Università di Genova

Scuola di Scienze Matematiche, Fisiche e Naturali

Dipartimento di Matematica

Corso di studi in

Matematica generale

Prova Finale

Sobolev spaces on metric measure spaces

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Anno accademico 2022/2023

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Abstract

In questo lavoro di tesi è presentata una possibile costruzione di spazi di Sobolev su spazi metrici di misura. Il primo capitolo è dedicato alla teoria classica degli spazi di Sobolev in $\mathbb{R}$ basata su [3], che in alcuni casi serviranno da ispirazione per motivare l'approccio seguito nel caso generale. L'idea è di generalizzare la definizione classica che utilizza l'integrazione per parti tramite le derivazioni.

Dopo aver introdotto lo spazio $L^0(\mathfrak{m})$ delle funzioni misurabili su uno spazio metrico di misura, nel secondo capitolo viene introdotta la teoria degli $L^0(\mathfrak{m})$-moduli $L^0(\mathfrak{m})$-normati, che con la loro nozione di dualità sono la struttura giusta con cui parlare di derivazioni, ovvero funzionali lineari su una classe di funzioni lipschitziane che verificano la regola di Leibniz e giocano il ruolo dei campi vettoriali nella definizione di spazio di Sobolev, ovvero le sezioni del fibrato tangente. Lo spazio delle derivazioni sarà chiamato modulo tangente. Introdurremo una nozione di divergenza per derivazioni con la quale definiremo gli spazi di Sobolev. Per dualità generalizzeremo le nozioni di 1-forme, ovvero le sezioni del fibrato cotangente.

Nell'ultimo capitolo concluderemo analizzando brevemente altri approcci presenti in letteratura ([7]) e vedremo che in realtà le definizioni sono equivalenti. La definizione proposta in questa tesi può essere considerata più naturale perché parte dalla costruzione di modulo tangente e cotangente per poi costruire gli spazi di Sobolev e non generalizza solo il modulo del gradiente.

Chapter 1

Preliminaries

1.1 A motivating example

Consider the following Cauchy problem on $I = [0, 1]$

$$\begin{cases} -u''(x) + u(x) = f(x) \\ u(0) = u(1) = 0 \end{cases},$$

where f is a continuous function over I. Our goal is to find a formulation of the problem which is well posed when u and u' are in $L^1([0, 1])$.

By multiplying the left hand side by any smooth compactly supported function on $(0, 1)$ and then integrating we obtain that

$$\text{for all } \varphi \in C_c^\infty(0,1) \quad \int_0^1 -\varphi(x)u''(x)\,dx + \int_0^1 \varphi(x)u(x)\,dx = \int_0^1 f(x)\varphi(x)\,dx. \tag{1.1.1}$$

Integrating the first summand by parts and recalling that $u(0) = u(1) = \varphi(0) = \varphi(1) = 0$ we get that

$$\text{for all } \varphi \in C_c^\infty(0,1) \quad \int_0^1 \varphi'(x)u'(x) + \varphi(x)u(x)\,dx = \int_0^1 f(x)\varphi(x)\,dx. \tag{1.1.2}$$

We now observe that the integral equation in 1.1.2 makes sense for $u, u' \in L^1(I)$. This formulation is called the *weak formulation* of the original problem and solutions to 1.1.2 are called *weak solutions*. To make this argument completely formal we need a notion of derivative for L^1 functions over I.

Remark 1.1.1. *The steps above show that a regular solution to the problem is also a weak solution, since the classical integration by parts formula holds.*

We will tackle the problem more in general, by defining a notion of derivative for some L^p functions over an interval and thus introducing the Sobolev space $W^{1,p}(I)$.

1.2 Classical Sobolev spaces

Definition 1.2.1. *Let $I \subseteq \mathbb{R}$ be an interval and $1 \leq p < \infty$. We say that a function $u \in L^p(I)$ is Sobolev is there exists a function $g \in L^p(I)$ such that for all $\varphi \in C_c^\infty(I)$*

$$\int_I u\varphi' dx = -\int_I g\varphi dx.$$

Such a function g behaves like the derivative in the integration by parts formula, and is therefore called the weak derivative of u. The space of $p-$integrable Sobolev functions over I is denoted by $W^{1,p}(I)$.

Definition 1.2.2. *We endow $W^{1,p}(I)$ with the norm*

$$\|u\|_{W^{1,p}(I)} := \|u\|_{L^p(I)} + \|u'\|_{L^p(I)}.$$

Observe that the definition of weak derivative extends the classical definition of the derivative. Moreover, the weak derivative is unique, as a consequence of the following lemma.

Lemma 1.2.3 (Fundamental Lemma of the Calculus of Variation)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $h \in L^1_{loc}(\Omega)$ be such that*

$$\text{for all } \varphi \in C_c^\infty(\Omega) \quad \int_\Omega h(x)\varphi(x)\, dx = 0.$$

Then $h = 0$ almost everywhere.

Proof. Suppose $h \neq 0$. Then there exists a ball $B = B(x_0, r) \subset \Omega$ where h restricted to $B(x_0, r)$ is not identically zero.. In particular.

$$\int_{B(x_0,r)} |h|\, dx > 0.$$

Let

$$g(x) = \begin{cases} 1 & x \in B, h(x) > 0 \\ -1 & x \in B, h(x) < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Clearly g is bounded, $\text{supp}(g) = \overline{B}$ and

$$\int_B h(x)g(x)\, dx = \int_B |h|\, dx > 0.$$

We can approximate g by a sequence of smooth and compactly supported functions $g_n \in C_c^\infty(\Omega)$ such that

- $\text{supp}(g_n) \subset B(x_0, r + \frac{1}{n})$,

- $g_n \to g$ almost everywhere.

By using our assumptions on $g_n \in C_c^\infty(\Omega)$ and the dominated convergence theorem we obtain a contradiction. Let $\delta > 0$ be such that $B = B(x_0, r) \subset B(x_0, r+\delta) \subset \Omega$. then, for n large enough we have that

$$\begin{aligned} 0 = \int_\Omega h(x) g_n(x)\, dx \quad &= \int_{B(x_0+\frac{1}{n})} g_n(x) h(x)\, dx = \int_{B(x_0+\delta)} g_n(x) h(x)\, dx \\ \overset{n}{\to} \int_{B(x_0+\delta)} g(x) h(x)\, dx \quad &= \int_{B(x_0,r)} g(x) h(x)\, dx = \int_B |h|\, dx > 0. \end{aligned}$$

□

Remark 1.2.4. *Let $f, g \in L^p(I)$ be weak derivatives for $u \in W^{1,p}(I)$. Then $\int_I (f-g)\varphi \mathrm{d}x = 0$ for all $\phi \in C_c^\infty(I)$ and by the Fundamental Lemma of the Calculus of Variation $f = g$ almost everywhere.*

Lemma 1.2.5. *Let $u \in L^1(I)$ be such that $\int_I u\varphi' = 0$ for all $\varphi \in C_c^\infty(I)$. Then u is constant almost everywhere.*

Proof. Let $\omega \in C_c^\infty(I)$ and $\psi \in C_c^\infty(I)$ be such that $\int_I \psi = 1$. There exist $a, b \in I$ such that

$$\mathrm{supp}(\omega) \cup \mathrm{supp}(\psi) \subseteq [a, b] \Subset I.$$

Now we define

$$\varphi(x) = \begin{cases} \int_a^x \omega(t) - \left(\int_I \omega\right) \psi(t)\, dt & x \in (a, b) \\ 0 & \text{otherwise} \end{cases}$$

and notice that φ is smooth and compactly supported with $\mathrm{supp}(\varphi) \subseteq [a, b]$, and

$$\varphi' = \omega - \left(\int_I \omega\right) \psi.$$

We then deduce that, for all $\omega \in C_c^\infty(I)$,

$$\begin{aligned} 0 = \int_I u \left(\omega - \left(\int_I \omega\right) \psi\right) dx &= \int_I u\omega\, dx - \int_I omega \int_I u\psi \\ &= \int_I \left(u - \int_I u\psi\right) \omega\, dx. \end{aligned}$$

By the Fundamental Lemma of the Calculus of Variations 1.2.3 we conclude that $u - \int_I u\psi = 0$ almost everywhere, hence u is constant. □

Remark 1.2.6. *By uniqueness of the weak derivative the expected linearity properties hold: $(u+v)' = u' + v'$, $(\lambda u)' = \lambda u'$ for all $u, v \in W^{1,p}(I), \lambda \in \mathbb{R}$.*

These lemmas are useful to characterize Sobolev functions, show some of their properties and find embedding of Sobolev spaces into known spaces.

Proposition 1.2.7. *$W^{1,p}(I)$ is a Banach space. Moreover, if $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(I)$ is such that $u_n \overset{L^p}{\rightharpoonup} u$ and $u_n' \overset{L^p}{\rightharpoonup} g$ converge weakly in $L^p(I)$, then $u \in W^{1,p}(I)$ and $u' = g$.*

Proof. It is clear that $\|\cdot\|_{W^{1,p}}$ is a norm, in particular the triangle inequality holds because $(u+v)' = u' + v'$. To show that $W^{1,p}(I)$ is complete, let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(I)$ be a Cauchy sequence. By definition of $\|\cdot\|_{W^{1,p}}$ this means that $(u_n)_{n\in\mathbb{N}}$ and $(u_n')_{n\in\mathbb{N}}$ are Cauchy sequences in $L^p(I)$, which is complete. Hence there exist $u, g \in L^p(I)$ such that

$$u_n \xrightarrow[L^p(I)]{n} u, \qquad u_n' \xrightarrow[L^p(I)]{n} g.$$

By dominated convergence, for all $\varphi \in C_c^\infty(I)$

$$\int_I u\varphi' \mathrm{d}x = \lim_n \int_I \varphi' u_n = \lim_n - \int \varphi u_n' \mathrm{d}x = -\int_I \varphi g \mathrm{d}x,$$

so we obtain that $u \in W^{1,p}(I)$ and $g = u'$. Finally,

$$\|u_n - u\|_{W^{1,p}} = \|u_n - u\|_{L^p} + \|u_n' - u'\|_{L^p} \overset{n}{\to} 0.$$

If $u_n \overset{L^p}{\rightharpoonup} u$ and $u_n' \overset{L^p}{\rightharpoonup} g$ then, since $C_c^\infty(I)$ is contained in $L^q(I)$,

$$\int_I \varphi' u \mathrm{d}x = \lim_n \int_I \varphi' u_n \mathrm{d}x = -\lim_n \int_I \varphi u_n' \mathrm{d}x = -\int_I \varphi g \mathrm{d}x \quad \forall \varphi \in C_c^\infty(I).$$

□

Proposition 1.2.8. *Let $u \in W^{1,p}(I)$. Then u has a continuous representative, i.e. there exists a continuous function $\hat{u}$ such that $u = \hat{u}$ almost everywhere.*

Proof. Let $x_0 \in I$ and define $v(x) = \int_{x_0}^x u'(y)\, dy$. The function v is continuous since, for any sequence $x_n \to x$,

$$v(x_n) = \int_{x_0}^{x_n} u'(y)\, dy = \int_I \chi_{[x_0,x_n]}(y) u'(y)\, dy \overset{n}{\to} \int_I \chi_{[x_0,x](y)} u'(y)\, dy = \int_{x_0}^x u'(y)\, dy = v(x)$$

by dominated convergence. Moreover,

$$\int_I v\varphi' = -\int_I u'\varphi \quad \forall \varphi \in C_c^\infty(I)$$

To see this, fix $\varphi \in C_c^\infty(I)$ and take $a, b \in I$ such that $\mathrm{supp}(\varphi) \subseteq (a,b)$. Then, by Fubini's theorem,

$$\begin{aligned}
\int_I v(x)\varphi'(x)\, dx &= \int_I \left(\int_{x_0}^x u'(y)\, dy \right) \varphi'(x)\, dx = \int_I \int_{x_0}^x u'(y)\varphi'(x)\, dy\, dx \\
&= \int_{x_0}^b u'(y) \int_y^b \varphi'(x)\, dx\, dy - \int_a^{x_0} u'(y) \int_a^y \varphi'(x)\, dx\, dy \\
&= -\int_{x_0}^b u'(y)\varphi(y)\, dy - \int_a^{x_0} u'(y)\varphi(y)\, dy \\
&= -\int_I u'(y)\varphi(y)\, dy.
\end{aligned}$$

Thus we obtain that

$$\int_I v\varphi' = -\int_I u'\varphi = \int_I y\varphi' \quad \forall \varphi \in C_c^\infty(I).$$

By lemma 1.2.5 we conclude that $u - v = c$ is constant almost everywhere. Since v is continuous, $v + c$ is a continuous representative for u. □

An immediate corollary to the last proposition is that Sobolev functions over bounded intervals are bounded.

Corollary 1.2.9. *Let I be a bounded interval and $u \in W^{1,p}(I)$. Then $u \in L^\infty(I)$.*

Proof. By proposition 1.2.8 u admits a continuous representative, which is bounded over I. □

By the above Corollary and Proposition we have an inclusion $W^{1,p} \hookrightarrow L^\infty(I)$ when I is a bounded interval. Since $C(I)$ is a closed subspace of $L^\infty(I)$, we can consider the inclusion $p : W^{1,p} \hookrightarrow C(I)$ given by Proposition 1.2.8 to study how $W^{1,p}(I)$ is embedded in $L^\infty(I)$.

Theorem 1.2.10. *Let $I \subseteq \mathbb{R}$ be any interval. Then*

(i) $W^{1,p}(I) \hookrightarrow L^\infty(I)$ is a continuous injection,

(ii) if I is bounded and $p > 1$ then $P : W^{1,p} \hookrightarrow C(I)$ is compact.

Proof. Suppose $I = \mathbb{R}$ and let $u \in W^{1,p}(\mathbb{R})$. For every interval of the form $J_a = (a - \frac{1}{2}, a + \frac{1}{2})$ there exists $x_0 \in J_a$ such that $|u(x_0)| \leq \|u\|_{L^p}$. Indeed, if $|u(x_0)| > \|u\|_{L^p}$ for all $x \in J_a$ then

$$\|u\|_{L^p}^p = \int_{\mathbb{R}} |u(x)|^p \mathrm{d}x \geq \int_{J_a} |u(x)|^p \mathrm{d}x > \int_{J_a} \|u\|_{L^p}^p \mathrm{d}x = \|u\|_{L^p}^p,$$

leading to a contradiction. Now consider the continuous representative $\tilde{u}$ of u. By Hölder's inequality

$$|\tilde{u}(x_0) - \tilde{u}(a)| = \left| \int_{x_0}^a u'(x)\mathrm{d}x \right| \leq |x_0 - a|^{1/q} \left(\int_{x^0}^a |u'(x)|\mathrm{d}x \right)^{1/p} = |x_0 - a|^{1-1/p} \|u'\|_{L^p}, \quad (1.2.1)$$

from which we deduce

$$|\tilde{u}(a)| \leq |\tilde{u}(x_0)| + |x_0 - a|^{1/q} \|u'\|_{L^p} =\leq 2^{p-1} \|u\|_{W^{1,p}},$$

and continuity of $W^{1,p}(I) \hookrightarrow L^\infty(I)$ follows by taking the supremum over $a \in \mathbb{R}$.

Compactness of $P : W^{1,p} \hookrightarrow C(I)$ when I is bounded is a consequence of the Ascoli-Arzelà theorem. Indeed $P\left(\overline{B(0,1)_{W^{1,p}(I)}}\right) \subset C(I)$ is

- bounded, since $\|P(u)\|_{L^\infty} \leq C\|u\|_{W^{1,p}(I)} \leq C$ for all $u \in \overline{B(0,1)_{W^{1,p}(I)}}$,

- equicontinuous by the same argument used in (1.2.1),

$$|u(x)-u(y)| = \left|\int_x^y u'(t)\mathrm{d}t\right| \leq |x-y|^{1-1/p}\|u'\|_{L^p} \leq |x-y|^{1-1/p} \qquad \forall u \in \overline{B(0,1)_{W^{1,p}(I)}}.$$

□

Remark 1.2.11. *Both (i) and (ii) in the theorem above can fail if we remove the respective assumptions.*

If $I = \mathbb{R}$, $W^{1,p}(I) \hookrightarrow L^\infty(I)$ is continuous but not compact, since the sequence of translations $(\tau_n u)_{n\in\mathbb{N}}$ of any nonzero $u \in C_c^\infty(\mathbb{R}) \cap \overline{B(0,1)_{W^{1,p}}}$ has no converging subsequences in $L^\infty(\mathbb{R})$.

If $I = [0,1]$ and $p = 1$, $W^{1,p}(I) \hookrightarrow L^\infty(I)$ is continuous but not compact. Let

$$W^{1,1}(I) \ni u_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right) & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}$$

Then

$$\|u_n\|_{W^{1,1}(I)} = \|u\|_{L^1} + \|u'\|_{L^1} \leq 1 + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} \frac{n}{2}\mathrm{d}x = 2$$

is bounded, but $\|u_n - u_m\|_{L^\infty} = 1$ whenever $n \neq m$.

1.3 Characterization of $W^{1,p}(I)$

We give a first characterization of $W^{1,p}(I)$ in the following proposition.

Proposition 1.3.1. . *Let $p > 1$, q be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in L^p(I)$. Then the following are equivalent*

(a) u is a Sobolev function, i.e. $u \in W^{1,p}(I)$,

(b) there exists $C > 0$ such that, for all $\varphi \in C_c^\infty(I)$, $|\int_I u\varphi'| \leq C\|\varphi\|_{L^q(I)}$,

(c) there exists $C > 0$ such that, for all $h \in \mathbb{R}$, $\|\tau_h u - u\|_{L^p(I\cap I_{-h})} \leq C|h|$.

Proof.
- $(a) \implies (b)$ The constant that does the job is the norm of the weak derivative of u, since by the definition of $W^{1,p}(I)$ and the Hölder inequality we have that

$$\left|\int_I u\varphi'\right| = \left|\int_I u'\varphi\right| \leq \|u'\|_{L^p(I)}\|\varphi\|_{L^q(I)}.$$

- $(b) \implies (a)$ Consider the linear functional

$$\mathcal{F} : C_c^\infty(I) \to \mathbb{R}, \quad \varphi \longmapsto \int_I u\varphi'$$

Since we are assuming $p > 1$, $C_c^\infty(I)$ is a dense subset of $L^q(I)$. Moreover $\mathcal{F}$ is bounded by hypothesis, hence it can be extended to a unique functional $\mathcal{F} : L^q(I) \to \mathbb{R}$. Recall that $L^q(I)^* \cong L^p(I)$, therefore there exists a function $f \in L^p(I)$ such that

$$\int_I u\varphi' = \mathcal{F}\varphi = \langle \varphi, f \rangle = \int_I \varphi f.$$

This shows that u is indeed a Sobolev function and $u' = -f$.

- $(a) \implies (c)$ Let $\tilde{u}$ be the continuous representative of u

$$\tilde{u} = u(y) + \int_x^y u' \mathrm{d}t \quad \forall x, y \in I,$$

then

$$\tau_h u(x) - u(x) = u(x+h) - u(x) - \int_x^{x+h} u'(t)\mathrm{d}t$$

and

$$\begin{aligned} |\tau_h u - u\|_{L^p(I\cap I_{-h})}^p &= \int_a^b |\tau_h u(x) - u(x)|^p \mathrm{d}x = \int_a^{b-h} \left| \int_x^{x+h} 1 \cdot u'(t)\mathrm{d}t \right|^p \mathrm{d}x \\ &\le \int_a^{b-h} \int_x^{x+h} |u'(t)|^p \mathrm{d}t \cdot h^{\frac{p}{q}} \mathrm{d}x \le (b-a) \int_a^b |u'(t)|^p \cdot h^{p/q} h = \|u'\|_{L^p}^p h^p. \end{aligned}$$

- $(c) \implies (a)$ For all $\varphi \in C_c^\infty(I)$ and h sufficiently small,

$$\begin{aligned} \int_I (\tau_h u(x) - u(x))\varphi(x) dx &= \int_I u(x+h)\varphi(x)\mathrm{d}x - \int_I u(x)\varphi(x)\mathrm{d}x \\ &= \int_I u(x)(\varphi(x-h) - \varphi(x))\mathrm{d}x. \end{aligned}$$

Now,

$$\begin{aligned} \left|\int_I u\varphi'\right| &= \left|\lim_{h\to 0} \int_I u(x) \frac{\varphi(x) - \varphi(x-h)}{h} \mathrm{d}x\right| = \lim_{h\to 0} \frac{1}{|h|} \left|\int_I (\tau_h u - u)\varphi \mathrm{d}x\right| \\ &\le \lim_{h\to 0} \frac{1}{|h|} \|\tau_h u - u\|_{L^p} \|\varphi\|_{L^q} \le \lim_{h\to 0} \frac{1}{|h|} C|h| \|\varphi\|_{L^q} = C|h|\|\varphi\|_{L^q}. \end{aligned}$$

□

Corollary 1.3.2. $W^{1,\infty}(I) = \mathrm{Lip}(I)$.

Proof. Let $\tilde{u}$ be a continuous representative of $u \in W^{1,\infty}(I) = \text{Lip}(I)$. Then $|\tilde{u}(x) - \tilde{u}(y)| = |\int_x^y u'(t)\mathrm{d}t| \leq \|u'\|_{L^\infty(I)}|x - y|$, hence u is Lipschitz with constant $\|u'\|_{L^\infty(I)}$. Conversely, suppose u is L-Lipschitz. Then $|\tau_h u(x) - u)x)| = |u(x+h) - u(x)| \leq L|h|$, hence $\|\tau_h u - u\|_{L^\infty} \leq L|h|$. By Corollary 1.3.1, $u \in W^{1,\infty}(I)$. □

Next we give a density result. In order to achieve it we need to show that there exists an extension operator $E : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$.

Proposition 1.3.3. *Let I be an interval. There exists a continuous linear operator $E : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ such that for all $u \in W^{1,p}(I)$*

- $(Eu)|_I = u$,
- $\|Eu\|_{L^p(\mathbb{R})} \leq C\|u\|_{L^p(I)}$, $\|(Eu)'\|_{L^p(\mathbb{R})} \leq C\|u'\|_{L^p(I)}$.

Proof. Let us show the proof for $I = (0, +\infty)$.

We can extend functions to $\mathbb{R}$ by reflection,

$$Eu(x) = \begin{cases} u(x) & x > 0 \\ u(-x) & x < 0 \end{cases}.$$

Clearly, $\|Eu\|_{L^p(\mathbb{R})} \leq 2\|u\|_{L^p(I)}$ and $\|Eu'\|_{L^p(\mathbb{R})} \leq 2\|u'\|_{L^p(I)}$.

For bounded intervals a similar argument can be obtained by introducing a mollifier η and reflecting $u = u\eta + u(1-\eta)$ around the interval endpoints to extend u to $\mathbb{R}$. □

Theorem 1.3.4. *Let $u \in W^{1,p}(I)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that $u_n|_I \xrightarrow[W^{1,p}]{n} u$. In other words, given the restriction operator*

$$\mathcal{R} : C_c^\infty(\mathbb{R}) \to W^{1,p}(I), \quad u \longmapsto u|_I$$

we have that its range $\mathcal{R}(C_c^\infty(\mathbb{R}))$ is dense in $W^{1,p}(I)$.

Proof. Suppose $I = \mathbb{R}$. We can assume $u \in C_c(\mathbb{R})$. Let $\eta =$ be a C^∞ cutoff which is also an even function and let $\eta_n(x) = \eta(\frac{|x|}{n})$ with $|\eta_n'| \leq \frac{C}{n}$. Then $\eta_n u \in C_c(\mathbb{R})$, by the chain rule, $(\eta_n u)' = \eta_n' u + \eta u'$ and

$$\|\eta_n u - u\|_{W^{1,p}} = \|\eta_n u - u\|_{L^p} + \|\eta_n' u + \eta u' - u'\|_{L^p} \leq \|\eta_n u - u\|_{L^p} + \frac{C}{n}\|\eta_n' u\| + \|\eta_n u' - u'\| \xrightarrow{n} 0$$

by dominated convergence.

Let $u \in C_c^\infty(\mathbb{R})$ and define $\eta_\varepsilon = \frac{1}{\varepsilon}\eta\left(\frac{|x|}{\varepsilon}\right)$, $u_\varepsilon = u * \eta_\varepsilon \in C_c^\infty(\mathbb{R})$. Then $u_\varepsilon \to u$, $(u * \eta_\varepsilon) = u' * \eta_\varepsilon \to u'$ in $L^p(\mathbb{R})$.

For a generic interval I we consider the extension operator $E : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ as in 1.3.3. There exists $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ that converges to Eu in $W^{1,p}(\mathbb{R})$. Then

$$\|u_n|_I - u\|_{W^{1,p}(I)} = \|u_n|_I - Eu|_I\|_{W^{1,p}(I)} \leq \|u_n|_I - Eu|_I\|_{W^{1,p}(\mathbb{R})} \xrightarrow{n} 0.$$

□

Remark 1.3.5. *The requirement for u_n to be compactly supported in $\mathbb{R}$ is crucial.; $C_c^\infty(I)$ is not dense in $W^{1,p}(I)$ unless $I = \mathbb{R}$, as we see in the following definition and proposition.*

Definition 1.3.6. $W_0^{1,p}(I) := \overline{C_c^\infty(I)}$, *where the closure is taken with respect to the $\|\cdot\|_{W^{1,p}(I)}$ topology.*

Proposition 1.3.7. *The following facts hold:*

(i) $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$,

(ii) $W_0^{1,p}(I) = \{u \in W^{1,p}(I) : u|_{\partial I} = 0\}$.

Remark 1.3.8. *The requirement that $u|_{\partial I} = 0$, i.e. u vanishes at the endpoints of the interval, would not make sense for a general function in L^p, which is defined almost everywhere. However, as shown in Proposition 1.2.8, $u \in W^{1,p}(I)$ has a continuous representative.*

Proof. We have already shown that (i) holds in Theorem 1.3.4. Observe that (i) also follows from (ii), since $\partial\mathbb{R} = \emptyset$.

Let $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(I)$ converge to $u \in W^{1,p}(I)$. Then, since $W^{1,p}(I) \hookrightarrow L^\infty(I)$ is continuous, $\|u_n - u\|_{L^\infty} = 0$ and for $a \in \partial a$

$$u(a) = \lim_n u_n(a) = 0.$$

Conversely, suppose $u \in W^{1,p}(I)$ is such that $u|_{\partial I} = 0$. Let G_ε be smooth with bounded derivative and such that

$$G_\varepsilon(t) = \begin{cases} t & |t| > \varepsilon \\ 0 & |t| < \frac{\varepsilon}{2} \end{cases}.$$

Moreover, assume $|G_\varepsilon(t) - t| \leq \varepsilon$ Observe that $G_\varepsilon \circ u \in W_0^{1,p}(I)$ since $\operatorname{supp}(G_\varepsilon \circ u) = \{|u| > \frac{\varepsilon}{2}\}$ is compact. Then

$$\|G_\varepsilon(u) - u\|_{L^p} \leq \varepsilon(b-a)^{1/p} \to 0$$

and, by lemma 1.3.9 and corollary 1.3.10

$$\|G_\varepsilon(u)' - u'\|_{L^p}^p = \int_I |G_\varepsilon'(u) - 1|^p |u'|^p \mathrm{d}x \to \int_{\{u=0\}} |u'|^p \mathrm{d}x = 0.$$

□

Lemma 1.3.9. *(algebra of Sobolev functions) Let I be any interval and $1 \leq p \leq \infty$, The following hold:*

(i) (Leibniz rule) if $u, v \in W^{1,p}(I)$, then $uv \in W^{1,p}(I)$ and $(uv)' = u'v + uv'$,

(ii) (chain rule) if $u \in W^{1,p}(I)$ and $G \in C^1(\mathbb{R})$ is Lipschitz and $G(0) = 0$, then $G \circ u \in W^{1,p}(I)$ and $(G \circ u)'(x) = G'(u(x))u'(x)$.

(iii) if $u \in W^{1,p}(I)$, then $u^+ = \max\{0, u\}, u^+ = \max\{0, -u\} \in W^{1,p}(I)$ with $(u^+)' = u'\mathbb{1}_{\{u>0\}}$ and $(u^-)' = u'\mathbb{1}_{\{u<0\}}$.

Corollary 1.3.10. *Let $u \in W^{1,p}(I)$. Then $u' = 0$ almost everywhere on $\{u = 0\}$.*

Proof. Observe that

$$u = u_+ + u_- \implies u' = (u_+ + u_-)' = u'(\mathbb{1}_{\{u>0\}} + \mathbb{1}_{\{u<0\}}) = u'(\mathbb{1}_{\{u\neq 0\}}).$$

Then $u'(1 - \mathbb{1}_{\{u\neq 0\}}) = 0$. □

We now give the definition of Sobolev space on an open subset of $\mathbb{R}^n$ and pose three questions which will be answered in the sequel.

Definition 1.3.11. *Let $\Omega \subseteq \mathbb{R}^n$ be open. A function $u \in L^p(\Omega)$ is said to be Sobolev ($u \in W^{1,p}(\Omega)$) if there exists $g \in L^p(\Omega, \mathbb{R}^n)$ such that*

$$\int_\Omega u \operatorname{div}(v) \mathrm{d}x = -\int_\Omega \langle g, v\rangle \, \mathrm{d}x \quad \text{for all } v \in C_c^\infty(\Omega, \mathbb{R}^n).$$

The function g is called the weak gradient of u and denoted by $g = \nabla u$.

Remark 1.3.12. *$W^{1,p}(\Omega)$ is a Banach space when endowed with the norm*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega,\mathbb{R}^n)}.$$

As we are interested in generalizing the concept of Sobolev space of $W^{1,p}$ to the context of metric (measure) spaces we will need to find suitable substitutes for the notions of

- vector fields $v \in C_c^\infty(\Omega, \mathbb{R}^n)$,
- divergence of $v \in C_c^\infty(\Omega, \mathbb{R}^n)$,
- duality in $\mathbb{R}^n$, the spaces involved will be a priori distinct and there will be no natural identification between them.

Chapter 2

The machinery of L^0-normed modules

2.1 Metric measure spaces

A metric measure space is a triple $(X, d, \mathfrak{m})$, where

- (X, d) is a complete and separable metric space
- $\mathfrak{m} \neq 0$ is a non-negative Borel measure which is finite on bounded sets.

Given a metric space (X, d) we will denote by $\mathscr{P}(X)$ the space of Borel probability measures on X, and by $C_b(X)$ the space of continuous and bounded functions $X \to \mathbb{R}$. Moreover, $B_r(x)$ will denote the open ball of center $x \in X$ and radius $r > 0$.

Remark 2.1.1. *The measure of a metric measure space is automatically σ-finite, since $X = \bigcup_{n=1}^{\infty} B_n(x)$ and $\mathfrak{m}(B_n(x)) < +\infty$.*

2.2 Lipschitz functions

Given two metric spaces (X, d_X), (Y, d_Y) and a map $f : X \to Y$ between them, we say that f is Lipschitz if there exists a constant $C > 0$ such that for all $x_1, x_2 \in X$

$$\mathsf{d}_Y(f(x_1), f(x_2)) \leq C \mathsf{d}_X(x_1, x_2),$$

equivalently if

$$\mathrm{Lip}(f) \coloneqq \sup_{x_1 \neq x_2} \frac{\mathsf{d}_Y(f(x_1), f(x_2))}{\mathsf{d}_X(x_1, x_2)} < \infty.$$

We denote by $\mathrm{Lip}(X, Y)$ the space of Lipschitz functions between X and Y. When $Y = \mathbb{R}$ we will denote $\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathbb{R})$

A function $f : X \to Y$ is said to be locally Lipschitz if for every $x \in X$ there exists $r > 0$ such that $f|_{B_r(x)}$ is Lipschitz.

Given a function $f : X \to Y$ and a Borel set E we will denote with $\mathrm{Lip}(f, E)$ the Lipschitz constant of f restricted to the Borel set E;

Two notion of local Lipschitz constant will be useful in the sequel.

Definition 2.2.1. *For a locally Lipschitz function $f : X \to Y$ we define the* asymptotic *Lipschitz constant at a point $x \in X$ as*

$$\mathrm{lip}_a f(x) \coloneqq \lim_{r \to 0} \mathrm{Lip}(f, B_r(x)).$$

Remark 2.2.2. *Such limit exists since $r \longmapsto \mathrm{Lip}(f, B_r(x)) = \sup\left\{\frac{|f(y)-f(z)|}{\mathsf{d}(y,z)} : y, z \in B_r(x), y \neq z\right\}$ decreases as r approaches 0.*

Definition 2.2.3. *For a locally Lipschitz function $f : X \to Y$ we define the* slope *(or local Lipschitz constant) at a point $x \in X$ as*

$$|Df|(x) \coloneqq \limsup_{y \to x} \frac{\mathsf{d}_Y(f(y), f(x))}{\mathsf{d}_X(y, x)}.$$

Remark 2.2.4. *Observe that*

$$|Df|(x) = \limsup_{y \to x} \frac{\mathsf{d}_Y(f(y) - f(x))}{\mathsf{d}_X(y, x)} = \lim_{r \to 0}\left(\sup_{y \in B_r(x) \setminus x} \frac{\mathsf{d}_Y(f(y) - f(x))}{\mathsf{d}_X(y, x)}\right) \leq \mathrm{lip}_a f(x).$$

Notice that for all $x \in X$ it holds

$$|D(\alpha f + \beta g)| \leq |\alpha||Df| + |\beta||Dg|, \forall \alpha, \beta \in \mathbb{R},$$

and a Leibniz-type inequality

$$|D(fg)| \leq |f||Dg| + |g||Df|. \tag{2.2.1}$$

The same estimate holds for the asymptotic Lipschtiz constant:

$$|\mathrm{lip}_a(fg)| \leq |f||\mathrm{lip}_a g| + |g||\mathrm{lip}_a f|. \tag{2.2.2}$$

Definition 2.2.5. *When $Y = \mathbb{R}, \mathsf{d}_Y = |\cdot|$ we can define the one-sided counterparts of $|Df|$, which are called respectively descending slope and ascending slope:*

$$|Df|^-(x) \coloneqq \limsup_{y \to x} \frac{[f(y) - f(x)]^-}{\mathsf{d}(y, x)}, \qquad |Df|^+(x) \coloneqq \limsup_{y \to x} \frac{[f(y) - f(x)]^+}{\mathsf{d}(y, x)},$$

which for all $x \in X$ statisfy

$$|Df|(x) = \max\{|D^- f|(x), |D^+ f|(x)|\}, |D^- f|(x) = |D^+(-f)|(x)$$

2.3 The space $L^0(X, \mathfrak{m})$

Definition 2.3.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We denote by $L^0(X, \mathfrak{m})$ the set of measurable functions $f : X \to \mathbb{R}$, up to equality $\mathfrak{m}$-almost everywhere.*

Clearly $L^0(X, \mathfrak{m})$ is a vector space over the real numbers. Given a probability measure $\mathfrak{m}'$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ we can make $L^0(X, \mathfrak{m})$ into a topological vector space by equipping it with the topology induced by the distance

$$\mathsf{d}_{L^0,\mathfrak{m}'}(f, g) := \int_X \inf\{|f - g|, 1\} d\mathfrak{m}'.$$

Observe that by the above definition

$$\mathsf{d}_{L^0,\mathfrak{m}'}(f, g) = \| \inf\{|f - g|, 1\}\|_{L^1(\mathfrak{m}')}. \tag{2.3.1}$$

Remark 2.3.2. *The requirement on* $\mathfrak{m}'$ *to be a probability measure ensures that* $\mathsf{d}_{L^0,\mathfrak{m}'}(f, g) \leq \int_X 1 d\mathfrak{m}' = 1$,*so that* $(L^0(\mathfrak{m}'), \mathsf{d}_{L^0,\mathfrak{m}'})$ *is a metric space of finite diameter. Since* $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$, *by the Radon-Nikodym theorem there are functions* $f \in L^1(\mathfrak{m})$ *and* $g \in L^0(\mathfrak{m}')$ *such that* $d\mathfrak{m} = f d\mathfrak{m}'$ *and* $d\mathfrak{m}' = g d\mathfrak{m}$. *Hence* $d\mathfrak{m} = fg d\mathfrak{m}$, *or more explicitly*

$$\mathfrak{m}(E) = \int_E 1 d\mathfrak{m} = \int_E fg d\mathfrak{m} \quad \textit{for all Borel sets } E \subseteq X.$$

Thus $fg = 1$ $\mathfrak{m}$*-almost everywhere.*

The the distance $\mathsf{d}_{L^0,\mathfrak{m}'}$ depends on the probability measure $\mathfrak{m}'$, however the topology it induces does not, as shown in the following proposition.

Proposition 2.3.3. *Fix a probability measure* $\mathfrak{m}'$ *such that* $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. *A sequence* (f_n) *of functions in* $L^0(X, \mathfrak{m})$ *is Cauchy with respect to* $\mathsf{d}_{L^0,\mathfrak{m}'}$ *if and only if for all Borel sets* $E \subseteq X$ *of finite measure* $\mathfrak{m}(E) < \infty$

$$\limsup_{n,m \to +\infty} \mathfrak{m}(E \cap \{|f_n - f_m| > \varepsilon\} = 0. \tag{2.3.2}$$

Proof. Suppose (f_n) is a sequence in $L^0(X, \mathfrak{m})$ that satisfies 2.3.2 and fix a point $x_0 \in X$. Since $\mathfrak{m}'$ is a probability measure,

$$1 = \mathfrak{m}'(X) = \mathfrak{m}\left(\bigcup_n B_n(x_0)\right) = \lim_{n \to +\infty} \mathfrak{m}'(B_n(x_0).$$

Thus there exists $N > 0$ such that $\mathfrak{m}'(B_N(x_0)) \geq 1 - \varepsilon$. By definition of metric measure space $\mathfrak{m}$ is finite on bounded sets, hence $\mathfrak{m}(B_N(x_0)) < +\infty$.

Let $g \in L^1(\mathfrak{m})$ be the Radon-Nikodym density of $\mathfrak{m}'$ with respect to $\mathfrak{m}$. Fix $\varepsilon > 0$ and define $A_{n,m}(\varepsilon) = B_N(x_0) \cap \{|f_n - f_m| > \varepsilon\}$. By assumption 2.3.2

$$\mathfrak{m}(A_{n,m}(\varepsilon)) \xrightarrow[n,m]{} 0,$$

so that

$$\chi_{A_{n,m}(\varepsilon)} \xrightarrow[n,m]{L^1(\mathfrak{m})} 0.$$

By dominated convergence we deduce that

$$\limsup_{m,n} \mathfrak{m}'(A_{m,n}(\varepsilon)) = \limsup_{m,n} \int_X \chi_{A_{n,m}(\varepsilon)} \mathrm{d}\mathfrak{m}' = \limsup_{m,n} \int_X (\chi_{A_{n,m}(\varepsilon)} \cdot g) \mathrm{d}\mathfrak{m} = 0. \tag{2.3.3}$$

Now observe that

$$\begin{aligned}
\mathsf{d}_{L^0,\mathfrak{m}'}(f_n, f_m) &= \int_X \inf(|f_n - f_m|, 1) \mathrm{d}\mathfrak{m}' \\
&= \int_{X \setminus B_N(x_0)} \inf(|f_n - f_m|, 1) \mathrm{d}\mathfrak{m}' + \int_{B_N(x_0)} \inf(|f_n - f_m|, 1) \mathrm{d}\mathfrak{m}' \\
&\leq \varepsilon + \int_{B_N(x_0) \setminus A_{n,m}(\varepsilon)} \inf(|f_n - f_m|, 1) \mathrm{d}\mathfrak{m}' + \int_{A_{n,m}(\varepsilon)} \inf(|f_n - f_m|, 1) \mathrm{d}\mathfrak{m}' \\
&\leq \varepsilon + \varepsilon \mathfrak{m}'(B_N(x_0) \setminus A_{n,m}) + \mathfrak{m}'(A_{n,m}(\varepsilon)) \leq 2\varepsilon + \mathfrak{m}'(A_{n,m}(\varepsilon)).
\end{aligned}$$

Combining this and 2.3.2 shows that $\limsup_{n,m} \mathsf{d}_{L^0,\mathfrak{m}'}(f_n, f_m) = 0$, i.e. (f_n) is a $\mathsf{d}_{L^0,\mathfrak{m}'}$-Cauchy sequence, since $\varepsilon > 0$ was chosen to be arbitrary.

Conversely, suppose (f_n) is a $\mathsf{d}_{L^0,\mathfrak{m}'}$-Cauchy sequence and fix $\varepsilon \in (0, 1)$. By Chebyshev's inequality,

$$\mathfrak{m}'(\{|f_n - f_m| > \varepsilon\}) = \mathfrak{m}'(\{\inf(|f_n - f_m|, 1) > \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X \inf(|f_n - f_m|, 1) \mathrm{d}\mathfrak{m} = \frac{1}{\varepsilon} \mathsf{d}_{L^0,\mathfrak{m}'}(f_n, f_m) \xrightarrow[n,m]{} 0.$$

Now fix a Borel set $E \subseteq X$ such that $\mathfrak{m}(E) < \infty$ and, like above, let $f \in L^1(\mathfrak{m}')$ be the Radon-Nikodym density of $\mathfrak{m}$ with respect to $\mathfrak{m}'$. Then $\chi_E f \in L^1(\mathfrak{m}')$ and

$$\mathfrak{m}(E \cap \{|f_n - f_m| > \varepsilon\}) = \int_X (\chi_{\{|f_n - f_m| > \varepsilon\}} \cdot \chi_E \cdot f) \mathrm{d}\mathfrak{m}'.$$

By dominated convergence, condition 2.3.2 holds. □

Remark 2.3.4. *Clearly condition* 2.3.2 *does not depend on* $\mathfrak{m}'$*, hence all metrics* $\mathsf{d}_{L^0,\mathfrak{m}'}$ *are equivalent, i.e. they induce the same topology on* $L^0(\mathfrak{m})$*, which justifies the notation* d_{L^0} *we will use from now on. We shall often deal with distances of similar form when speaking of* L^0*-modules.*

Remark 2.3.5. *One might see the definition of* d_{L^0} *as an attempt to adapt the* L^1 *norm to measurable functions. It's clear, however, that* d_{L^0} *is not induced by a norm, since it doesn't behave well with scalars: even for a constant function* $f = 1$ *and for* $\lambda > 1$

$$\mathsf{d}_{L^0}(f, 0) = \int_X \inf\{|f|, 1\} \mathrm{d}\mathfrak{m}' = \mathfrak{m}'(X) = 1, \ \textit{but}$$
$$\mathsf{d}_{L^0}(\lambda f, 0) = \int_X \inf\{\lambda, 1\} \mathrm{d}\mathfrak{m}' = \mathfrak{m}'(X) = 1 \neq \lambda \mathsf{d}_{L^0}(f, 0).$$

To show that $L^0(\mathfrak{m}, \mathsf{d}_{L^0})$ is a complete and separable space we first prove some equivalent characterization of converging sequences with respect to d_{L^0}, which is inspired by Proposition 2.3.3.

Proposition 2.3.6. *Let $(X, d, \mathfrak{m})$ be a metric measure space and $\mathfrak{m}' \in \mathscr{P}(X)$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. Fix a sequence $(f_n)_n \subset L^0(X)$ and a function $f \in L^0(X)$. The following are equivalent:*

(1) f_n converges to f with respect to d_{L^0},

(2) any subsequence of (f_n) has a further subsequence which converges to f $\mathfrak{m}$-almost everywhere,

(3) $\limsup_n \mathfrak{m}(E \cap \{|f_n - f| > \varepsilon\}) = 0$ for all $\varepsilon > 0$ and for all Borel sets $E \subseteq X$ such that $\mathfrak{m}(E) < \infty$,

(4) $\limsup_n \mathfrak{m}'(\{|f_n - f| > \varepsilon\}) = 0$ for all $\varepsilon > 0$.

Remark 2.3.7. *Condition* (4) *amounts to saying that f_n converges to f in measure with respect to $\mathfrak{m}'$. Since $\mathfrak{m}'$ is a probability measure, convergence in measure implies convergence $\mathfrak{m}'$-a.e. for a subsequence.*

Proof. (1) $\implies$ (2) Any subsequence $(f_{n_k})_k$ of $(f_n)_n$ converges to f by assumption. By remark 2.3.1 this means that $\inf\{|f_{n_k} - f|, 1\} \xrightarrow[L^1(\mathfrak{m}')]{k} 0$. This implies that there exists a further subsequence which converges $\mathfrak{m}'$-almost everywhere.
(2) $\implies$ (3) By contradiction suppose there exist $\varepsilon > 0$ and a Borel set $E \subseteq X$ of finite measure $\mathfrak{m}$ such that $\limsup_n \mathfrak{m}(E \cap \{|f_n - f| > \varepsilon\}) > 0$. In particular there exists a subsequence $(f_{n_k})_k$ such that

$$\lim_k \mathfrak{m}(E \cap \{|f_{n_k} - f| > \varepsilon\}) = r > 0. \tag{2.3.4}$$

By assumption (2), (f_{n_k}) admits a further subsequence $(g_\ell)_\ell$ which converges to f in the $\mathfrak{m}$-a.e. sense. Applying the dominated convergence theorem to $\mathbb{1}_{\{|g_\ell - f| > \varepsilon\} \cap E}$ we obtain

$$\mathfrak{m}(\{|g_\ell - f| > \varepsilon\} \cap E) = \int_X \underbrace{\mathbb{1}_{\{|g_\ell - f|,1 > \varepsilon\} \cap E}}_{\leq \mathbb{1}_E \in L^1(\mathfrak{m})} \mathrm{d}\mathfrak{m} \xrightarrow[\ell]{} 0,$$

which contradicts 2.3.4 since $(g_\ell)_\ell$ is a subsequence of $(f_{n_k})_k$.

(3) $\implies$ (4) Similarly to the proof of Proposition 2.3.3, let $x_0 \in X$ and fix $\delta > 0$ and $N > 0$ such that $\mathfrak{m}'(X \setminus B_N(x_0)) < \delta$. Just like in that proof, our assumption that $\limsup_n \mathfrak{m}(\{|f_n - f| > \varepsilon\} \cap B_N(x_0)) = 0$ implies that the same holds for $\mathfrak{m}'$. Therefore,

$$\limsup_n \mathfrak{m}'(\{|f_n - f| > \varepsilon\}) \leq \mathfrak{m}'(X \setminus B_N(x_0)) + \underbrace{\limsup_n \mathfrak{m}(\{|f_n - f| > \varepsilon\} \cap B_N(x_0))}_{=0} < \delta.$$

Since δ is arbitrary we conclude.

(4) $\implies$ (1) It suffices to adapt the argument used to prove proposition 2.3.3 to this situation. Fix $\varepsilon > 0$. Then

$$\mathsf{d}_{L^0}(f_n, f) = \int_X \inf\{|f - f_n|, 1\} \mathrm{d}\mathfrak{m}' \leq \mathfrak{m}'(\{|f_n - f| > \varepsilon\}) + \varepsilon,$$

thus $\limsup_n \mathsf{d}_{L^0}(f_n, f) < \varepsilon$ and we conclude. □

Observe that there are natural inclusions $L^p(\mathfrak{m}) \hookrightarrow L^0(\mathfrak{m})$ and that they are continuous thanks to Hölder's inequality. Indeed, given $\mathfrak{m}' \in \mathscr{P}(X)$ like above and $f_n \xrightarrow[n]{L^p(\mathfrak{m})} f$,

$$\mathsf{d}_{L^0}(f_n, f) \leq \int_X |f_n - f| \mathrm{d}\mathfrak{m}' \leq \|f_n - f\|_{L^p(\mathfrak{m}')} \mathfrak{m}'(X) = \|f_n - f\|_{L^p(\mathfrak{m}')} \to 0.$$

Moreover, this inclusion is dense. Indeed, let $f \in L^0(\mathfrak{m})$ and $x_0 \in X$, then

$$f_n(x) = \begin{cases} f(x) & x \in B_n(x_0) \text{ and } |f(x)| \leq n \\ n & x \in B_n(x_0) \text{ and } f(x) > n \\ -n & x \in B_n(x_0) \text{ and } f(x) < -n \\ 0 & x \notin B_n(x_0) \end{cases}$$

is bounded and has bounded support inside $B_n(x_0)$, hence $f_n \in L^p(\mathfrak{m})$ for all $p \in [1, \infty]$. Clearly f_n converges pointwise $\mathfrak{m}$-almost everywhere and condition (3) of proposition 2.3.6 holds. Therefore f_n converges to f with respect to d_{L^0}.

Theorem 2.3.8. $(L^0(\mathfrak{m}), \mathsf{d}_{L^0})$ *is a complete and separable metric space.*

Proof. By what we have just shown, $L^1(\mathfrak{m})$ is embedded in $L^0(\mathfrak{m})$ continuously and densely. Since $L^1(\mathfrak{m})$ is separable, $L^0(\mathfrak{m})$ is separable as well.

To show completeness, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^0(\mathfrak{m})$. By proposition 2.3.3

$$\limsup_{n,m\to+\infty} \mathfrak{m}(E \cap \{|f_n - f_m| > \varepsilon\} = 0$$

for all $\mathfrak{m}(E) < \infty$, in particular $\limsup_{m,n} \mathfrak{m}'(\{|f_n - f_m| > \varepsilon\}) = 0$. Therefore there exists a subsequence $(f_{n_k})_k$ such that $\mathfrak{m}'(\{|f_{n_{k+1}} - f_{n_k}| > \varepsilon 2^{-k}\}) < 2^{-k}$. Fix representatives for f_{n_k} and let

$$N_k := \{|f_{n_{k+1}} - f_{n_k}| > \varepsilon 2^{-k}\},$$

and

$$N := \bigcap_n \bigcup_{k\geq n} N_k.$$

Observe that

$$\mathfrak{m}'(N) \le \inf_n \left\{ \mathfrak{m}'\left(\bigcup_{k\ge n} N_k \right) \right\} \le \inf_n \left\{ \sum_{k\ge n} \mathfrak{m}'(N_k) \right\} \le \inf_n \left\{ \sum_{k\ge n} 2^{-k} \right\} = 0.$$

Let $g(x) = \sum_k |f_{n_k}(x) - f_{n_{k+1}}(x)|$. If $g(x) < \infty$ then $(f_{n_k}(x))_k$ is a Cauchy sequence and therefore converges. By construction, $g(x) < \infty$ for $x \in X \setminus N$, hence $f_{n_k}(x)$ converges for almost every x, denote the pointwise limit by f, Then $f \in L^0(\mathfrak{m})$ and we can extend f to vanish on N.

□

2.4 L^0-modules

Definition 2.4.1. *Let $(X, \mathfrak{m})$ be a measure space. An $L^0(X,\mathfrak{m})$-normed $L^0(X,\mathfrak{m})$-module is a quadruple $(\mathscr{M}, \tau, \cdot, |\cdot|)$ where*

(i) $(\mathscr{M}, \tau)$ is a complete topological vector space.

(ii) The bilinear map $\cdot : L^0(X,\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ satisfies

$$\begin{aligned} f\cdot(g\cdot v) &= (fg)\cdot v \qquad && \forall f,g \in L^0(X,\mathfrak{m}), \forall v \in \mathscr{M} \\ 1\cdot v &= v && \forall v \in \mathscr{M}. \end{aligned}$$

(iii) The map $|\cdot| : \mathscr{M} \to L^0(X,\mathfrak{m})$, called the pointwise norm, satisfies the following properties

$$\begin{aligned} |v| &\ge 0 \qquad && \mathfrak{m}\text{-}a.e. \qquad && \forall v \in \mathscr{M} \\ |f\cdot v| &= |f||v| && \mathfrak{m}\text{-}a.e. && \forall f \in L^0(X,\mathfrak{m}), \forall v \in \mathscr{M} \\ |v+w| &\le |v| + |w| && \mathfrak{m}\text{-}a.e. && \forall v,w \in \mathscr{M}. \end{aligned}$$

(iv) The topology τ is the one induced by the distance $\mathsf{d}_{\mathscr{M}}$ on $\mathscr{M}$ defined by

$$\mathsf{d}_{\mathscr{M}}(v,w) = \int_X \inf\{|v-w|, 1\}\, \mathrm{d}\mathfrak{m}' \qquad \text{for some } \mathfrak{m}' \in \mathscr{P}(X) \text{ such that } \mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$$

Remark 2.4.2. *The name L^0-module is justified by axiom (ii) in our definition, which ensures that $(\mathscr{M}, \tau, \cdot, |\cdot|)$ is indeed a module over the ring of measurable functions. As seen in the definition of the pointwise norm, measurable functions behave like scalars in this setting. This idea also inspires the following definition of dual L^0-module as the set of linear maps into $L^0(X,\mathfrak{m})$, with an additional boundedness assumption as one usually does in analysis.*

Proposition 2.4.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $\mathscr{M}$ be a $L^0(\mathfrak{m})$-normed module. Then:*

(1) $\lambda v = \hat{\lambda} v$ for $\lambda \in \mathbb{R}$, where $\hat{\lambda}$ denotes the (equivalence class of the) function identically λ on X.

(2) $|\lambda v| = |\lambda||v|$ holds $\mathfrak{m}$-a.e. for all $v \in \mathscr{M}$ and $\lambda \in \mathbb{R}$.

(3) the pointwise norm is uniformly continuous.

Proof. (1) Given $\lambda \in \mathbb{R}$ and $v \in \mathscr{M}$, by using the bilinearity of multiplication we have that $\hat{\lambda} v = (\lambda \hat{1}) v = \lambda(\hat{1} v) = \lambda v$.

(2) Fix $\lambda \in R$ and $v, w \in \mathscr{M}$. By the definition of L^0-module and (1) we have that $|\lambda v| = |\hat{\lambda} v| = |\hat{\lambda}||v| = |\lambda||v|$ holds $\mathfrak{m}$-a.e..

(3) Thanks to the triangle inequality for the pointwise norm we have that

$$d_{L^0}(|v|, |w|) = \int_X \inf\{||v| - |w||, 1\} \mathrm{d}\mathfrak{m}' \leq \int_X \inf\{|v - w|, 1\} \mathrm{d}\mathfrak{m}' = d_{\mathscr{M}}(v, w),$$

so the pointwise norm is uniformly continuous.

□

We will now give the notions of submodule and dual for an $L^0(X, \mathfrak{m})$-normed $L^0(X, \mathfrak{m})$-module.

Definition 2.4.4. *Let $\mathscr{M}$ be an $L^0(X, \mathfrak{m})$-module. A closed linear subspace $\mathscr{N} \subseteq \mathscr{M}$ is a submodule if it closed by multiplication by measurable functions, i.e.*

$$f \cdot v \in \mathscr{N} \quad \textit{for all } f \in L^0(\mathfrak{m}), v \in \mathscr{N}.$$

Given a subset $S \subset \mathscr{M}$, the submodule $\mathcal{G}(S, \mathscr{M})$ generated by S is defined as the smallest (with respect to inclusion) submodule of $\mathscr{M}$ containing S.

Equivalently, $\mathcal{G}(S, \mathscr{M}) = \overline{G(S)}$, where

$$G(S) = \left\{ \sum_{i=1}^{n} f_i \cdot s_i : n \in \mathbb{N}, f_i \in L^0(\mathfrak{m}), s_i \in S \right\} = \mathrm{span}_{L^0}(S)$$

is the set of all (finite) L^0-linear combinations of elements of S.

Remark 2.4.5. *It is straightforward to see that the intersection of submodules is itself a submodule.*

2.5 Duality for L^0-modules

Definition 2.5.1. *Given a measure space $(X, \mathfrak{m})$ and an $L^0(X, \mathfrak{m})$-normed $L^0(X, \mathfrak{m})$-module $\mathscr{M}$, we define its dual module $\mathscr{M}^*$ as the set of continuous linear maps $\omega : \mathscr{M} \to L^0(X, \mathfrak{m})$ such that*

(i) $\omega(f \cdot v) = f\omega(v)$, for every $v \in \mathscr{M}$ and for every $f \in L^0(X, \mathfrak{m})$

(ii) there exists $g \in L^0(X, \mathfrak{m})$ such that

$$|\omega(v)| \leq |g||v| \qquad \mathfrak{m}\text{-}a.e. \qquad \forall v \in \mathscr{M}. \tag{2.5.1}$$

We then define $|\omega|_*$ as the least (positive) function $g \in L^0(X, \mathfrak{m})$ for which (ii) is satisfied.

Remark 2.5.2. *Observe that the definition of $|\omega|_*$ is well posed, since whenever two functions g and h satisfy condition (ii) their infimum, which is still measurable, also does.*

The dual $(\mathscr{M}^*, |\cdot|_*)$ has a natural structure of $L^0(X, \mathfrak{m})$-normed $L^0(X, \mathfrak{m})$-module given by

- $(f \cdot \omega)(v) = f \cdot (\omega(v))$
- $|(f \cdot \omega)(v)| = |f(\omega(v)| = |f||\omega(v)|$ hence $|f \cdot \omega|_* = |f||\omega|_*$,
- $|(\omega_1 + \omega_2)(v)| \leq |\omega_1(v)| + |\omega_2(v)| \leq (|\omega_1|_* + |\omega_1|_*)|v|$, hence $|\omega_1 + \omega_2|_* \leq |\omega_1|_* + |\omega_2|_*$.
- $\mathsf{d}_{\mathscr{M}^*}(\omega_1, \omega_2) = \int_X \inf\{|\omega_1 - \omega_2|_*, 1\} \mathrm{d}\mathfrak{m}'$.

We can adapt the definition of $L^0(X, \mathfrak{m})$-normed $L^0(X, \mathfrak{m})$-module to include integrability requirements of the norm, as seen in the following definition.

Definition 2.5.3. *An $L^p(X, \mathfrak{m})$-normed $L^\infty(X, \mathfrak{m})$-module is a triple $(\mathscr{M}_p, \|\cdot\|, |\cdot|)$ where*

(i) $(\mathscr{M}_p, \|\cdot\|)$ is a Banach space.

(ii) The bilinear map $\cdot : L^\infty(X, \mathfrak{m}) \times \mathscr{M}_p \to \mathscr{M}_p$ satisfies

$$\begin{aligned} f \cdot (g \cdot v) &= (fg) \cdot v \qquad && \forall f, g \in L^\infty(X, \mathfrak{m}), \forall v \in \mathscr{M}_p \\ 1 \cdot v &= v && \forall v \in \mathscr{M}_p. \end{aligned}$$

(iii) The map $|\cdot| : \mathscr{M} \to L^p(X, \mathfrak{m})$, called the pointwise norm, satisfies the following properties

$$\begin{aligned} |v| &\geq 0 \qquad && \mathfrak{m}\text{-}a.e. \qquad && \forall v \in \mathscr{M}_p \\ |f \cdot v| &= |f||v| && \mathfrak{m}\text{-}a.e. && \forall f \in L^\infty(X, \mathfrak{m}), \forall v \in \mathscr{M} \\ |v + w| &\leq |v| + |w| && \mathfrak{m}\text{-}a.e. && \forall v, w \in \mathscr{M}_p. \end{aligned}$$

(iv) We have $\|v\| = (\int_X |v|^p \mathrm{d}\mathfrak{m})^{1/p}$.

Remark 2.5.4. *The definition differs from the* L^0 *case in that we require* L^p*-normed* L^∞*-modules to be Banach spaces, not just topological vector spaces. This allows us to drop condition (iv) in 2.5.1 and avoids the dependence on a probability measure.*

We can also adapt the definition of dual module.

Definition 2.5.5. *Given a measure space* $(X, \mathfrak{m})$ *and an* $L^p(X, \mathfrak{m})$*-normed* $L^\infty(X, \mathfrak{m})$*-module* $\mathscr{M}_p$*, we define* $(\mathscr{M}_p)^*$*, its dual module, as the set of continuous linear maps* $\omega : \mathscr{M}_p \to L^1(X, \mathfrak{m})$ *such that*

(i) $\omega(f \cdot v) = f\omega(v)$*, for every* $v \in \mathscr{M}_p$ *and for every* $f \in L^\infty(X, \mathfrak{m})$

(ii) there exists $g \in L^q(X, \mathfrak{m})$ *such that*

$$|\omega(v)| \leq |g||v| \qquad \mathfrak{m}\text{-}a.e. \qquad \forall v \in \mathscr{M}$$

We then define $|\omega|_*$ *as the least function* $g \in L^q(X, \mathfrak{m})$ *for which (ii) is satisfied. Then* $((\mathscr{M}_p)^*, |\cdot|_*)$ *has a natural structure of* $L^q(X, \mathfrak{m})$*-normed* $L^\infty(X, \mathfrak{m})$*-module.*

Theorem 2.5.6. *Let* $p \in [1, \infty]$*. Given* $\mathscr{M}_p$ *an* $L^p(X, \mathfrak{m})$*-normed* $L^\infty(X, \mathfrak{m})$*-module, there exists a unique* $L^0(X, \mathfrak{m})$*-normed* $L^0(X, \mathfrak{m})$*-module* $\mathscr{M}$ *(up to isomorphisms) such that*

$$\mathscr{M}_p = L^p(\mathscr{M}, \mathfrak{m}) := \{\omega \in \mathscr{M} : |\omega| \in L^p(\mathfrak{m})\}.$$

Moreover, we have the following Banach spaces are canonically isomorphic:

(i) $(\mathscr{M}_p)^*$ *the dual module of* $\mathscr{M}_p$ *considered as* $L^p(X, \mathfrak{m})$*-normed* $L^\infty(X, \mathfrak{m})$*-module;*

(ii) $L^q(\mathscr{M}^*, \mathfrak{m})$*.*

Moreover if $p < \infty$ *we have also* $L^q(\mathscr{M}^*, \mathfrak{m}) = (\mathscr{M}_p)'$*, the Banach dual of* $\mathscr{M}_p$ *considered as Banach space.*

Proof. It is clear that $(\mathscr{M}_p)^*$ and $L^q(\mathscr{M}^*, \mathfrak{m})$ are canonically isomorphic. Moreover it is also trivial that for every p $L^q(\mathscr{M}^*, \mathfrak{m}) \subseteq (\mathscr{M}_p)'$. Let us prove the reverse inclusion in the case $p < \infty$. Let $\ell \in (\mathscr{M}_p)'$. Fix $v \in \mathscr{M}_p$ and consider the map

$$\mu_v : E \mapsto \ell(\chi_E \cdot v).$$

Then μ_v is a measure and moreover

$$|\mu_v(E)| \leq \|\ell\|_{\mathscr{M}_p'} \|\chi_E \cdot v\|_{\mathscr{M}_p} \leq \|\ell\|_{\mathscr{M}_p'} \left(\int_E |v|^p \mathrm{d}\mathfrak{m} \right)^{1/p},$$

thus $\mu_v \ll \mathfrak{m}$. We denote its density by $\omega(v) \in L^1(\mathfrak{m})$. Via a density argument it can be shown that ω is $L^\infty(\mathfrak{m})$-linear.

Since $\ell \in (\mathscr{M}_p)'$ there exists $C > 0$ such that

$$\int_X \omega(v)\mathrm{d}\mathfrak{m} = \ell(v) \le C\|v\|_p. \tag{2.5.2}$$

Let us consider two functionals in the Banach space $Y = L^p(\mathfrak{m})$:

$$\Psi_2(h) = C\|h\|_{L^p(\mathfrak{m})} \tag{2.5.3}$$

$$\Psi_1(h) = \sup\left\{\int_X \omega(v)\,\mathrm{d}\mathfrak{m} : |v| \le h\ ,\ v \in \mathscr{M}_p\right\} \tag{2.5.4}$$

where the supremum of the empty set is meant to be $-\infty$. Equation (2.5.2) guarantees that

$$\Psi_1(h) \le \Psi_2(h) \qquad \forall h \in Y. \tag{2.5.5}$$

Moreover Ψ_2 is convex and continuous while we claim that Ψ_1 is concave: it is clearly positive 1-homogeneus and so it is sufficient to show that

$$\Psi_1(h_1 + h_2) \ge \Psi_1(h_1) + \Psi_1(h_2).$$

We can assume that $\Psi_1(h_i) > -\infty$ for $i = 1, 2$ because otherwise the inequality is trivial. In this case for every $\varepsilon > 0$ we can pick $v_i \in \mathscr{M}_p$ such that

$$\int_X \omega(v_1)\,\mathrm{d}\mathfrak{m} \ge \Psi_1(h_1) - \varepsilon \qquad |v_1| \le h_1$$

$$\int_X \omega(v_2)\,\mathrm{d}\mathfrak{m} \ge \Psi_1(h_2) - \varepsilon \qquad |v_2| \le h_2$$

and so we can consider $v_1 + v_2 \in \mathscr{M}_p$, clearly $|v_1 + v_2| \le |v_1| + |v_2| \le (h_1 + h_2)$ and so

$$\Psi_1(h_1 + h_2) \ge \int_X (v_1 + v_2)(f)\,\mathrm{d}\mathfrak{m} \ge \Psi_1(h_1) + \Psi_1(h_2) - 2\varepsilon,$$

and we get the desired inequality letting $\varepsilon \to 0$. By Hahn-Banach theorem we can find a continuous linear functional L on $L^p(\mathfrak{m})$ such that

$$\Psi_1(h) \le L(h) \le \Psi_2(h).$$

Since $p < \infty$ we know that $(L^p)^* = L^q$ and so we can find $g \in L^q$ such that $L(h) = \int_X gh\,\mathrm{d}\mathfrak{m}$. This proves in particular that

$$\int_X \omega(v)\mathrm{d}\mathfrak{m} \le \int_X g|v|\mathrm{d}\mathfrak{m}.$$

We can localize the above inequality and get $\omega(v) \le g|v|$ almost everywhere and thus $\omega \in L^q(\mathscr{M}^*, \mathfrak{m})$. Moreover we have that $|\omega|_* \le g$ and so $\|\omega\|_{L^q} \le \|g\|_p \le C$.

□

Both in the L^0 and the L^p case, when speaking of duals we have defined the pointwise norm of a functional ωas the *least* among functions g satisfying $|\omega(v)| \leq |g||v|$ for all elements of the module. To see that such a function actually exists, we give the following lemma.

Lemma 2.5.7. *Let $\mathscr{M}$ be a $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and $\omega : \mathscr{M} \to L^0(\mathfrak{m})$ be a continuous linear functional. Then there exists a least positive function in*

$$Y = \{g \in L^0(\mathfrak{m}) : g \geq 0 \quad \mathfrak{m}\text{-}a.e, |\omega(v)| \leq g|v| \quad \mathfrak{m}\text{-}a.e \quad \forall v \in \mathscr{M}\}.$$

Proof. Consider $\mathfrak{m}' \in \mathscr{P}(X)$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ and let g_n be a minimizing sequence in Y for $\underset{Y}{\operatorname{argmin}}\{\int_X \arctan(|g|)\mathrm{d}\mathfrak{m}'\}$ and define $\bar{g}(x) = \inf_n g_n(x)$. Then $\bar{g} \leq g \quad \mathfrak{m}\text{-}a.e$ for all $g \in Y$. □

2.6 Hahn-Banach theorem for L^∞-modules

We want to prove an extension theorem for linear functional with pointwise estimates.

Lemma 2.6.1. *Let M_q be an $L^q(X, \mathfrak{m})$-normed $L^\infty(X, \mathfrak{m})$-module. Let $W \subset V$ be a sub-module, and let $L : W \to L^1$ be an L^∞-linear functional such that $L(w) \leq |w|g$ for some $g \in L^p(X, \mathfrak{m})$. Then L can be extended to a L^∞-linear functional $\tilde{L} : V \to L^1(X, \mathfrak{m})$ without increasing the pointwise norm, namely such that*

$$\tilde{L}(v) \leq |v|g \qquad \forall v \in V.$$

Proof. With an application of Zorn Lemma, we can reduce ourselves to extend the functional to $\operatorname{span}\{x_0, W\}$, for some $x_0 \notin W$. Any element in $\operatorname{span}\{x_0, W\}$ is of the form $\lambda x_0 + w$ where $\lambda \in L^\infty(X, \mathfrak{m})$ and $w \in W$. Since the extension $\tilde{L}$ has to be L^∞-linear, it must satisfy the equation

$$\tilde{L}(\lambda x_0 + w) = \lambda r_0 + \tilde{L}(w) = \lambda r_0 + L(w)$$

for some $r_0 \in L^1$. However we need to chose r_0 in such a way that we do not increase the norm. Then we need to choose r_0 so that

$$|\tilde{L}(\lambda x_0 + w)| \leq g|\lambda x_0 + w| \qquad \mathfrak{m}\text{-a.e.}$$
$$-g|\lambda x_0 + w| - L(w) \leq \lambda r_0 \leq g|\lambda x_0 + w| - L(w) \qquad \mathfrak{m}\text{-a.e.} \tag{2.6.1}$$

Denoting $\lambda_\varepsilon(x) = \sup\{\varepsilon, \lambda_+\} - \sup\{\varepsilon, \lambda_-\}$ we have that if (2.6.1) is verified for λ_ε for all $\varepsilon > 0$ then it is verified also for λ since in the set $\lambda = 0$ the inequality is trivial. In particular we have to check (2.6.1) only when $\lambda, \lambda^{-1} \in L^\infty$; so we can multiply by λ^{-1} the whole inequality (and pay attention to the signs) to get

$$-g\left|x_0 + \frac{w}{\lambda}\right| - L\Big(\frac{w}{\lambda}\Big) \leq r_0 \leq g\left|x_0 + \frac{w}{\lambda}\right| - L\Big(\frac{w}{\lambda}\Big) \qquad \mathfrak{m}\text{-a.e.}$$

Since $\lambda^{-1}w$ is an arbitrary element of W we can finally rewrite this inequality as

$$-g|x_0 + w| - L(w) \le r_0 \le g|x_0 + w| + L(w) \quad \mathfrak{m}\text{-a.e.} \qquad \forall w \in W.$$

Now, if $w_1, w_2 \in W$ we have that

$$\begin{aligned} L(w_1) - L(w_2) &= L(w_1 - w_2) \\ &\le g|w_1 - w_2| = g|(w_1 + x_0) - (w_2 + x_0)| \\ &\le g|w_1 + x_0| + g|w_2 + x_0| \qquad \mathfrak{m}\text{-a.e.} \end{aligned}$$

so that

$$-g|w_2 + x_0| - L(w_2) \le L(w_1) + g|w_1 + x_0| \quad \mathfrak{m}\text{-a.e.} \qquad \forall w_1, w_2 \in W,$$

Now we take the essential infimum of the right hand side and the essential supremum of the left hand side to obtain

$$a(x) = \operatorname*{esssup}_{w_2 \in W} \big\{-g|w_2 + x_0| - L(w_2)\big\} \le \operatorname*{essinf}_{w_1 \in W} \big\{g|w_1 + x_0| - L(w_1)\big\} = b(x);$$

so it is sufficient to take $a(x) \le r_0(x) \le b(x)$ for $\mathfrak{m}$-almost every x in order to extend the functional without increasing the pointwise estimate. $\square$

Chapter 3

Derivations

In this chapter we introduce derivations, which will play a key role in the construction of the tangent module.

3.1 Notation

Given a function $f : X \to \mathbb{R}$ and a Borel set E we will denote with $\mathrm{Lip}(f, E)$ the Lipschitz constant of f restricted to the Borel set E; if the set E is not indicated it is assumed to be $E = X$. A function f is said to be Lipschitz if $\mathrm{Lip}(f) < \infty$, and the set of Lipschitz functions is denoted by $\mathrm{Lip}(X, \mathsf{d})$. Other spaces that will be used in the sequel are:

- $\mathrm{Lip}_0(X, \mathsf{d})$, the set of Lipschitz functions with bounded support: the support of a continuous function f is defined as $\mathrm{supp}(f) = \overline{\{f \neq 0\}}$;
- $\mathrm{Lip}_b(X, \mathsf{d})$, the set of bounded Lipschitz functions;
- $\mathrm{Lip}_{loc}(X, \mathsf{d})$, the set of locally Lipschitz functions, that is those functions f such that for any x there exists $r > 0$ such that $f|_{B_r(x)}$ is Lipschitz.

We have the obvious inclusions $\mathrm{Lip}_0(X, \mathsf{d}) \subseteq \mathrm{Lip}_b(X, \mathsf{d}) \subseteq \mathrm{Lip}(X, \mathsf{d}) \subseteq \mathrm{Lip}_{loc}(X, \mathsf{d})$. Recall that for a locally Lipschitz function f asymptotic Lipschitz constant is defined as

$$\mathrm{lip}_a f(x) = \lim_{r \to 0} \mathrm{Lip}(f, B_r(x)).$$

3.2 Weaver derivations

Definition 3.2.1. *A Weaver derivation on a metric measure space* $(X, d, \mathfrak{m})$ *is a linear map* $\boldsymbol{b} : \mathrm{Lip}_0(X, \mathsf{d}) \to L^0(X, \mathfrak{m})$ *such that*

(i) (Leibniz rule) for every $f, g \in \mathrm{Lip}_0(X, \mathsf{d})$*, we have* $\boldsymbol{b}(fg) = \boldsymbol{b}(f)g + f\boldsymbol{b}(g)$*;*

(ii) (Locality) There exists some function $g \in L^0(X, \mathfrak{m})$ such that

$$|\boldsymbol{b}(f)|(x) \leq g(x) \cdot \mathrm{lip}_a f(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x, \forall f \in \mathrm{Lip}_0(X, \mathsf{d}).$$

The smallest function g with this property is denoted by $|\boldsymbol{b}|$.

(iii) (weak continuity) whenever $f_n \xrightarrow{n} f$ pointwise with uniformly bounded Lipschitz constants (i.e. $\sup_n \mathrm{Lip}(f_n) < \infty$), $\boldsymbol{b}(f_n) \xrightarrow{*} \boldsymbol{b}(f)$ in duality with $L^1(|\boldsymbol{b}|\mathrm{d}\mathfrak{m})$.*

We will denote by TX the space of Weaver derivations on X.

We will see that when $\boldsymbol{b}$ admits divergence weak* continuity follows from the first two properties.

Remark 3.2.2. *Derivations vanish at constant functions. Indeed, by the Leibniz rule,*

$$\boldsymbol{b}(1) = \boldsymbol{b}(1 \cdot 1) = 1 \cdot \boldsymbol{b}(1) + 1 \cdot \boldsymbol{b}(1) = 2\boldsymbol{b}(1).$$

Clearly the set of derivations TX has a vector space structure over $\mathbb{R}$, our main goal now is to endow it with the structure of $L^0(X, \mathfrak{m})$-normed $L^0(X, \mathfrak{m})$-module. According to the definition we then need to define the following:

- multiplication of derivations by measurable functions, this can be done naturally by setting
$$(u \cdot \boldsymbol{b})(f) \coloneqq u\boldsymbol{b}(f)$$
for $u \in L^0(\mathfrak{m}), \boldsymbol{b} \in TX, f \in \mathrm{Lip}_0(X, \mathsf{d})$, it is straightforward to see that with this definition $u \cdot \boldsymbol{b}$ is indeed a derivation,
- a pointwise norm, which can be given by choosing $|\boldsymbol{b}|$ as in the *locality* section of definition 3.2.1,
- a topology on TX which is compatible with the pointwise norm.

What we do is essentially adapting the d_{L^0} distance to TX.

Lemma 3.2.3. *Given $\mathfrak{m} \in \mathscr{P}(X)$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ consider*

$$\mathsf{d}_{TX}(\boldsymbol{b}_1, \boldsymbol{b}_2) \coloneqq \int_X \inf\{|\boldsymbol{b}_1 - \boldsymbol{b}_2|, 1\} \mathrm{d}\mathfrak{m}'$$

and let τ be the topology on TX induced by d_{TX}. Then $(TX, \tau, \cdot, |\cdot|)$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module.

Remark 3.2.4. *Just like in the case of d_{L^0} the topology τ does not depend on the specific probability measure $\mathfrak{m}'$ chosen to define d_{TX}.*

Proof. (Lemma) The only nontrivial condition to check in 2.4.1 is homogeneity, i.e.

$$|u \cdot \boldsymbol{b}| = |u||\boldsymbol{b}| \quad \mathfrak{m}\text{-}a.e. \qquad \text{for all } u \in L^0(\mathfrak{m}), \boldsymbol{b} \in TX. \tag{3.2.1}$$

Fix $u \in L^0(\mathfrak{m})$. By definition $|u \cdot \boldsymbol{b}|(x) \leq |\boldsymbol{b}|(x)|u(x)|$ almost everywhere. The reverse inequality is obviously true where u vanishes. Let us define

$$g(x) = \begin{cases} u(x)^{-1} & u(x) \neq 0 \\ 0 & \text{otherwise} \end{cases} \in L^0(\mathfrak{m})$$

Notice that $\boldsymbol{b}(f) = (g \cdot u)\boldsymbol{b}(f) = g \cdot (u \cdot \boldsymbol{b})$ in $\{u(x) \neq 0\}$. By 3.2.1 we obtain

$$|g \cdot (u \cdot \boldsymbol{b})| \leq |g||u \cdot \boldsymbol{b})| \leq |g||u||\boldsymbol{b}| = |\boldsymbol{b}| \quad \text{in } \{u \neq 0\}$$

and so we deduce the reverse inequality on $\{u \neq 0\}$ as well.

It remains to show that d_{TX} is complete. Let $(\boldsymbol{b}_n)_n$ be a d_{TX}-Cauchy sequence. For all $f \in \mathrm{Lip}_0(X, \mathsf{d})$ $(\boldsymbol{b}_n(f))_n$ is a d_{L^0}-Cauchy sequence in L^0, which is complete (lemma 2.3.8), therefore there exists $\boldsymbol{b}(f)$ such that

$$\boldsymbol{b}_n(f) \xrightarrow[n]{L^0} \boldsymbol{b}(f),$$

The Leibniz rule and locality property hold for $\boldsymbol{b}$ with $|\boldsymbol{b}_n| \xrightarrow[n]{L^0} |\boldsymbol{b}|$. Let us now discuss weak* continuity. Suppose $f_k \xrightarrow{k} f$ pointwise with uniformly bounded Lipschitz constants. We need to show that for all $g \in L^1(|\boldsymbol{b}|, \mathfrak{m})$

$$\int_X g\boldsymbol{b}(f_k)\mathrm{d}\mathfrak{m} \xrightarrow[k]{} \int_X g\boldsymbol{b}(f_k)\mathrm{d}\mathfrak{m}. \tag{3.2.2}$$

By assumption,

$$\int_X \inf\{|\boldsymbol{b} - \boldsymbol{b}_n|, 1\}\mathrm{d}\mathfrak{m}' \xrightarrow[n]{} 0.$$

Up to passing to a subsequence we can suppose $|\boldsymbol{b} - \boldsymbol{b}_n| \xrightarrow[n]{} 0$ $\mathfrak{m}$-almost everywhere, in particular $g|\boldsymbol{b}|\mathfrak{m}$-almost everywhere. Therefore there exists a Borel set E^ε such that

$$\int_{X \setminus E^\varepsilon} g|\boldsymbol{b}|\mathrm{d}\mathfrak{m} < \varepsilon$$

and

$$\begin{aligned} \limsup_k \left| \int_X g\boldsymbol{b}(f_k) - g\boldsymbol{b}(f)\mathrm{d}\mathfrak{m}' \right| &\leq \limsup_k \left| \int_{E^\varepsilon} g\boldsymbol{b}(f_k)\mathrm{d}\mathfrak{m}' - \int_{E^\varepsilon} g\boldsymbol{b}(f)\mathrm{d}\mathfrak{m}' \right| + 2\varepsilon \sup_k \mathrm{Lip}(f_k) \\ &\leq \limsup_k \left| \int_{E^\varepsilon} g\boldsymbol{b}_k(f_k) - g\boldsymbol{b}_k(f)\mathrm{d}\mathfrak{m}' \right| + 2\varepsilon \sup_k \mathrm{Lip}(f_k) + 2\varepsilon \int g|\boldsymbol{b}|\mathrm{d}\mathfrak{m}', \end{aligned}$$

which yields (3.2.2).

□

We will refer to TX as the *tangent module* to $(X, \mathsf{d}, \mathfrak{m})$, where derivations are to be thought as vector fields. The elements of the dual module, as constructed in 2.5.1, will play the role of 1-forms, i.e. sections of the cotangent bundle.

Moreover we will denote by $L^p(TX, \mathfrak{m})$ the space of derivations $\boldsymbol{b}$ such that $|\boldsymbol{b}| \in L^p(\mathfrak{m})$.

3.3 Equivalent locality conditions for derivations

According to definition to definition 3.2.1 a derivation $\boldsymbol{b} : \mathrm{Lip}_0(X, \mathsf{d}) \to L^0(\mathfrak{m})$ on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies $|\boldsymbol{b}(f)| \leq |\boldsymbol{b}| \mathrm{lip}_a(f)$ $\mathfrak{m}$-almost everywhere for all $f \in \mathrm{Lip}_0(X, \mathsf{d})$. We will now show that this in actually equivalent to a seemingly weaker condition,

$$|\boldsymbol{b}(f)| \leq |\boldsymbol{b}| \mathrm{lip}_a(f) \text{ a.e. } \forall f \in \mathrm{Lip}_0(X, \mathsf{d}) \iff |\boldsymbol{b}(f)| \leq |\boldsymbol{b}| \,\mathrm{Lip}(f) \text{ a.e. } \forall f \in \mathrm{Lip}_0(X, \mathsf{d}). \tag{3.3.1}$$

Sufficiency is clear, since $\mathrm{lip}_a(x) = \lim_{r \to 0} \mathrm{Lip}(f, B_r(x)) \leq \mathrm{Lip}(f)$.

Conversely, suppose $|\boldsymbol{b}(f)| \leq |\boldsymbol{b}| \,\mathrm{Lip}(f)$ a.e. $\forall f \in \mathrm{Lip}_0(X, \mathsf{d})$.

Step 1. If $f(x) = 0$ for all $x \in B_\varepsilon(x_0)$ then $\boldsymbol{b}(f)(x) = 0$ for $\mathfrak{m}$-almost everywhere in $B_\varepsilon(x_0)$. Let χ_ε be a Lipschitz function such that $\chi_\varepsilon > 0$ on $B_\varepsilon(x_0)$ and $\chi_\varepsilon = 0$ in $X \setminus \chi_\varepsilon$ (e.g. $\chi_\varepsilon(x) = [\varepsilon - \mathsf{d}(x, x_0)]^+$). Then $f\chi \equiv 0$, thus $\boldsymbol{b}(f\chi_\varepsilon) = 0$ almost everywhere. By the Leibniz rule we obtain that

$$0 = \boldsymbol{b}(f\chi_\varepsilon) = f\boldsymbol{b}(\chi_\varepsilon) + \chi_\varepsilon \boldsymbol{b}(f).$$

Then for almost every $x \in B_\varepsilon(x_0)$

$$\boldsymbol{b}(f)(x) \cdot \chi_\varepsilon(x) = -f(x) \cdot \boldsymbol{b}(\chi_\varepsilon)(x) = 0, \quad \chi_\varepsilon(x) > 0 \implies \boldsymbol{b}(f)(x) = 0.$$

Step 2. For all $x_0 \in X$ we have $\boldsymbol{b}(f)(x) \leq |\boldsymbol{b}|(x) \,\mathrm{Lip}(f, B_\varepsilon(x_0))$ almost everywhere in $B_\varepsilon(x_0)$.

Let $L = \mathrm{Lip}(f, B_\varepsilon(x_0))$ and consider the McShane extension

$$f_L^{B_\varepsilon(x_0)}(x) = \sup_{y \in B_\varepsilon(x_0)} \{f(y) - L\mathsf{d}(x, y)\}.$$

We have that

- if $x \in B_\varepsilon(x_0)$ then $f_L^{B_\varepsilon(x_0)} \geq f(x) - L\mathsf{d}(x, x) = f(x)$,
- for any $x \in X$ and $y \in B_\varepsilon(x_0)$

$$\begin{aligned} f(y) - L\mathsf{d}(x, y) - f(x) &\leq L\mathsf{d}(x, y) - L\mathsf{d}(x, y) = 0 \\ f(y) - L\mathsf{d}(x, y) &\leq f(x) \end{aligned}$$

Therefore $f_L^\varepsilon(x) = f(x)$ when $x \in B_\varepsilon(x_0)$. By claim 1 we know that $\boldsymbol{b}(f) = \boldsymbol{b}(f_L^\varepsilon)$ almost everywhere on $B_\varepsilon(x_0) \subset \{f = f_L^\varepsilon\}$, thus

$$|\boldsymbol{b}(f_L^\varepsilon)|(x) = |\boldsymbol{b}(f)|(x) \leq |\boldsymbol{b}|(x) \cdot L = |\boldsymbol{b}|(x) \cdot \mathrm{Lip}(f, B_\varepsilon(x_0)) \text{ for all } x \in B_\varepsilon(x_0).$$

Step 3. We now use the previous result to show that

$$\boldsymbol{b}(f)(x) \leq |\boldsymbol{b}|(x) \cdot \mathrm{Lip}(f, B_\varepsilon(x)) \quad \text{for almost every } x \in X. \tag{3.3.2}$$

Assume by contradiction that (3.3.2) does not hold. Then there exist $\bar{x} \in X$ and $0 < \tilde{\varepsilon} < \frac{\varepsilon}{2}$ such that (3.3.2) does not hold in a subset $E \subset B_{\tilde{\varepsilon}}(\bar{x})$ of positive measure. By applying step 2 to $\tilde{\varepsilon}$ we obtain that for $x \in E$ $B_{\tilde{\varepsilon}}(\bar{x}) \subset B_\varepsilon(x)$ and

$$\boldsymbol{b}(f)(x) \leq |\boldsymbol{b}|(x)\,\mathrm{Lip}(f, B_{\tilde{\varepsilon}}(\bar{x})) \leq |\boldsymbol{b}|(x)\,\mathrm{Lip}(f, B_\varepsilon(x))$$

and thus a contradiction.

Step 4. By letting $\varepsilon \downarrow 0$ in 3.3.2 we obtain that

$$\boldsymbol{b}(f) \leq |\boldsymbol{b}| \cdot \mathrm{lip}_a f \quad \mathfrak{m}\text{-almost everywhere}$$

and 3.3.1 is proved.

In a similar spirit we will also show that

$$\boldsymbol{b}(f) \leq |\boldsymbol{b}|(f)\mathrm{lip}_a f \quad \forall f \in \mathrm{Lip}_0(X, \mathsf{d}) \iff \boldsymbol{b}(f) \leq |\boldsymbol{b}||Df| \quad \forall f \in \mathrm{Lip}_0(X, \mathsf{d}), \tag{3.3.3}$$

and that therefore definition 3.2.1 and definition 4.3.8 have the same locality requirements.

As shown in 2.2.4, $|Df| \leq \mathrm{lip}_a f$. Necessity is then clear.

As for sufficiency, suppose $\boldsymbol{b}(f) \leq |\boldsymbol{b}|(f)\mathrm{lip}_a f$, let $\ell, r > 0$ and consider

$$K_r^\ell := \left\{ x \in X : \sup_{x \neq y \in B_r(x)} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)} \right\}.$$

Then define

$$K^\ell := \bigcup_{r > 0} K_r^\ell = \{x \in X : |Df|(x) < \ell\}.$$

To show that $\boldsymbol{b}(f) \leq |\boldsymbol{b}||Df|$ it is enough to prove that $\boldsymbol{b}(f) \leq |\boldsymbol{b}|\ell$ holds almost everywhere on K^ℓ or, equivalently, almost everywhere on K_r^ℓ for all $r > 0$.

Fix $\varepsilon > 0$. By inner regularity there exists a compact set $K \subset K_r^\ell$ such that $\mathfrak{m}'(K_r^\ell \backslash K) < \varepsilon$. Consider the open cover $K \subset \bigcup_{x \in K} B_{\frac{r}{2}}(x)$. Since K is compact there exists a finite collection $x_1, \ldots, x_N \in K$ such that $K \subset \bigcup_{i=1}^N B_{\frac{r}{2}}(x_i)$.

For $i = 1, \ldots, N$ consider

$$f_i(x) = \sup_{y \in B_{\frac{r}{2}}(x_i)} \{f(y) - \ell \mathsf{d}(x, y)\}.$$

Then

(i) f_i is ℓ-Lipschitz,

(ii) $f_i(x) \geq f(x)$ for $x \in B_{\frac{r}{2}}(x_i)$,

(iii) $f_i(x) \leq f(x)$ for $x \in B_{\frac{r}{2}}(x_i) \cap K$.

Claims (i) and (ii) are clear and can be shown as in section 4.

As for (iii), notice that if $y \in B_{\frac{r}{2}}(x_i)$ and $x \in B_{\frac{r}{2}}(x_i) \cap K$, then $y \in B_r(x)$ and $f(y) - f(x) \leq \ell \mathsf{d}(x, y)$. Therefore

$$f(y) - \ell \mathsf{d}(x, y) \leq f(x).$$

By taking the supremum over $y \in B_{\frac{r}{2}}(x_i)$ we obtain $f_i(x) \leq f(x)$.

3.4 Derivations with divergence

Definition 3.4.1. *Let $\boldsymbol{b} \in TX$ such that $|\boldsymbol{b}| \in L^1_{loc}(X)$; then we say that $\boldsymbol{b} \in D(\mathrm{div})$ if there exists a measure $\mu_{\boldsymbol{b}}$ (its divergence), finite on bounded sets, such that*

$$-\int_X \boldsymbol{b}(f)\, \mathrm{d}\mathfrak{m} = \int_X f\, \mathrm{d}\mu_{\boldsymbol{b}} \qquad \forall f \in \mathrm{Lip}_0(X, \mathsf{d}).$$

If $\mu_{\boldsymbol{b}} \ll \mathfrak{m}$ we will say that $\boldsymbol{b} \in \mathrm{Div}_a$ and denote $\mu_{\boldsymbol{b}} = \mathrm{div}\, \boldsymbol{b} \cdot \mathfrak{m}$; in particular if $\boldsymbol{b} \in \mathrm{Div}_a$ and $\mathrm{div}\, \boldsymbol{b} \in L^p(\mathfrak{m})$ we will say that $\boldsymbol{b} \in \mathrm{Div}_p$.

Lemma 3.4.2. *Let $u \in \mathrm{Lip}_0(X, \mathsf{d})$ and $\boldsymbol{b} \in L^{p_1}(\mathfrak{m})$ be a derivation with $|\boldsymbol{b}| \in L^{p_1}(\mathfrak{m}) \cap \mathrm{Div}_{\mathrm{p}_2}$. Then $u\boldsymbol{b} \in L^1(TX, \mathfrak{m}) \cap \mathrm{Div}_{\mathrm{p}_3}$ and*

$$\mathrm{div}(u\boldsymbol{b}) = u\, \mathrm{div}\, \boldsymbol{b} + \boldsymbol{b}(u),$$

where $p_3 = \max\{p_1, p_2\}$.

Proof. Let $f \in \mathrm{Lip}_0(X, \mathsf{d})$. By the Leibniz rule for derivations, $\boldsymbol{b}(fu) = u\boldsymbol{b}(f) + f\boldsymbol{b}(u)$. Then

$$\begin{aligned} -\int_X u\boldsymbol{b}(f)\mathrm{d}\mathfrak{m} &= -\int_X \boldsymbol{b}(fu)\mathrm{d}\mathfrak{m} + \int_X f\boldsymbol{b}(u)\mathrm{d}\mathfrak{m} \\ &= \int_X fu \cdot \underbrace{\mathrm{div}(\boldsymbol{b})\mathrm{d}\mathfrak{m}}_{\mathrm{d}\mu_{\boldsymbol{b}}} + \int_X f\boldsymbol{b}(u)\mathrm{d}\mathfrak{m} \\ &= \int_X f \cdot (u\, \mathrm{div}(\boldsymbol{b}) + \boldsymbol{b}(u))\mathrm{d}\mathfrak{m}. \end{aligned}$$

Since f is arbitrary we obtain that $\mathrm{div}(u\boldsymbol{b}) = u\, \mathrm{div}(\boldsymbol{b}) + \boldsymbol{b}(u)$. □

Definition 3.4.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $q \in [1, \infty]$. *We define* $T_q X$ *as the submodule of* TX *generated by* $L^q(TX, \mathfrak{m}) \cap \mathrm{Div_q}$, *i.e. the smallest submodule containing derivations with* q*-integrable pointwise norm and divergence. We then define* $T_q^* X$ *to be the dual module of* $T_q X$.

Lemma 3.4.4. *(Strong locality for weak-continuos derivations) Let* $\boldsymbol{b} \in TX$. *Then for every* $f \in \mathrm{Lip}_0(X, \mathsf{d})$

(i) $\boldsymbol{b}(f) = 0$ $\mathfrak{m}$*-almost everywhere in* $\{f = 0\}$,

(ii) $\boldsymbol{b}(f) \leq |\boldsymbol{b}| \mathrm{lip}_a(f|_C)$ $\mathfrak{m}$*-almost everywhere on every closed subset* C.

Corollary 3.4.5. *Let* $\boldsymbol{b} \in TX$. *Then for every* $f, g \in \mathrm{Lip}(X, \mathsf{d})$ $\boldsymbol{b}(f) = \boldsymbol{b}(g)$ *on* $\{f = g\}$.

Lemma 3.4.4. Let $\phi_\varepsilon(x) = (x - \varepsilon)_+ - (x + \varepsilon)_-$. Then ϕ_ε is 1-Lipschitz, $|\phi_\varepsilon(x) - x| \leq \varepsilon$ and $\phi_\varepsilon(x) = 0$ for $|x| \leq \varepsilon$. Given $f \in \mathrm{Lip}_0(X, \mathsf{d})(X\mathsf{d})$, define $f_\varepsilon := \phi_\varepsilon \circ f$. Then $\boldsymbol{b}(f_\varepsilon) \leq |\boldsymbol{b}| \mathrm{lip_a(f_\varepsilon)} \leq |\boldsymbol{b}| \mathrm{lip_a(f)}$. By assumption

$$|f_\varepsilon(x) - f(x)| = |\phi_\varepsilon(f(x)) - f(x)| \leq \varepsilon \quad \forall x \in X \implies f_\varepsilon \rightrightarrows f.$$

Moreover $\mathrm{Lip}(f_\varepsilon) \leq \mathrm{Lip}(f)$ are uniformly bounded. It follows that

$$\int_X g \boldsymbol{b}(f_\varepsilon) \mathrm{d}\mathfrak{m} \underset{\varepsilon}{\to} \int_X \boldsymbol{b}(f) \mathrm{d}\mathfrak{m}.$$

Notice that f_ε is ($\mathfrak{m}$-a.e.) constantly zero where $|f| < \varepsilon$, so that $\mathrm{lip}_a = 0$ a.e. in the same region. Let h be a positive function in $L^1(\mathfrak{m})$ and define $g = \mathbb{1}_{\{f=0\}} \mathrm{sgn}(\boldsymbol{b}(f))$. Observe that

$$\int_{\{f=0\}} h \cdot |\boldsymbol{b}(f)| \mathrm{d}\mathfrak{m} = \int_X h \cdot g \boldsymbol{b}(f) \mathrm{d}\mathfrak{m} = \lim_{\varepsilon \to 0} \int_X h \cdot g \boldsymbol{b}(f_\varepsilon) \mathrm{d}\mathfrak{m} = 0.$$

Therefore $h \cdot |\boldsymbol{b}(f)| = 0$ $\mathfrak{m}$-a.e. on $\{f = 0\}$. Since h was chosen positive a.e. we conclude $|\boldsymbol{b}(f)| = 0$ $\mathfrak{m}$-a.e. and (i) is proved.

To prove (ii) we use the McShane extension of f restricted to closed balls. For all $y \in X$ and $r > 0$ let us denote

$$g_y^r(x) := \sup\{f(x') - L\mathsf{d}(x, x') : x' \in C \cap \bar{B}_r(y)\}, \quad L = \mathrm{Lip}(f, C \cap \bar{B}_r(y)),$$

which extends $f|_{C \cap \bar{B}_r(y))}$ to X preserving its Lipschitz constant, i.e.

$$f = g_y^r \text{ on } C \cap \bar{B}_r(y)) \text{ and } \mathrm{Lip}(f, C \cap \bar{B}_r(y))) = \mathrm{Lip}(g_y^r, C \cap \bar{B}_r(y)).$$

By corollary 3.4.5 (which follows from (i)) $\boldsymbol{b}(f) = \boldsymbol{b}(g_y^r)$ $\mathfrak{m}$-a.e. on $C \cap \bar{B}_r(y))$. In particular, since $B_r(y) \subset B_{2r}(x)$ for $x \in B_r(y)$,

$$|\boldsymbol{b}(f)|(x) \leq |\boldsymbol{b}| \,\mathrm{Lip}(f|_C, \bar{B}_r(y)) \leq |\boldsymbol{b}| \,\mathrm{Lip}(f|_C, B_{2r}(x))) \xrightarrow[r \to 0]{} |\boldsymbol{b}| \mathrm{lip}_a(f|_C)(x) \quad \mathfrak{m}\text{-a.e. on } C \cap B_r(y).$$

Since y is arbitrary, the thesis follows.

□

3.5 A look at the smooth case

Let $X = (M, \mathfrak{g}, vol{\cdot}g)$ be a smooth manifold. We will give an analysis of the correspondence between vector fields in $L^0(TM)$, where TM is the classical tangent bundle of differential geometry, and our notion of TX (Weaver derivations) when X is seen as a metric measure space.

In the standard case, every point $x \in M$ comes with a tangent space T_xM and a measurable vector field $v \in L^0(TM)$ is a measurable function $v : M \to TM$ such that $v(x) \in T_xM$ for a.e. $x \in M$.

We will focus on the case $M = \mathbb{R}^n$ for simplicity, but a similar result holds for any smooth manifold. With this choice of M vector fields can act as directional derivatives. The following correspondence holds.

Theorem 3.5.1. *Let $M = \mathbb{R}$. In this case derivations can be identified as vector fields.*

(i) Given $v \in L^0(TM)$ there is a unique $\boldsymbol{b}_v \in TX$ such that $\boldsymbol{b}_v(f) = \frac{\partial}{\partial v} f$.

(ii) Given $\boldsymbol{b} \in TX$, can we find $v_{\boldsymbol{b}} \in L^0(TM)$ such that $\boldsymbol{b}(f) = \frac{\partial}{\partial v_{\boldsymbol{b}}} f$.

Proof. Let us address question (i) first. By Rademacher's theorem Lipschitz functions on $\mathbb{R}^n$ are Lebesgue-a.e. differentiable. Therefore, given $f \in \mathrm{Lip}_0(X, \mathsf{d})$ we can speak of its differential $\mathrm{d}f$ and define

$$\boldsymbol{b}_v(f) \coloneqq \frac{\partial}{\partial v} f = \langle v, \mathrm{d}f \rangle \quad \text{(defined a.e.)},$$

which is linear in f, satisfies the Leibniz rule and $|\boldsymbol{b}_v(f)| \le |v||\nabla f| \le |v|\mathrm{lip}_a(f)$. To see that weak* continuity also holds let $f_k \xrightarrow{k} f$ pointwise with $\sup_k \{\mathrm{Lip}(f_k)\} < \infty$).

It is enough to show that

$$\int \left(\frac{\partial f_k}{\partial x_i} \cdot h \right) \mathrm{d}x \xrightarrow{k} \int \left(\frac{\partial f}{\partial x_i} \cdot h \right) \mathrm{d}x \qquad \text{for all } \cdot h \in L^1(\mathbb{R}^n). \tag{3.5.1}$$

Indeed, if one assumes (3.5.1), then for all $h \in L^1(\mathbb{R}^n)$

$$\begin{array}{ccc} \int (\langle \mathrm{d}f_k, v\rangle \cdot h)\,\mathrm{d}x & = & \sum_{i=1}^n \int \left(\frac{\partial f_k}{\partial x_i} \cdot v_i \cdot h \right) \mathrm{d}x \\ \big\downarrow k & & \big\downarrow k \\ \int (\langle \mathrm{d}f, v\rangle \cdot h)\mathrm{d}x & & \sum_{i=1}^n \int \left(\frac{\partial f}{\partial x_i} \cdot v_i \cdot h \right) \mathrm{d}x = \int \frac{\partial f}{\partial v} \cdot h \mathrm{d}x \end{array}$$

Let us prove that 3.5.1 holds. If $h \in C_c^\infty(\mathbb{R}^n)$ then by integration by parts and dominated convergence

$$\int \left(\frac{\partial f_k}{\partial x_i} \cdot h \right) \mathrm{d}x = -\int f_k \frac{\partial h}{\partial x_i} \underset{k}{\to} -\int f \frac{\partial h}{\partial x_i} \mathrm{d}x = \int \frac{\partial f}{\partial x_i} h \mathrm{d}x$$

By density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ we conclude. Let $h \in L^1(\mathbb{R}^n)$ and $(h_\varepsilon) \subset C_c^\infty(\mathbb{R}^n)$ be such that $h_\varepsilon \xrightarrow{\varepsilon\to 0} h$. Then

$$\begin{aligned}
&\left|\int \left(\frac{\partial f_k}{\partial x_i}\cdot h\right)\mathrm{d}x - \int\left(\frac{\partial f}{\partial x_i}\cdot h\right)\mathrm{d}x\right| \\
&\le \left|\int \frac{\partial f_k}{\partial x_i}\cdot (h-h_\varepsilon)\mathrm{d}x\right| + \left|\int\left(\frac{\partial f_k}{\partial x_i} - \frac{\partial f}{\partial x_i}\right)\cdot h_\varepsilon \mathrm{d}x\right| + \left|\int \frac{\partial f}{\partial x_i}(h-h_\varepsilon)\mathrm{d}x\right| \\
&\le \left(\sup_m \{\mathrm{Lip}(f_m)\} + \mathrm{Lip}(f)\right)\|h-h_\varepsilon\|_{L^1(\mathbb{R}^n)} + \left|\int\left(\frac{\partial f_k}{\partial x_i} - \frac{\partial f}{\partial x_i}\right)\cdot h_\varepsilon \mathrm{d}x\right| \\
&\underset{k}{\longrightarrow} \left(\sup_m \{\mathrm{Lip}(f_m)\} + \mathrm{Lip}(f)\right)\|h-h_\varepsilon\|_{L^1(\mathbb{R}^n)} \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}$$

This shows that $L^0(TM) \ni v \longmapsto \boldsymbol{b}_v \in TX$ is well defined and gives a positive answer to question (i). It is not hard to see that this map preserves sums, multiplication by scalars and that it is injective.

Conversely, to answer question (ii) let $\boldsymbol{b} \in TX$ and let $f_i(x) = x_i$ be the i-th coordinate. We define

$$\mathbf{v}_b := \sum_{i=1}^n \boldsymbol{b}(f_i)e_i.$$

According to definition to definition 3.2.1 a derivation $\boldsymbol{b} : \mathrm{Lip}_0(X,\mathsf{d}) \to L^0(\mathfrak{m})$ on a metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies $|\boldsymbol{b}(f)| \le |\boldsymbol{b}|\mathrm{lip}_a(f)$ $\mathfrak{m}$-almost everywhere for all $f \in \mathrm{Lip}_0(X,\mathsf{d})$.

We would like to show that

$$\boldsymbol{b}(f) = \frac{\partial}{\partial v_b}f.$$

Since f is Lipschitz, almost everywhere we have $|\mathrm{d}f| = |Df| = |\nabla f|$.

Let $\bar{q} = (q_1,\dots,q_n) \in \mathbb{Q}^n$. Then

$$\begin{aligned}
\boldsymbol{b}(f) - \sum_{i=1}^n \boldsymbol{b}(x_i)q_i &= \boldsymbol{b}\left(f - \sum_{i=1}^n q_i x_i\right) \\
\left|\boldsymbol{b}(f) - \sum_{i=1}^n \boldsymbol{b}(x_i)q_i\right| &\le |\boldsymbol{b}|\left|D(f - \sum_{i=1}^n q_i x_i)\right| = |\boldsymbol{b}||\mathrm{d}f - \bar{q}|.
\end{aligned}$$

Therefore for all $\bar{q} \in \mathbb{Q}^n$

$$\begin{aligned}
\left|\boldsymbol{b}(f) - \frac{\partial}{\partial v_b}f\right| &\le \left|\boldsymbol{b}(f) - \sum_{i=1}^n \boldsymbol{b}(x_i)q_i\right| + \left|\sum_{i=1}^n \boldsymbol{b}(x_i)q_i - \sum_{i-=1}^n \boldsymbol{b}(x_i)\frac{\partial f}{\partial x_i}\right| \\
&\le |\boldsymbol{b}||\mathrm{d}f - \bar{q}| + \left|\sum_{i=1}^n \boldsymbol{b}(x_i)\left(q_i - \frac{\partial f}{\partial x_i}\right)\right| \le (1+\sqrt{n})|\boldsymbol{b}||\mathrm{d}f - \bar{q}|.
\end{aligned} \tag{3.5.2}$$

Hence for any $\bar{q} \in \mathbb{Q}^n$,

$$\left| \boldsymbol{b}(f) - \frac{\partial}{\partial v_b} f \right| \leq (1 + \sqrt{n}) |\boldsymbol{b}| |\mathrm{d}f - \bar{q}| \qquad \mathfrak{m}\text{-almost everywhere.} \tag{3.5.3}$$

Since $\mathbb{Q}^n$ is countable, $\mathfrak{m}$-almost everywhere the following holds

$$\left| \boldsymbol{b}(f) - \frac{\partial}{\partial v_b} f \right| \leq (1 + \sqrt{n}) |\boldsymbol{b}| |\mathrm{d}f - \bar{q}| \qquad \text{for all } \bar{q} \in \mathbb{Q}^n. \tag{3.5.4}$$

By taking the infimum over $\bar{q} \in \mathbb{Q}^n$ we obtain $\left| \boldsymbol{b}(f) - \frac{\partial}{\partial v_b} f \right| = 0$. □

3.6 Sobolev spaces via derivations

The goal of this section is to define a notion of Sobolev space $W^{1,p}$ over a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ which uses derivations.

Definition 3.6.1. *Let $f \in L^p(X, \mathfrak{m})$. then $f \in W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ if, setting $p = q/(q-1)$, there exists $df \in L^p(T_q^* X, \mathfrak{m})$ satisfying*

$$\int_X df \cdot \boldsymbol{b} \, d\mathfrak{m} = - \int_X f \operatorname{div} \boldsymbol{b} \, d\mathfrak{m} \qquad \textit{for all } \boldsymbol{b} \in L^q(TX, \mathfrak{m}) \cap \mathrm{Div}_q. \tag{3.6.1}$$

In analogy with the characterization of classical Sobolev spaces seen in 1.3.1 we give the following

Definition 3.6.2. *Let $f \in L^p(X, \mathfrak{m})$. Then, setting $p = q/(q-1)$, the following are equivalent:*

(i) There exists $df \in L^p(T_q^* X, \mathfrak{m})$ satisfying

$$\int_X df \cdot \boldsymbol{b} \, d\mathfrak{m} = - \int_X f \operatorname{div} \boldsymbol{b} \, d\mathfrak{m} \qquad \textit{for all } \boldsymbol{b} \in L^q(TX, \mathfrak{m}) \cap \mathrm{Div}_q. \tag{3.6.2}$$

(ii) There exists a linear map $L_f : L^q(TX, \mathfrak{m}) \cap \mathrm{Div}_q \to L^1(X, \mathfrak{m})$ which is Lip_b-linear, such that

$$\int_X L_f(\boldsymbol{b}) \, d\mathfrak{m} = - \int_X f \operatorname{div} \boldsymbol{b} \, d\mathfrak{m} \qquad \textit{for all } \boldsymbol{b} \in L^q(TX, \mathfrak{m}) \cap \mathrm{Div}_q. \tag{3.6.3}$$

(iii) Only for $p > 1$: there exists a constant $C_f < \infty$ such that

$$\left| \int_X f \operatorname{div} \boldsymbol{b} \, d\mathfrak{m} \right| \leq C_f \| |\boldsymbol{b}| \|_{L^q} \qquad \textit{for all } \boldsymbol{b} \in L^q(TX, \mathfrak{m}) \cap \mathrm{Div}_q. \tag{3.6.4}$$

If one of the above is satisfied we say the $f \in W^{1,p}(X, \mathsf{d}, \mathfrak{m})$, we say that df is the differential *of the function f. Moreover we have that the best C_f in Equation* (3.6.4) *is $\| |df|_* \|_{L^p}$.*

Remark 3.6.3. *In the above definition it is clear that (i) $\implies$ (ii) and (ii) $\implies$ (iii). To see that*

3.7 An example in $\mathbb{R}$

In this example we will consider the metric measure space $(X, \mathsf{d}, \mathfrak{m}) = (\mathbb{R}, |\cdot|, \rho \mathrm{d}x)$ for a function $\rho \in L^0(\mathbb{R})$. In this setting TX can be identified with $\mathbb{R}$ and T_qX is a linear subspace of TX, so the only options are $T_qX = 0$ and $T_qX = \mathbb{R}$.

Theorem. *Let E be the set of points around which ρ^{-1} is not $q-1$-integrable. Then $T_q\mathbb{R} = 0$ $\rho-$almost everywhere on E.*

Proof. Let us assume that ρ is bounded. We would like to show that if $v \in L^q(\rho)$ and $\mathrm{div}_\rho(v) \in L^\infty(\rho)$, then $v = 0$ on E. By assumption for every $\varphi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} \varphi' v \rho \mathrm{d}x = -\int \varphi \, \mathrm{div}_\rho(v) \rho \mathrm{d}x.$$

But then $v\rho$ is Sobolev in the classical sense on $\mathbb{R}$ and therefore it has a continuous representative. Suppose $v(x_0)\rho(x_0) \neq 0$ for some $x_0 \in E$, then in a neighbourhood of x_0 we have

$$|v(x)| \geq \frac{c}{\rho(x)} \implies v \notin L^q(\rho).$$

Since $\rho > 0$ ρ-almost everywhere on E we get $v(x) = 0$ ρ-almost everywhere on E. □

Question: Does there exist a function ρ which is almost everywhere positive and such that $T_qX = 0$ almost everywhere?

For instance, if $\rho(x) = \min\{|x|, 1\}$, $T_q\mathbb{R} = \begin{cases} \mathbb{R} & x \neq 0 \\ 0 & x = 0 \end{cases}$, but we would like to have this behaviour everywhere.

If ρ were not integrable over any interval of $\mathbb{R}$, then this behaviour ($T_x\mathbb{R} = 0$) would occur at (a.e.) point. We will give an explicit example of such a function in the case $p = 2$.

Let $(q_n)_n$ be an enumeration of the rational numbers and

$$\rho(x) = \frac{1}{1 + \sum_{n=0}^\infty \frac{1}{|x-q_n|} \mathbb{1}_{B(q_n, 2^{-n})}}.$$

Let $C_n := \mathbb{R} \setminus \bigcup_{k \geq n} B(q_k, 2^{-k})$ and consider

$$\bigcup_n C_n = \mathbb{R} \setminus \bigcap_n \bigcup_{k\geq n} B(q_k, 2^{-k}).$$

Observe that $|\bigcap_n \bigcup_{k\geq n} B(q_k), 2^{-k}| \leq \inf_k c2^{-k} = 0$ (for some $c > 0$) so that $\bigcup_n C_n$ has full measure.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ it is clear that

$$\rho^{-1}(x) = 1 + \sum_{n=0}^\infty \frac{1}{|x-q_n|} \mathbb{1}_{B(q_n, 2^{-n})}$$

is nowhere integrable.

Our goal now is to show that if $v \in L^2(\rho)$ and $\operatorname{div}_\rho(v) \in L^\infty(\rho)$, then $v = 0$.

Indeed, suppose $v\rho$ has a continuous nonzero representative around x_0. Then in a neighbourhood of x_0 we have

$$|v(x)| \geq \frac{c}{\rho(x)} \implies v \notin L^2(\rho).$$

As a consequence of this observation we have the following lemma.

Lemma. *Let $f \in L^2(\rho)$. Then there exists a sequence $(f_n) \subset C_c^\infty(\mathbb{R})$ such that*

1. *$f_n \to f$ in $L^2(\rho)$*
2. *$\nabla f_n \to \nabla f = 0$ in $L^2(\rho)$.*

Chapter 4

Other definitions of Sobolev spaces

In this chapter we give a brief overview of other approaches to the definition of Sobolev spaces over metric measure spaces that can be found in literature, see [7]. Thanks to what we have showed in section we obtain that these notions of Sobolev spaces are equivalent to the one described in section 3.6.

4.1 Cheeger energy and minimal relaxed slope

Definition 4.1.1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and fix* $p > 1$. *The* Cheeger energy *is the convex and lower semicontinuous functional* $\mathrm{Ch} : L^p(X, \mathfrak{m}) \to [0, +\infty]$ *defined as*

$$\mathrm{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X \mathrm{lip}_a^p(f_n) : f_n \in \mathrm{Lip}_0(X, \mathsf{d}) \cap L^p(X, \mathfrak{m}), f_n \xrightarrow[L^p(X,\mathfrak{m})]{n} f \right\}.$$

The Sobolev space $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ *is then defined as the finiteness domain of* Ch.

Recall the definition of slope for a Lipschitz function $g : X \to \mathbb{R}$ seen in 2.2.3:

$$|Dg|(x) = \limsup_{y\to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)}.$$

Definition 4.1.2. *(Relaxed gradients) Given* $f \in L^p(X, \mathfrak{m})$ *we say* $G \in L^p(X, \mathfrak{m})$ *is* a *relaxed gradient of* f *if there exists a sequence of Borel* d*-Lipschitz functions such that*

$f_n \to f$ *in* $L^p(X, \mathfrak{m})$ *and the asymptotic Lipschitz constants* $\mathrm{lip}_a f_n$ *converge weakly to a function* $\tilde{G}$,

- $\tilde{G} \leq G$ $\mathfrak{m}$*-almost everywhere in* X.

We say that G *is the minimal relaxed gradient of* f *if its* $L^p(X, \mathfrak{m})$ *norm is minimal among relaxed gradients. We shall denote by* $|\mathrm{d}f|_*$ *the minimal relaxed gradient.*

The main result we want to achieve is the following.

Theorem 4.1.3. *For all $f \in W^{1,p}(X)$ one has*

$$\mathrm{Ch}(f) = \int_X |\mathrm{d}f|_*^p \mathrm{d}\mathfrak{m}$$

and there exists $f_n \in \mathrm{Lip}_0(X, \mathsf{d}) \cap L^p(X, \mathfrak{m})$ with $f_n \to f$ in $L^p(X, \mathfrak{m})$ and $\mathrm{lip_a f_n} \to |\mathrm{Df}|$ in $L^p(X, \mathfrak{m})$. In particular, if $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ is reflexive, there exists $f_n \in \mathrm{Lip}_0(X, \mathsf{d}) \cap L^p(X, \mathfrak{m})$ satisfying $f_n \to f$ in $L^p(X, \mathfrak{m})$ and $|D(f_n - f)| \to 0$ in $L^p(X, \mathfrak{m})$.

Lemma 4.1.4. *(i) If $G \in L^p(X, \mathfrak{m})$ is a relaxed gradient of $f \in L^p(X, \mathfrak{m})$, then there exist Borel d-Lipschitz functions f_n converging to f in $L^p(X, \mathfrak{m})$ and $g_n \in L^p(X, \mathfrak{m})$ strongly convergent to $\tilde{G}$ in $L^p(X, m)$ with $|Df_n| \leq G_n$ and $\tilde{G} \leq G$.*

(ii) If $G_n \in L^p(X, \mathfrak{m})$ is a relaxed gradient of $f_n \in L^p(X, \mathfrak{m})$ and $f_n \rightharpoonup f$, $g_n \rightharpoonup g$ weakly in $L^p(X, \mathfrak{m})$, then g is a relaxed gradient of f.

(iii) In particular, the collection of all the relaxed gradients of f is closed in $L^p(X, \mathfrak{m})$ and there exist bounded Borel d-Lipschitz functions $f_n \in L^p(X, \mathfrak{m})$ such that

$$f_n \to f, \quad |Df_n| \to |Df|_* \qquad \text{strongly in } L^p(X, \mathfrak{m}).$$

Remark 4.1.5. *Observe that (ii) is in analogy with 1.2.7 in the classical theory of Sobolev spaces.*

Proof. (i) Since g is a relaxed gradient, we can find Borel d-Lipschitz functions $g_i \in L^p(X, \mathfrak{m})$ such that $g_i \to f$ in $L^p(X, \mathfrak{m})$ and $|Dg_i|$ weakly converges to $\tilde{g} \leq g$ in $L^p(X, \mathfrak{m})$; by Mazur's lemma we can find a sequence of convex combinations g_n of $|Dg_i|$, strongly convergent to $\tilde{g}$ in $L^p(X, \mathfrak{m})$; the corresponding convex combinations of g_i, that we shall denote by f_n, still converge in $L^p(X, \mathfrak{m})$ to f and $|Df_n|$ is bounded from above by g_n.

(ii) We will show that the set

$$S \coloneqq \{(f, G) \in L^p(X, \mathfrak{m}) \times L^p(X, \mathfrak{m}) : G \text{ is a relaxed gradient of } f\}$$

is weakly closed in $L^p(X, \mathfrak{m}) \times L^p(X, \mathfrak{m})$. Since S is convex, it is sufficient to prove that it is strongly closed. Consider a sequence $(f^i, G^i) \in S$ and suppose $(f^i, G^i) \to (f, G)$ in $L^p(X, \mathfrak{m}) \times L^p(X, \mathfrak{m})$. For each (f^i, G^i) we can find a sequence of Borel Lipschitz functions (f^i_n) and of nonnegative $L^p(X, \mathfrak{m})$ functions (G^i_n) such that $|Df^i_n| \rightharpoonup_n \tilde{G}^i$, $\tilde{G}^i \leq G^i$ and

$$f^i_n \xrightarrow{n} f^i, \quad G^i_n \xrightarrow{n} \text{ strongly in } L^p(X, \mathfrak{m}).$$

Up to taking a subsequence we can assume $\tilde{G}^i \rightharpoonup \tilde{G}$ weakly in $L^p(X, \mathfrak{m})$. By a diagonal argument we can find $f^i_{n(i)} \to f$, $G^i_{n(i)} \rightharpoonup \tilde{G}$ in $L^p(X, \mathfrak{m})$ and such that $|Df^i_{n(i)}|$ is bounded in $L^p(X, \mathfrak{m})$. Since $L^p(X, \mathfrak{m})$ is reflexive, $|Df^i_{n(i)}|$ has a subsequence that converges weakly, so it is not restrictive to suppose $|Df^i_{n(i)}| \rightharpoonup H$. It follows that $H \leq \tilde{G} \leq G$ and G is a relaxed gradient for f.

(*iii*) Now consider the minimal relaxed gradient $G := |Df|_*$ and let f_n, G_n be sequences in $L^p(X, \mathfrak{m})$ as in (*i*). Again, since $|Df_n|$ is uniformly bounded in $L^p(X, \mathfrak{m})$ it is not restrictive to assume that it is weakly convergent to some limit $H \in L^p(X, \mathfrak{m})$ with $0 \leq H \leq \tilde{G} \leq G$. Since G is minimal this implies that $H = \tilde{G} = G$ and $|Df_n n|$ weakly converges to $|Df|_*$ (because any limit point in the weak topology of $|Df_n|$ is a relaxed gradient with minimal norm) and that the convergence is strong, since

$$\limsup_n \int |Df_n|^p \mathrm{d}\mathfrak{m} \leq \limsup_n \int G_n^p \mathrm{d}\mathfrak{m} = \int G^p \mathrm{d}\mathfrak{m} = \int H^p \mathrm{d}\mathfrak{m}.$$

Finally, replacing f_n by suitable truncations $\tilde{f_n}$, made in such a way that $\tilde{f} \to f$ in $L^p(X, \mathfrak{m})$, we can achieve the boundedness property retaining the strong convergence of $|D\tilde{f_n}|$ to $|Df|_*$, since $|D\tilde{f_n}| \leq |Df|$ and any weak limit point of $|D\tilde{f_n}|$ is a relaxed gradient.
□

Remark 4.1.6. *The minimal relaxed gradient satisfies a Leibniz inequality: if $f, g \in L^p(X, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ have relaxed gradients, then fg has a relaxed gradient and*

$$|D(fg)|_* \leq |f||D(g)|_* + |g||D(f)|_*,$$

as a consequence of the properties of the asymptotic Lipschitz constant (2.2.1).

Lemma 4.1.7. *Let G_1, G_2 be relaxed gradients of f. Then* $\min\{G_1, G_2\}$ *and* $\mathbb{1}_B G_1 + \mathbb{1}_{X \setminus B} G_2, B \in B(X)$*, are relaxed gradients of f as well. In particular, for any relaxed gradient G of f it holds*

$$|Df|_* \leq G \quad \mathfrak{m} - a.e. \text{ in } X.$$

Theorem 4.1.8. *Cheeger's functional*

$$\mathrm{Ch}(f) = \int_X |Df|_*^p \mathrm{d}\mathfrak{m},$$

(set equal to $+\infty$ if f has no relaxed slope), is convex and lower semicontinuous in $L^p(X, m)$.

Proof. Recall $|\mathrm{lip_a}(\alpha f + \beta g)| \leq |\alpha||\mathrm{lip_a} f| + |\beta||\mathrm{lip_a} g|$. If F is a relaxed gradient for F and G is a relaxed gradient for g, then $\alpha F + \beta G$ is a relaxed gradient for $\alpha f + \beta g$ when α and β are nonnegative. Taking the minimal relaxed gradients $F = |Df|_*$ and $G = |Dg|_*$ yields

$$|D(\alpha f + \beta g)|_* \leq |\alpha||Df|_* + |\beta||Dg|_* \quad \text{for every } \alpha, \beta \in \mathbb{R}.$$

This proves the convexity. □

4.2 Weak upper gradient

In [7] the weak upper gradient is defined as follows. The Sobolev class $S^p_{loc}(X)$ is introduced as the set of measurable functions f which admit a weak upper gradient which is $L^p_{loc}(\mathfrak{m})$ and $|Df|_w$ is chosen to be the minimal weak upper gradient in the $\mathfrak{m}$-almost everywhere sense. It can be shown that this notion of gradient is local and satisfies the chain rule and the Leibniz rule. The Sobolev class $S^p(X)$ is then defined as the space of functions in $S^2_{loc}(X)$ whose minimal weak upper gradient $|Df|_w$ is in $L^p(X)$.

More precisely, $S^2(X)$ is defined as follows.

Definition 4.2.1. *A measure $\pi \in \mathscr{P}(C([0,1],X))$ is said to be a q-test plan on X if it satisfies the following properties:*

(1) there exists a constant $C > 0$ such that $(e_t)_\#\pi \le C\mathfrak{m}$ for every $t \in [0,1]$,

(2) it holds that $\iint_0^1 |\dot\gamma|^q \mathrm{d}\pi(\gamma)\mathrm{d}t < \infty$.

Definition 4.2.2. *The Sobolev class $S^p_{loc}(X)$ is defined as the space of functions $f \in L^0(\mathfrak{m})$ for which there exists a function $G \in L^p_{loc}(\mathfrak{m})$ with $G \ge 0$ such that*

$$\int |f(\gamma_1) - f(\gamma_0)|\mathrm{d}\pi(\gamma) \le \iint_0^1 G(\gamma_t)|\dot\gamma_t|\mathrm{d}t\ \mathrm{d}\pi(\gamma) \quad \textit{for every q-test plan } \pi \textit{ on } X.$$

Sobolev spaces are then defined as $W^{1,p}(X) := L^p(\mathfrak{m}) \cap S^p(X)$ with the norm given by

$$\|f\|_{W^{1,p}(X)} = \left(\|f\|^p_{L^p(\mathfrak{m})} + \||Df|_w\|^p_{L^p(\mathfrak{m})}\right)^{1/p}, \tag{4.2.1}$$

it is shown in **??** that $(W^{1,p}(X), \|\cdot\|_{W^{1,p}(X)})$ is a Banach space.

4.3 Construction of the cotangent module

Theorem 4.3.1. *There exists a unique pair $(L^0(T^*X), \mathrm{d})$ where $L^0(T^*X)$ is a L^0-module and $\mathrm{d} : S^2(X) \to L^0(T^*X)$ is linear and such that*

1. $|\mathrm{d}f| = |Df|$ $\mathfrak{m}$-almost everywhere and for every $f \in S^2(X)$,

2. $L^0(T^*X)$ is generated by $\mathrm{d}f : f \in S^2_{loc}(X)$.

Remark 4.3.2. *Uniqueness is intended up to existence of a unique isomorphism.*

Definition 4.3.3. *The module $L^0(T^*X)$ is called the $L^0(\mathfrak{m})$-cotangent module of X and the map d the differential.*

We will show the construction of the cotangent module with this definition, i.e. the existence part of Theorem 4.3.1.

Proof. (Theorem 4.3.1)

(Uniqueness) Suppose $(L^0(T^*X), \mathrm{d})$ and $(\mathscr{M}, \mathrm{d}')$ verify the conditions in 4.3.1. In particular for all f, g and $E \subset X$ Borel

$$\mathrm{d}f = \mathrm{d}g \mathfrak{m}\text{-a.e. on } E \iff |D(f-g)| = 0\text{-a.e. on } E \iff \mathrm{d}'f = \mathrm{d}'g\text{-a.e. on } E.$$

Our goal is to show that $\Phi(\mathrm{d}f) = \mathrm{d}'f$ defines a well posed map which is an isomorphism between $(L^0(T^*X), \mathrm{d})$ and $(\mathscr{M}, \mathrm{d}')$. By L^0-linearity it must hold

$$\Phi\left(\sum_{i=1}^n \chi_{E_i} \mathrm{d}f_i\right) = \sum_{i=1}^n \chi_{E_i} \mathrm{d}' f_i$$

for every simple function $\sum_{i=1}^n \chi_{E_i} \mathrm{d}f_i : X \to \mathrm{d}[S^2_{loc}(X)]$. Moreover, pointwise norms are preserved, since

$$\left|\sum_{i=1}^n \chi_{E_i} \mathrm{d}f_i\right| = \sum_{i=1}^n \chi_{E_i} |\mathrm{d}f_i| = \sum_{i=1}^n \chi_{E_i} |Df_i| = \sum_{i=1}^n \chi_{E_i} |\mathrm{d}f_i| = \sum_{i=1}^n \chi_{E_i} |\mathrm{d}'f_i| = \left|\sum_{i=1}^n \chi_{E_i} \mathrm{d}'f_i\right|.$$

Since $\mathrm{d}f : f \in S^2_{loc}(X)$ generates $L^0(T^*X)$ (i.e. simple functions $\sum_{i=1}^n \chi_{E_i} \mathrm{d}f_i : X \to \mathrm{d}[S^2_{loc}(X)]$ are dense in $L^0(T^*X)$) we can uniquely extend Φ to a linear, continuous isometry $\Phi : L^0(T^*X) \to \mathscr{M}$. It is also surjective since simple functions $|\sum_{i=1}^n \chi_{E_i} \mathrm{d}'f_i|$ are dense in $\mathscr{M}$. L^0-linearity follows from the definition over simple functions, every measurable function can be approximated by simple functions.

By construction this is the unique isomorphism between $(L^0(T^*X), \mathrm{d})$ and $(\mathscr{M}, \mathrm{d}')$, thus 4.3.2 holds.

Existence We define the "Pre-Cotangent Module" as the set of finite measurable partitions of X, where each component carries a function of class S^2:

$$\mathrm{Pcm} \coloneqq \{(E_i, f_i)_{i=1,\dots,n} : (E_i)_i \text{ is a Borel partition of } X, f_i \in S^2_{loc}(X)\}.$$

We introduce an equivalence relation on Pcm by declaring $(E_i, f_i)_i \sim (F_j, g_j)_j$ whenever $|D(fi - gj)| = 0$ $\mathfrak{m}$-a.e.in $E_i \cap F_j$ forevery i, j. We denote by $[E_i, f_i]_i \in \mathrm{Pcm}/\sim$ the equivalence class of $(E_i, f_i)_i \in \mathrm{Pcm}$. We can endow the quotient with a structure of vector space by defining a sum and a scalar multiplication naturally by restricting to intersections:

$$\lambda[E_i, f_i]_i + \mu[F_j, g_j]_j \coloneqq [E_i \cap F_j, \lambda f_i + \mu g_j]_{g,j}.$$

Moreover we can define a multiplication by simple functions (measurable functions attaining finitely many values) as $\cdot : S(X) \times \mathrm{Pcm}/\sim \to \mathrm{Pcm}/\sim$, where

$$\left(\sum_{j=1}^n \alpha_j \chi_j\right) \cdot [E_i, f_i]_i \coloneqq [E_i \cap F_j, \alpha_j f_i]_{i,j} \qquad \text{for all } [E_i, f_i]_i \in \mathrm{Pcm}/\sim, \sum_{j=1}^n \alpha_j \chi_j \in S(X).$$

Finally, we endow $\mathrm{Pcm}/\sim$ with a pointwise norm $|\cdot| : \mathrm{Pcm}/\sim \to L^0(\mathfrak{m})$ by setting

$$|[E_i, f_i]_i| \coloneqq \sum_{i=1}^{n} \chi_{E_i}|Df_i| \qquad \text{for all } [E_i, f_i]_i \in \mathrm{Pcm}/\sim$$

and a distance

$$\mathsf{d}_{\mathrm{Pcm}/\sim}([E_i, f_i]_i, [F_j, g_j]_j) \coloneqq \sum_{i,j} \int_{E_i \cap F_j} \inf\{D(f_i - g_j), 1\} \mathrm{d}\mathfrak{m}',$$

where $\mathfrak{m}' \in \mathscr{P}(X)$ is such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$.

The completion of $\mathrm{Pcm}/\sim$ with respect to $\mathsf{d}_{\mathrm{Pcm}/\sim}$ is our candidate for $L^0(T^*X)$. The pointwise norm and the product by simple functions are Cauchy continuous with respect to $\mathsf{d}_{\mathrm{Pcm}/\sim}$ (2.4.3) and can therefore be extended to maps

$$|\cdot| : L^0(T^*X) \to L^0(\mathfrak{m}), \qquad \cdot : L^0(\mathfrak{m}) \times L^0(T^*X) \to L^0(T^*X).$$

This endows $L^0(T^*X)$ with the structure of an L^0-normed L^0-module. The differential operator is defined as $\mathrm{d} : S^2_{loc}(X) \to L^0(T^*X)$ as

$$\mathrm{d}f \coloneqq [X, f] \in \mathrm{Pcm}/\sim\, \subset L^0(T^*X),$$

where X is the trivial partition.

The definitions above ensure that d is linear,

$$\mathrm{d}(\alpha f + \beta g) = [X, \alpha f + \beta g] = [X \cap X, \alpha f + \beta g] = \alpha[X, f] + \beta[X, g] = \alpha \mathrm{d}f + \beta]dg.$$

By the definition of pointwise norm in $\mathrm{Pcm}/\sim$, $|\mathrm{d}f| = |[X, f]| = |Df|$ holds almost everywhere for functions in $S^2_{loc}(X)$ and, by extension, on $L^0(T^*X)$. To conclude, $\mathrm{d}[S^2_{loc}]$ coincides exactly with $\mathrm{Pcm}/\sim$ and is thus dense in $L^0(T^*X)$ by construction. □

Proposition 4.3.4. *Let $(f_n)_n \subset S^2_{loc}(X)$ be a sequence mm-a.e. converging to a some function $f \in L^0(\mathfrak{m})$. Assume that $(\mathrm{d}f_n)_n$ converges to some $\omega \in L^0(T^*X)$ in $L^2l_{loc}(\mathfrak{m})$, i.e. for every $B \subseteq X$ bounded*

$$\lim_n \int_B |\mathrm{d}f_n - \omega|^2 \mathrm{d}\mathfrak{m} = 0. \tag{4.3.1}$$

Then $f \in S^2_{loc}(X)$ and $\mathrm{d}f = \omega$.

Remark 4.3.5. *Again, notice the similarity with 1.2.7.*

Observe that the differential can be restricted to $W^{1,2}(X)$ as defined in (4.2.1) and the restriction is continuous in the sense that if $f_n \to f$ in $W^{1,2}(X)$ then $\mathsf{d}_{L^0(T^*X)}(\mathrm{d}f_n, \mathrm{d}f) \xrightarrow{n} 0$. Actually, a stronger fact holds.

Lemma 4.3.6. *The cotangent module $L^0(T^*X)$ is generated by*

$$\mathrm{d}W^{1,2} \coloneqq \{\mathrm{d}f : d \in W^{1,2}(X)\}.$$

Proof. By construction of the cotangent module, it is enough to show that $\mathrm{d}f \in G(\mathrm{d}W^{1,2})$ for every $f \in S^2_{loc}(X)$. $\square$

After constructing the cotangent module as described, the tangent module is defined as follows.

Definition 4.3.7. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. The tangent module $L^0(TX)$ is defined as the dual as module of $L^0(T^*X)$. Its elements are called vector fields.*

In this setting, derivations are defined as follows.

Definition 4.3.8. *A linear map $L : S^2_{loc} \to L^0(\mathfrak{m})$ is an $L^0(\mathfrak{m})$ derivation if there exists $g \in l^0(\mathfrak{m})$ such that*

$$|L(f) \leq g|Df| \qquad \text{for all } f \in S^2_{loc}(X). \tag{4.3.2}$$

Recall that in the construction we showed in Chapter 2, the space of derivations was taken as the definition of the tangent module. There is a natural identification between these two notions of tangent module which connects derivations and vector fields.

Theorem 4.3.9. *For any vector field $X \in L^0(TX)$ the map $X \circ \mathrm{d} : S^2_{loc}(X) \to L^0(\mathfrak{m})$ is a derivation. Conversely, given a derivation L there exists a unique vector field X in $L^0(TX)$ such that the diagram*

$$\begin{array}{ccc} S^2_{loc}(X) & \xrightarrow{\mathrm{d}} & L^0(T^*X) \\ & \searrow^{L} & \downarrow X \\ & & L^0(\mathfrak{m}) \end{array}$$

commutes. Moreover, $|X|$ is the minimal g that satisfies condition (4.3.2).

The differences between the derivations considered here and those defined in chapter 2 3.2.1are the following:

- in 3.2.1 a derivation $\boldsymbol{b} : \mathrm{Lip}_0(X, \mathsf{d}) \to L^0(\mathfrak{m})$ takes Lipschitz functions as opposed to Sobolev class functions,
- the Leibniz rule is not explicitly stated in definition 4.3.8. However, as we have seen, both the chain rule and Leibniz rule follow from locality.

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Ringraziamenti

Ringrazio il moi relatore, Simone Di Marino, per la sua guida e il suo ascolto. Ringrazio Emanuela per avermi aiutato a orientarmi in un momento di smarrimento. A tutti i miei compagni di studi va un grande ringraziamento per aver reso questi anni belli non solo per la matematica. Ho la fortuna di condividere questa passione con tutta la mia famiglia. A loro va il ringraziamento più grande per aver sempre sostenuto il mio percorso, stimolato la mia curiosità ed essere stati fonte di grande ispirazione.