provided that this choice indeed leads to $p \leq 1/2$ and $\alpha \leq 1$.

To complete the proof, we first observe that, if $\tau \geq e^2$, then

$$p = \frac{1}{\log\tau} \leq \frac{1}{2} \quad \text{and} \quad \alpha = \frac{1}{\log\big((1/p)+1\big)} \leq \frac{1}{\log(2+1)} \leq 1$$

are valid choices for p and α. On the other hand, if $\tau \leq e^2$ we choose $p = 1/2$ and $\alpha = 1$ in (5.21) to get

$$\|H\|_p^p \lesssim C_K^{3p/2} C_\gamma^\beta C_\Omega D_K^{p/2} A^{\gamma+3} \cdot 2^{\beta(\gamma+2)} \lesssim C_K C_\gamma^\beta C_\Omega D_K^{1/4} A^{\gamma+3}. \tag{5.22}$$

Note that $\tau \geq 2$, as either $\delta \leq 1/2$ or $1-\delta \leq 1/2$. Combining this observation, the assumption $\tau \leq e^2$, (5.19) and (5.22), we conclude

$$\begin{aligned}\big|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\big| &\lesssim C_K C_\gamma^\beta C_\Omega D_K^{1/4} A^{\gamma+3} \cdot \tau^{1/4} \\ &\leq C_K C_\gamma^\beta C_\Omega D_K^{1/4} A^{\gamma+3} \cdot \frac{\sqrt{e}(\log\tau)^{\beta(\gamma+1)} \log\big[\log(\tau)+1\big]}{(\log 2)^{\beta(\gamma+1)} \log\big[\log(2)+1\big]} \\ &\leq C_K C_\gamma^\beta C_\Omega D_K^{1/4} A^{\gamma+3} \cdot (\log\tau)^{\beta(\gamma+1)} \log\big[\log(\tau)+1\big].\end{aligned}$$

Hence, we have proved (5.18) for all possible values of τ. □

Remark 5.3.3. The requirements $D_K, A, \gamma \geq 1$ are not restrictive.

Remark 5.3.4. Assumptions [**GS**] and [**C2**] are somehow related. We wish to give a second version of the theorem where to use only one constant instead of C_K and D_K, and to separate the contribution of the kernel decay from the geometric integral property of the distance function over the metric space X.

If we assume the following condition:

[**CD**] there exist $C_d > 0$, $\gamma \geq 1$ such that

$$\int_X \frac{1}{(1+d(z,z'))^{\gamma+2}}\, d\mu(z') \leq C_d\,, \qquad z \in X,$$

then, together with the Gelfand-Shilov-type condition **[GS]**, assumption **[C2]** follows with constant $C_K \coloneqq C_d D_K A^{\gamma+3}(\gamma+3)^{\beta(\gamma+3)}$:

$$\begin{aligned}\|K_z\|_1 &= \int_X \frac{(1+d(z,z'))^{\gamma+2}|K(z,z')|}{(1+d(z,z'))^{\gamma+2}}\, d\mu(z') \\ &\le C_d \cdot \sup_{z'\in X} \Big((1+d(z,z'))^{\gamma+2}|K(z,z')| \Big) \\ &\le C_d D_K A^{\lceil \gamma+2\rceil} \lceil \gamma+2\rceil !^{\beta} \le C_d D_K A^{\gamma+3}(\gamma+3)^{\beta(\gamma+3)}.\end{aligned}$$

Using this information in (5.21), we can rewrite the theorem as follows.

Theorem 5.3.5 (Threshold-robust spectral bounds, Gelfand-Shilov-type condition – II)**.** *Let (X,d) be a locally compact metric space and μ be a positive, σ-finite, Borel measure on X that is finite on compact sets. Moreover, suppose that*

[CD] *there exist $C_d > 0$, $\gamma \ge 1$ such that*

$$\int_X \frac{1}{(1+d(z,z'))^{\gamma+2}}\, d\mu(z') \le C_d\,, \qquad z \in X. \tag{5.23}$$

Let $\mathbb{H} \subseteq L^2(X,\mu)$ be a RKHS whose reproducing kernel $K_y(x) = K(x,y)$ satisfies the following conditions:

[C1] *$\|K_y\|_2 = 1$ for every $y \in X$;*

[GS] *a Gelfand-Shilov-type condition with parameter $\beta > 0$: there exist constants $D_K, A \ge 1$ such that for every $m \in \mathbb{N}_0$*

$$|K(z,z')| \le D_K A^m m!^{\beta} (1+d(z,z'))^{-m}\,, \qquad z,z' \in X. \tag{5.24}$$

Let $\Omega \subseteq X$ be a compact set such that

[C3] *there exists $C_\Omega > 0$ such that*

$$\mu(\{x \in X \mid n \le d(x,\partial\Omega) < n+1\}) \le C_\Omega (1+n)^{\gamma}\,, \qquad n \in \mathbb{N}_0. \tag{5.25}$$

For $\delta \in (0,1)$, set $\tau \coloneqq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$. Then

$$\big|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\big| \le C'_K \cdot C_\Omega \cdot (\log\tau)^{\beta(\gamma+1)} \log\big[\log(\tau)+1\big], \tag{5.26}$$

where $C'_K \coloneqq C_d \cdot D_K \cdot C_\gamma^{\beta} \cdot A^{2\gamma+6}$, and C_γ depends only on γ.

5.3.2 Polynomial decay

We now apply the polynomial decay condition to the kernel.

Theorem 5.3.6 (Threshold-robust spectral bounds, Polynomial decay – I)**.** *Let* (X,d) *be a locally compact metric space and* μ *be a positive,* σ*-finite, Borel measure on* X *that is finite on compact sets. Let* $\mathbb{H} \subseteq L^2(X,\mu)$ *be a RKHS whose reproducing kernel* $K_y(x) = K(x,y)$ *satisfies the following conditions:*

[C1] $\|K_y\|_2 = 1$ *for every* $y \in X$*;*

[C2] *there exists* $C_K > 0$ *such that* $\|K_y\|_1 \leq C_K$ *for every* $y \in X$*;*

[PI] *there exist* $D_K > 0$*,* $s \geq 1$ *such that*

$$\int_X (1+d(z,z'))^s |K(z,z')|^2 \, d\mu(z') \leq D_K\,, \qquad z \in X. \tag{5.27}$$

Let $\Omega \subseteq X$ *be a compact set such that*

[C3] *there exist* $C_\Omega > 0$*,* $\gamma \geq 1$ *such that*

$$\mu(\{x \in X \mid n \leq d(x,\partial\Omega) < n+1\}) \leq C_\Omega (1+n)^\gamma\,, \qquad n \in \mathbb{N}_0. \tag{5.28}$$

Assume also $s \geq \gamma + 2$*. For* $\delta \in (0,1)$*, set* $\tau \coloneqq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$*. Then*

$$\left|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\right| \leq C'_K \cdot C_\Omega \cdot \tau^{\frac{\gamma+1}{s-1}} \cdot (\log(D_K \tau))^{\frac{s-\gamma-2}{s-1}}\,, \tag{5.29}$$

where $C'_K \coloneqq C_K^2 \cdot C_\gamma \cdot D_K^{\frac{\gamma+1}{s-1}}$*, and* C_γ *depends only on* γ*.*

Proof. We want to continue the estimates for the right hand side of inequality (5.7) by using the polynomial decay **[PI]**. We choose

$$p = 2\frac{\alpha+\gamma+1}{\alpha+s-1}, \qquad \text{so that} \qquad s = \gamma + 2 + \frac{(\alpha+\gamma+1)(2-p)}{p}\,.$$

Note that $p \leq 2$ since $s - 1 \geq \gamma + 1$. Then

$$\begin{aligned}
\int_\Omega \int_{\Omega^c} &(1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p}} |K(z',z)|^2 \, d\mu(z') \, d\mu(z) \\
&\leq D_K \int_\Omega \sup_{z' \in \Omega^c} \left((1+d(z,z'))^{(\alpha+\gamma+1)\frac{2-p}{p} - s}\right) d\mu(z) \\
&= D_K \int_\Omega \sup_{z' \in \Omega^c} \left((1+d(z,z'))^{-(\gamma+2)}\right) d\mu(z) \\
&\leq D_K \sum_{n \in \mathbb{N}} (1+n)^{-(\gamma+2)} \mu(\{z \in \Omega \mid n \leq d(z,\partial\Omega) < n+1\}) \\
&\leq C_\Omega D_K \sum_{n \in \mathbb{N}} (1+n)^{-2} \lesssim C_\Omega D_K\,.
\end{aligned}$$

Applying Proposition 5.2.2 we obtain

$$\|H\|_p^p \lesssim C_K^p C_\Omega \frac{D_K^{p/2}}{\alpha^{1-\frac{p}{2}}}\,.$$

As in the proof of Theorem 5.3.2, by Lemma 5.3.1, we derive (5.19). In combination with the previous estimate this leads to the bound

$$\big|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\big| \lesssim C_K^p C_\Omega \frac{(D_K\tau)^{p/2}}{\alpha^{1-\frac{p}{2}}}\,. \tag{5.30}$$

We now choose α so that the right hand side of (5.30) is small. Note that $D_K\tau \geq 2$ because $\tau = \max\{\delta^{-1}, (1-\delta)^{-1}\} \geq 2$ and, by assumption **[C1]**, given $z \in X$

$$1 = \|K_z\|_2^2 \leq \int_X (1 + d(z,z'))^s |K(z,z')|^2 \, d\mu(z') \leq D_K\,.$$

Therefore, we can choose

$$\alpha = \frac{\log 2}{\log(D_K\tau)} \leq 1\,.$$

Let us now estimate (5.30) for this choice of α. First note that

$$\frac{p}{2} = \frac{\alpha+\gamma+1}{\alpha+s-1} = \frac{\gamma+1}{\alpha+s-1} + \frac{\alpha}{\alpha+s-1} \leq \frac{\gamma+1}{s-1} + \frac{\log 2}{\log(D_K\tau)}\,,$$

and, consequently,

$$(D_K\tau)^{p/2} \leq 2(D_K\tau)^{\frac{\gamma+1}{s-1}}\,. \tag{5.31}$$

In addition, we have,

$$1 - \frac{p}{2} = \frac{s-\gamma-2}{\alpha+s-1} \leq \frac{s-\gamma-2}{s-1} \leq 1\,.$$

Hence, we obtain:

$$\left(\frac{\log(D_K\tau)}{\log 2}\right)^{1-p/2} \leq \left(\frac{\log(D_K\tau)}{\log 2}\right)^{\frac{s-\gamma-2}{s-1}} \leq \frac{(\log(D_K\tau))^{\frac{s-\gamma-2}{s-1}}}{\log 2}\,. \tag{5.32}$$

Finally, (5.29) follows by applying the estimates (5.31) and (5.32) to (5.30). □

Remark 5.3.7. Assumptions **[PI]** and **[C2]** are somehow related. We wish to give a second version of the theorem where we use only one constant replacing C_K and D_K. Consider the following condition:

[PII] there exist $D_K > 0$, $s \geq 1$ such that

$$\|K_w\|_1^2 \cdot \int_X (1 + d(z,z'))^s |K(z,z')|^2 \, d\mu(z') \leq D_K\,, \qquad z,w \in X. \tag{5.33}$$

We show that, if **[C1]** holds, then **[PII]** is equivalent to **[PI]** and **[C2]** with appropriate constants.

Conditions **[PI]** and **[C2]** clearly imply **[PII]**. Conversely, if **[PII]** holds, by assumption **[C1]**, for every $w \in X$ we have

$$\|K_w\|_1^2 = \|K_w\|_1^2 \cdot \|K_w\|_2^2 \leq \|K_w\|_1^2 \cdot \int_X (1 + d(w, z'))^s |K(w, z')|^2 \, d\mu(z') \leq D_K \, .$$

Then **[C2]** follows, and we can consider the constant $C_K \coloneqq \sup_{w \in X} \|K_w\|_1 \leq D_K^{1/2}$. Taking the supremum over $w \in X$ in (5.33), we get

$$C_K^2 \cdot \int_X (1 + d(z, z'))^s |K(z, z')|^2 \, d\mu(z') \leq D_K \, , z \in X,$$

which is **[PI]** with bound D_K / C_K^2. Therefore, we can rewrite the theorem as follows.

Theorem 5.3.8 (Threshold-robust spectral bounds, polynomial decay – II)**.** *Let (X, d) be a locally compact metric space and μ be a positive, σ-finite, Borel measure on X. Let $\mathbb{H} \subseteq L^2(X, \mu)$ be a RKHS whose reproducing kernel $K_y(x) = K(x, y)$ satisfies the following conditions:*

[C1] $\|K_y\|_2 = 1$ *for every* $y \in X$*;*

[PII] *there exist* $D_K > 0$*,* $s \geq 1$ *such that*

$$\|K_w\|_1^2 \cdot \int_X (1 + d(z, z'))^s |K(z, z')|^2 \, d\mu(z') \leq D_K \, , \qquad z, w \in X. \quad (5.34)$$

Let $\Omega \subseteq X$ be a compact set such that

[C3] *there exist* $C_\Omega > 0$*,* $\gamma \geq 1$ *such that*

$$\mu(\{x \in X \mid n \leq d(x, \partial\Omega) < n + 1\}) \leq C_\Omega (1 + n)^\gamma \, , \qquad n \in \mathbb{N}_0. \quad (5.35)$$

Assume also $s \geq \gamma + 2$. For $\delta \in (0, 1)$, set $\tau \coloneqq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$. Then

$$\left|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\right| \leq C_K' \cdot C_\Omega \cdot \tau^{\frac{\gamma+1}{s-1}} \cdot (\log(D_K \tau))^{\frac{s-\gamma-2}{s-1}} \, , \qquad (5.36)$$

where $C_K' \coloneqq C_\gamma \cdot D_K^{\frac{\gamma+1}{s-1}}$, and C_γ depends only on γ.

Remark 5.3.9. In the end, we notice that Condition **[PII]** can be relaxed by considering a different scale for the constant D_K. Assume that

[PIII] there exist $D_K > 0$, $s \geq 1$ such that

$$\|K_z\|_1^2 \cdot \int_X (1 + d(z, z'))^s |K(z, z')|^2 \, d\mu(z') \leq D_K \, , \qquad z \in X.$$

As shown in Remark 5.3.7 by assumption **[C1]** we get

$$\|K_z\|_1^2 \leq D_K\,, \qquad z \in X.$$

Moreover, by inequality (5.4)

$$\int_X (1+d(z,z'))^s |K(z,z')|^2 \, d\mu(z') \leq D_K\,, \qquad z \in X.$$

Therefore,

$$\|K_w\|_1^2 \cdot \int_X (1+d(z,z'))^s |K(z,z')|^2 \, d\mu(z') \leq D_K^2 \qquad z, w \in X,$$

which is assumption **[PII]** with bound D_K^2. Conversely, clearly **[PII]** implies **[PIII]**. Therefore, we can rewrite the theorem as follows.

Theorem 5.3.10 (Threshold-robust spectral bounds, polynomial decay – III). *Let (X,d) be a locally compact metric space and μ be a positive, σ-finite, Borel measure on X that is finite on compact sets. Let $\mathbb{H} \subseteq L^2(X,\mu)$ be a RKHS whose reproducing kernel $K_y(x) = K(x,y)$ satisfies the following conditions:*

[C1] $\|K_y\|_2 = 1$ *for every* $y \in X$*;*

[PIII] *there exist* $D_K > 0$, $s \geq 1$ *such that*

$$\|K_z\|_1^2 \cdot \int_X (1+d(z,z'))^s |K(z,z')|^2 \, d\mu(z') \leq D_K\,, \qquad z \in X. \tag{5.37}$$

Let $\Omega \subseteq X$ *be a compact set such that*

[C3] *there exist* $C_\Omega > 0$, $\gamma \geq 1$ *such that*

$$\mu(\{x \in X \mid n \leq d(x,\partial\Omega) < n+1\}) \leq C_\Omega (1+n)^\gamma\,, \qquad n \in \mathbb{N}_0. \tag{5.38}$$

Assume also $s \geq \gamma + 2$*. For* $\delta \in (0,1)$*, set* $\tau := \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\}$*. Then*

$$\left|\#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega)\right| \leq C_K' \cdot C_\Omega \cdot \tau^{\frac{\gamma+1}{s-1}} \cdot \left(\log(D_K^2 \tau)\right)^{\frac{s-\gamma-2}{s-1}}\,, \tag{5.39}$$

where $C_K' := C_\gamma \cdot D_K^{2\frac{\gamma+1}{s-1}}$*, and* C_γ *depends only on* γ*.*

5.4 Eigenvalue estimates

The Toeplitz operator is compact, self-adjoint, positive semi-definite and contractive. Its spectrum is contained in the interval $[0,1]$ and we can consider its eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ in decreasing order. Estimates for the individual eigenvalues can be deduced from the spectral deviation bounds. Set $A_\Omega = \lceil \mu(\Omega)\rceil$: as seen in Section 3.3.3, the inequalities (5.26) and (5.39) suggests that A_Ω is the index k where the eigenvalue λ_k crosses the threshold $1/2$. We first prove a propaedeutic lemma; compare it to [28, v5, Lemma 4.3].

Lemma 5.4.1. *Let $\Omega \subseteq X$ be a compact set and denote*

$$L_K := 2\int_\Omega \int_{\Omega^c} |K(z,z')|^2 \, d\mu(z')\, d\mu(z). \tag{5.40}$$

Assume **[C1]**. *Then*

(i) $\lambda_k \le 1/2$ *for every* $k \ge A_\Omega + L_K$,

(ii) $\lambda_k \ge 1/2$ *for every* $1 \le k \le A_\Omega - L_K$.

Proof. Without loss of generality we can assume that $L_K < +\infty$, otherwise the statements are vacuous. By Lemma 5.1.1 and Condition **[C1]**,

$$0 \le \operatorname{trace}(T_\Omega) - \operatorname{trace}(T_\Omega^2) = \int_\Omega \Big(1 - \int_\Omega |K(z,z')|^2 \, d\mu(z')\Big)\, d\mu(z) = \frac{1}{2}\cdot L_K. \tag{5.41}$$

If $L_K = 0$, then 1 is the only nonzero eigenvalue: its multiplicity (that is finite) is equal to $\mu(\Omega)$ by (5.5), and the claim follows. Suppose now $0 < L_K$; we proceed as in [28, v5, Lemma 4.3]. By definition,

$$\begin{aligned}
\operatorname{trace}(T_\Omega) - \operatorname{trace}(T_\Omega^2) &= \sum_{n=1}^{\infty} \lambda_n(1-\lambda_n) = \sum_{n=1}^{A_\Omega} \lambda_n(1-\lambda_n) + \sum_{n=A_\Omega+1}^{\infty} \lambda_n(1-\lambda_n) \\
&\ge \lambda_{A_\Omega} \sum_{n=1}^{A_\Omega}(1-\lambda_n) + (1-\lambda_{A_\Omega}) \sum_{n=A_\Omega+1}^{\infty} \lambda_n \\
&= \lambda_{A_\Omega} A_\Omega - \lambda_{A_\Omega} \sum_{n=1}^{A_\Omega} \lambda_n + (1-\lambda_{A_\Omega})\Big(\mu(\Omega) - \sum_{n=1}^{A_\Omega} \lambda_n\Big) \\
&= \lambda_{A_\Omega} A_\Omega - \sum_{n=1}^{A_\Omega} \lambda_n + (1-\lambda_{A_\Omega})\mu(\Omega) \\
&= \mu(\Omega) - \sum_{n=1}^{A_\Omega} \lambda_n + \lambda_{A_\Omega}(A_\Omega - \mu(\Omega)).
\end{aligned}$$

Combining this with (5.41) we obtain

$$\mu(\Omega) - \sum_{n=1}^{A_\Omega} \lambda_n + \lambda_{A_\Omega}(A_\Omega - \mu(\Omega)) \leq \operatorname{trace}(T_\Omega) - \operatorname{trace}(T_\Omega^2) \leq \frac{1}{2} \cdot L_K.$$

As a consequence,

$$\sum_{n=A_\Omega+1}^{\infty} \lambda_n = \mu(\Omega) - \sum_{n=1}^{A_\Omega} \lambda_n \leq \frac{1}{2} \cdot L_K\,, \tag{5.42}$$

and

$$\begin{aligned}\sum_{n=1}^{A_\Omega-1} (1-\lambda_n) &= \mu(\Omega) - \sum_{n=1}^{A_\Omega} \lambda_n + \lambda_{A_\Omega}(A_\Omega - \mu(\Omega)) - (1+\mu(\Omega) - A_\Omega)(1-\lambda_{A_\Omega}) \\ &\leq \frac{1}{2} \cdot L_K\,. \end{aligned} \tag{5.43}$$

To prove (i), let $k \geq A_\Omega + L_K$. We can write

$$k = A_\Omega + j,$$

with $j \geq L_K$, and use (5.42) to estimate

$$L_K \cdot \lambda_k \leq j \cdot \lambda_{A_\Omega+j} \leq \sum_{n=A_\Omega+1}^{\infty} \lambda_n \leq \frac{1}{2} \cdot L_K.$$

Since $0 < L_K < +\infty$, it follows that $\lambda_k \leq 1/2$, as claimed.

We proceed similarly to prove (ii). Let $1 \leq k \leq A_\Omega - L_K$. We write

$$k = A_\Omega - j,$$

with $j \geq L_K$, and use (5.43) to estimate

$$L_K \cdot (1-\lambda_k) \leq j \cdot (1-\lambda_{A_\Omega-j}) \leq \sum_{n=1}^{A_\Omega-1} (1-\lambda_n) \leq \frac{1}{2} \cdot L_K.$$

Again since $0 < L_K < +\infty$, it follows that $\lambda_k \geq 1/2$ as claimed. □

5.4.1 Gelfand-Shilov-type condition

We can now prove eigenvalue asymptotics in the context of Theorem 5.3.5.

First, redefine

$$C'_K := C_d \cdot D_K^2 \cdot C_\gamma^\beta \cdot A^{2\gamma+6}\,,$$

which still satisfies (5.26) since we also require $D_K \geq 1$ without loss of generality. Consider also C_γ sufficiently large such that the following result holds.

Corollary 5.4.2. *Consider $\mathbb{H} \subseteq L^2(X, \mu)$ and $\Omega \subseteq X$ as in the setting of Theorem 5.3.5, and write*

$$C := C'_K \cdot C_\Omega\,. \tag{5.44}$$

Then the following statements hold.

(i) For every $k = A_\Omega + Ch \in \mathbb{N}$ with $h \geq 1$,

$$\lambda_k \leq e^{-\left(\frac{h}{e(1+\log h)}\right)^{1/\beta(\gamma+1)}}. \tag{5.45}$$

(ii) For every $k = A_\Omega - Ch \in \mathbb{N}$ with $h \geq 1$,

$$\lambda_k \geq 1 - e^{-\left(\frac{h}{e(1+\log h)}\right)^{1/\beta(\gamma+1)}}. \tag{5.46}$$

Proof. Recall L_K defined in (5.40). By the Gelfand-Shilov type condition **[GS]** with $m_0 := \lceil \gamma + 2 \rceil$, and by Condition **[C2]** with $C_K = C_d D_K A^{\gamma+3}(\gamma+3)^{\beta(\gamma+3)}$ (see Remark 5.3.4), we get

$$\begin{aligned}
L_K &= 2\int_\Omega \int_{\Omega^c} |K(z', z)|^2 \, d\mu(z')\, d\mu(z) \\
&\leq 2C_K D_K A^{m_0} m_0!^\beta \int_\Omega \sup_{z' \in \Omega^c} (1 + d(z, z'))^{-(\gamma+2)} d\mu(z) \\
&\leq 2C_K D_K A^{m_0} m_0!^\beta C_\Omega \sum_{n \in \mathbb{N}} \frac{1}{(1+n)^2} \\
&\leq C_d D_K^2 A^{2\gamma+6} C_\gamma^\beta \cdot C_\Omega = C'_K \cdot C_\Omega = C
\end{aligned} \tag{5.47}$$

for C_γ large enough. Then the proof continues as in [28, v5, Corollary 1.7].

To prove (i), let $k = A_\Omega + Ch \in \mathbb{N}$ with $h \geq 1$. By (5.47),

$$A_\Omega + L_K \leq A_\Omega + C \leq k.$$

Thus, by Lemma 5.4.1, $\lambda_k \leq 1/2$. We may assume that $\lambda_k > 0$, since otherwise (5.45) holds trivially. Applying Theorem 5.3.5 for $0 < \delta \nearrow \lambda_k \leq 1/2$, we obtain

$$\begin{aligned}
k - A_\Omega &\leq \lim_{\delta \nearrow \lambda_k} \left| \#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega) \right| \\
&\leq C \lim_{\delta \nearrow \lambda_k} (\log(1/\delta))^{\gamma+1} \log\left[\log(1/\delta) + 1\right] \\
&= C(\log(1/\lambda_k))^{\gamma+1} \log\left[\log(1/\lambda_k) + 1\right] \leq \frac{C}{\varepsilon}(\log(1/\lambda_k))^{\gamma+1+\varepsilon},
\end{aligned}$$

for every $0 < \varepsilon \leq 1$, where we used that $\log(x+1) \leq x^\varepsilon/\varepsilon$ for $x \geq 0$. Rearranging the last expression, we arrive at

$$(\varepsilon h)^{\frac{1}{\gamma+1+\varepsilon}} \leq \log(1/\lambda_k). \tag{5.48}$$

Choosing $\varepsilon = 1/(1+\log h) \le 1$,

$$\begin{aligned}\Big(\frac{h}{1+\log h}\Big)^{\frac{1}{\gamma+1}} &= (\varepsilon h)^{\frac{1}{\gamma+1+\varepsilon}} (\varepsilon h)^{\frac{1}{\gamma+1}-\frac{1}{\gamma+1+\varepsilon}} = (\varepsilon h)^{\frac{1}{\gamma+1+\varepsilon}} (\varepsilon h)^{\frac{\varepsilon}{(\gamma+1)(\gamma+1+\varepsilon)}} \\ &\le (\varepsilon h)^{\frac{1}{\gamma+1+\varepsilon}} h^{\frac{1}{(1+\log h)(\gamma+1)^2}} \le (\varepsilon h)^{\frac{1}{\gamma+1+\varepsilon}} e^{\frac{1}{\gamma+1}}\,.\end{aligned}$$

Combining this with (5.48) yields

$$\begin{aligned}\Big(\frac{h}{e(1+\log h)}\Big)^{\frac{1}{\gamma+1}} &\le \log(1/\lambda_k)\,, \\ e^{\big(\frac{h}{e(1+\log h)}\big)^{\frac{1}{\gamma+1}}} &\le \frac{1}{\lambda_k}\,,\end{aligned}$$

and (5.45) follows.

Towards (ii), let $k = A_\Omega - Ch \in \mathbb{N}$ with $h \ge 1$. By (5.47),

$$k = A_\Omega - Ch \le A_\Omega - C \le A_\Omega - L_K\,.$$

From Lemma 5.4.1 it follows that $\lambda_k \ge 1/2$. Applying Theorem 5.3.5, we obtain

$$\begin{aligned}A_\Omega - k \le \mu(\Omega) - (k-1) &\le \big|\mu(\Omega) - \#\{\lambda \in \sigma(T_\Omega) : \lambda > \lambda_k\}\big| \\ &\le C(\log(1/(1-\lambda_k)))^{\gamma+1} \log\big[\log(1/(1-\lambda_k)) + 1\big] \\ &\le \frac{C}{\varepsilon}(\log(1/(1-\lambda_k)))^{\gamma+1+\varepsilon}\,, \qquad 0 < \varepsilon \le 1.\end{aligned}$$

Proceeding exactly as before, we arrive at

$$\begin{aligned}\Big(\frac{h}{e(1+\log h)}\Big)^{\frac{1}{\gamma+1}} &\le \log(1/(1-\lambda_k))\,, \\ e^{\big(\frac{h}{e(1+\log h)}\big)^{\frac{1}{\gamma+1}}} &\le \frac{1}{1-\lambda_k}\,,\end{aligned}$$

which proves (5.46). □

5.4.2 Polynomial decay

We can now prove eigenvalue asymptotics in the context to Theorem 5.3.10.

First, recall that $C'_K = C_\gamma \cdot D_K^{2^{\frac{\gamma+1}{s-1}}}$, where we consider C_γ sufficiently large such that the following corollary holds.

Corollary 5.4.3. *Consider $\mathbb{H} \subseteq L^2(X,\mu)$ and $\Omega \subseteq X$ as in the setting of Theorem 5.3.10, and write*

$$C := C'_K \cdot C_\Omega\,. \tag{5.49}$$

Then the following statements hold.

(i) For every $k = A_\Omega + Ch \in \mathbb{N}$ *with* $h \geq 1$*,*

$$\lambda_k \leq e^{\left(\frac{s-1}{\gamma+1}\right)^2} \cdot h^{-\frac{s-1}{\gamma+1}} \cdot (1 + \log(D_K^2 h))^{\frac{s-\gamma}{\gamma+1}}. \tag{5.50}$$

(ii) For every $k = A_\Omega - Ch \in \mathbb{N}$ *with* $h \geq 1$*,*

$$\lambda_k \geq 1 - e^{\left(\frac{s-1}{\gamma+1}\right)^2} \cdot h^{-\frac{s-1}{\gamma+1}} \cdot (1 + \log(D_K^2 h))^{\frac{s-\gamma}{\gamma+1}}. \tag{5.51}$$

Proof. Recall L_K defined in (5.40). Since $s \geq \gamma + 2$ we have that

$$\gamma + 1 \leq s - 1 \leq (\gamma + 1)(s - (\gamma + 1)).$$

By Conditions [**C1**] and [**PIII**], Jensen's inequality gives the following bound:

$$\begin{aligned}
L_K &= 2 \int_\Omega \int_{\Omega^c} |K(z', z)|^2 \, d\mu(z') \, d\mu(z) \\
&\leq 2 \int_\Omega \sup_{z' \in \Omega^c} (1 + d(z, z'))^{-1} \int_{\Omega^c} (1 + d(z, z'))|K(z', z)|^2 \, d\mu(z') \, d\mu(z) \\
&\leq 2 \int_\Omega \sup_{z' \in \Omega^c} (1 + d(z, z'))^{-1} \left(\int_{\Omega^c} (1 + d(z, z'))^{\frac{s-1}{\gamma+1}} |K(z', z)|^2 \, d\mu(z') \right)^{\frac{\gamma+1}{s-1}} d\mu(z) \\
&\leq 2 \int_\Omega \sup_{z' \in \Omega^c} (1 + d(z, z'))^{-1} \left(\int_{\Omega^c} (1 + d(z, z'))^{s-(\gamma+1)} |K(z', z)|^2 \, d\mu(z') \right)^{\frac{\gamma+1}{s-1}} d\mu(z) \\
&\leq 2 D_K^{\frac{\gamma+1}{s-1}} \int_\Omega \sup_{z' \in \Omega^c} (1 + d(z, z'))^{-1} \sup_{z' \in \Omega^c} (1 + d(z, z'))^{-(\gamma+1)} \, d\mu(z) \\
&\leq 2 D_K^{2\frac{\gamma+1}{s-1}} C_\Omega \sum_{n \in \mathbb{N}} \frac{1}{(1 + n)^{-2}} \\
&\leq D_K^{2\frac{\gamma+1}{s-1}} C_\gamma C_\Omega = C'_K \cdot C_\Omega = C
\end{aligned} \tag{5.52}$$

for C_γ large enough. Then, the proof continues as in [28, v5, Corollary 1.8].

To prove (i), we let $k = A_\Omega + \gamma h \in \mathbb{N}$ with $h \geq 1$ and use (5.52) to estimate

$$A_\Omega + L_K \leq A_\Omega + C \leq A_\Omega + Ch = k.$$

Thus, by Lemma 5.4.1, $\lambda_k \leq 1/2$. As before, we may assume that $\lambda_k > 0$, since otherwise (5.50) holds trivially. Applying Theorem 5.3.10 for $0 < \delta \nearrow \lambda_k \leq 1/2$ and denoting $\theta = (\gamma + 1)/(s - 1)$, we obtain

$$\begin{aligned}
k - A_\Omega &\leq \lim_{\delta \nearrow \lambda_k} \left| \#\{\lambda \in \sigma(T_\Omega) : \lambda > \delta\} - \mu(\Omega) \right| \\
&\leq C \lim_{\delta \nearrow \lambda_k} \delta^{-\theta} (\log(D_K^2/\delta))^{1-\theta} \\
&= C \lambda_k^{-\theta} (\log(D_K^2/\lambda_k))^{1-\theta} \leq \frac{C}{\varepsilon^{1-\theta} D_K^{2\theta}} (D_K^2/\lambda_k)^{\theta + \varepsilon(1-\theta)},
\end{aligned}$$

for every $\varepsilon > 0$, where we used that $\log(x) \leq x^{\varepsilon}/\varepsilon$ for $x > 0$. Rearranging the last expression, we arrive at

$$\left(\varepsilon^{1-\theta} D_K^{2\theta} h\right)^{\frac{1}{\theta+\varepsilon(1-\theta)}} \leq \frac{D_K^2}{\lambda_k}. \tag{5.53}$$

Choosing $\varepsilon = 1/(1+\log(D_K^2 h)) \leq 1$,

$$\begin{aligned} D_K^2 \left(\frac{h}{(1+\log(D_K^2 h))^{1-\theta}}\right)^{\frac{1}{\theta}} &= \left(\varepsilon^{1-\theta} D_K^{2\theta} h\right)^{\frac{1}{\theta+\varepsilon(1-\theta)}} \left(\varepsilon^{1-\theta} D_K^{2\theta} h\right)^{\frac{\varepsilon(1-\theta)}{\theta(\theta+\varepsilon(1-\theta))}} \\ &\leq \left(\varepsilon^{1-\theta} D_K^{2\theta} h\right)^{\frac{1}{\theta+\varepsilon(1-\theta)}} \left(D_K^2 h\right)^{\frac{1-\theta}{(1+\log(D_K^2 h))\theta^2}} \\ &\leq \left(\varepsilon^{1-\theta} D_K^{2\theta} h\right)^{\frac{1}{\theta+\varepsilon(1-\theta)}} e^{\frac{1-\theta}{\theta^2}} \\ &\leq \left(\varepsilon^{1-\theta} D_K^{2\theta} h\right)^{\frac{1}{\theta+\varepsilon(1-\theta)}} e^{\frac{1}{\theta^2}} . \end{aligned}$$

Combining the last estimate with (5.53) yields (5.50).

Towards (ii), let $k = A_\Omega - Ch \in \mathbb{N}$ with $h \geq 1$ and use (5.52) to estimate

$$k = A_\Omega - Ch \leq A_\Omega - C = A_\Omega - C'_K \cdot C_\Omega \leq A_\Omega - L_K .$$

From Lemma 5.4.1 it follows that $\lambda_k \geq 1/2$. Applying Theorem 5.3.10, we get

$$\begin{aligned} A_\Omega - k &\leq \mu(\Omega) - (k-1) \leq \left|\mu(\Omega) - \#\{\lambda \in \sigma(T_\Omega) : \lambda > \lambda_k\}\right| \\ &\leq C(1-\lambda_k)^{-\theta}(\log(D_K^2/(1-\lambda_k)))^{1-\theta} \\ &\leq \frac{C}{\varepsilon^{1-\theta} D_K^{2\theta}}(D_K^2/(1-\lambda_k))^{\theta+\varepsilon(1-\theta)}, \qquad \varepsilon > 0. \end{aligned}$$

Proceeding exactly as in Corollary 5.4.2 we arrive at

$$\left(\frac{h}{(1+\log(D_K^2 h))^{1-\theta}}\right)^{\frac{1}{\theta}} \leq \frac{e^{\frac{1}{\theta^2}}}{1-\lambda_k},$$

which proves (5.51). ☐

Notice that all the results proved in Chapter 5 recover both the estimates for the concentration operator and Gabor multipliers.

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