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The geometries dual to heavy states of maximally supersymmetric Yang-Mills theory

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Contents

Introduction

The two biggest problems of theoretical physics nowadays concerns the research for a theory of quantum gravity and the study of strong coupling field theories.

The first problem originates in the late 1970 when the Standard Model of particle physics was decisively confirmed by experiments. The key theoretical concept underlying this theory was gauge symmetry, i.e. the idea according to which the symmetry transformations act independently at each point of spacetime. In the Standard Model three of the four fundamental forces of nature are described; the fourth force, gravity, is the weakest one and does not fit into the same quantum description. The incompatibility of gravity with the other three forces of nature comes from a theoretical problem concerning the theory of general relativity, which actually provides our deepest understanding of gravitational physics at the classical level. One of the biggest problem about this theory regard its apparent incompatibility with quantum mechanics: when one tries to include quantum corrections in general relativity one obtains naively divergent answers, and these divergences cannot be cured with standard renormalisation techniques. In brief, the quantisation of general relativity unfortunately does not lead to a renormalisable theory. It is generally believed that the correct, fundamental description of all physical fields should undergo the general framework of quantum mechanics. Gravitational physics should make no exception and thus we have to look for a theory of quantum gravity that reduces to general relativity in the infrared. The quantum gravity effects naively show up only at the Plank length, which is a scale defined exclusively in terms of three universal physical constants $c, \hbar$ and the gravitational constant G_N

$$l_P = \sqrt{\frac{\hbar G_N}{c^3}} \sim 10^{-33}\text{cm}.$$

The smallness of l_P (at the present time we are able to probe physics up to a scale of 10^{-17} cm) is related to the weakness of the gravitational force, which is about 40 orders of magnitude weaker than the electromagnetic force. The fact that l_P is so remote makes it difficult to gain any experimental evidence of quantum gravity. Of course one may ask why then it can be useful to study quantum gravity; many of the answers for this question are contained in the theory of general relativity and in the holographic principle which is the subject of this thesis. In general relativity there are a lot of important solutions that have singularities, namely regions of infinite curvature. The most famous example is the Schwarzschild black hole ($G_N = 1$)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2,$$

where M is the mass of the black hole and $d\Omega^2$ is the differential angular element. This geometry has a real singularity in $r = 0$, which is not related to a possible choice of inappropriate coordinates, and an in-falling observer can reach that point in a finite proper time. Even more surprisingly at the Schwarzschild radius $r = 2M$ the coordinates break down but the underlying spacetime manifold remains perfectly smooth as is clear in other coordinates. This means that crossing the Schwarzschild radius from the exterior region is an irreversible process or, in other words, there is a one-way membrane that casually divides spacetime into the external universe and the black hole interior; this is the black

hole event horizon. One might not be very troubled that singular solutions to Einstein's equations exist but recent observations have shown that such objects are actually present in nature and therefore we cannot simply turn a blind eye to the singularities that arise in general relativity. Another very important singularity that we only mention and that quantum gravity could explained is at the original Big Bang event in the Friedmann-Robertson-Walker metric which best describes our universe on large scales.
In order to find a quantum theory of gravity over time many different solutions have been proposed. It has been realised that a natural way to avoid the pathologies between quantum mechanics and general relativity is to think of point particle as a non fundamental but derived concept. The simple way of doing this is to consider extended one dimensional objects, strings, as the fundamental ones. The dynamics of quantum relativistic strings contains pointlike particles as oscillations modes of the strings among which is the graviton: in this sense, string theory is a theory of quantum gravity. The propagation of bosonic strings is free of quantum anomalies only in 26 spacetime dimensions. In order to include fermions in the string spectrum, one can introduce into the theory the supersymmetry which is an enlargement of the symmetry group of spacetime obtained by including spinorial generators in the Poincaré algebra. Supersymmetric string theories are well defined "only" in 10 dimensions and the four dimensional physics is usually recovered through reduction over a compact six dimensional manifold.
The second of the problems with which we opened the introduction is about the study of strong coupling field theories. In general we are able to solve field theories exactly only when they are free theories. However, when we introduce interactions the only way to find an analytical solution is to use the perturbation theory which relies on expansions in the interaction coupling constant. Some interactions, like strong interactions described by QCD, are asymptotically free, meaning that they do not have a small coupling parameter at large distances (or low energy) and therefore for these theories in general one cannot perform a perturbative expansion. One possible approach is to use numerical simulations on the lattice which, at the present time, is the best available tool to do calculations in QCD at low energies.

A connection between the two introductory problems and, in particular, between string theories and strong coupling field theories was first discovered by Gerard 't Hooft in 1973 [1]. In order to study QCD with the canonical perturbation theory tools, 't Hooft pointed out that the gauge theories based on the group $G = SU(N)$ simplifies in the limit $N \to \infty$ and to recover the physics one could perform a $1/N$ expansion down to $N = 3$ in the case of QCD. The simplification rely on the fact that among all the possible Feynman diagrams, a subset dominate in this limit. In particular the dominant diagrams are those which can be drawn flat on a plane, which are referred to as planar diagrams. This new diagrammatic expansion of the field theory based on the topology of the Riemann surface on which the diagram lies, suggests that the large N theory is indeed a string theory for the following reason. Our best description of string theory comes from the worldsheet formalism of a single string propagating through a fixed background. This is analogous to the quantum mechanical description of a single particle, however the worldsheet description contains a lot more physics. Because the string worldsheet can have many different topologies and excitations, we can actually describe interactions between many different string states studying how the topology of the worldsheet changes. String theories are thus defined by a sum over topologies, with the sphere (or plain) giving the dominant contribution at weak coupling, in analogy with what happens in the large N limit in the $SU(N)$ field theories. However, a deeper connection between the two theories was found by Maldacena many years later.

The connection between gauge theories and string theories was made explicit by the so called gauge/gravity duality. Gauge/gravity duality, as first realized by the AdS/CFT correspondence of Maldacena in the late 1997 [2], is a very special holographic duality that relates a string theory on spacetimes that asymptote $AdS_n \times X$, where AdS_n stands for Anti de Sitter space in n dimension (which is the maximally symmetric space with negative curvature) and X is a generic compact manifold, with a $n-1$ dimensional conformal field theory (CFT), i.e. a quantum field theory with the conformal symmetry, defined on the conformal boundary of AdS_n. The correspondence is a realisation of the holographic principle because the two theories live in different dimensions and it is a duality in the sense that everything in one theory corresponds to something else in the other according to the holographic dictionary. Despite not yet having a formal demonstration, the correspondence is one of the greatest achievements of string theory since it provides new methods to investigate strongly coupled systems. In fact, one crucial aspect of the correspondence is that the perturbative regimes in the two dual theories are perfectly incompatible in the sense that the limit in which one becomes hard to compute is the limit in which the other simplifies, therefore in general it gives to us the possibility of computing observables in a strongly coupled field theory using a classical gravitational theory. For this reason holography became one of the most important discoveries of high-energy theoretical physics in recent years. The fact that the field theory lives in a lower dimensional space blends in perfectly with some previous speculation about quantum gravity. In fact this "holographic'' principle comes from thinking about the Bekenstein bound, which states that the maximum amount of entropy in some region is given by the area of the region in Planck units instead of its volume; the reason for this bound is that otherwise black holes formation could violate the second law of thermodynamics. Furthermore, the fact that string theory blends in naturally with holography is one of the most valid reasons that drives us to study it regardless the problem about quantum gravity. This theory was fundamental in the development of the AdS/CFT correspondence because, besides closed and open strings, it contains other multi-dimensional extended objects, called D-branes, which have two dual descriptions. From the point of view of the open strings, D-branes are D-dimensional objects on which their end-points are confined to move. The oscillations of the ends of the open strings give rise to gauge fields and it is in fact possible to construct a CFT on the worldvolume of the D-branes. On the other hand, from the point of view of the closed strings, D-branes are massive D-dimensional objects and therefore they source the gravitational field curving the spacetime and giving us a gravitational theory. In the so called decoupling limit the two theories decouple and Maldacena realized that the two different descriptions of the D-branes have to be equivalent.

Numerous dual theories have been found over the years. The one on which the thesis is focused, as well as the first to be discovered historically, is that between a gravitational string theory on spacetimes which asymptote $AdS_5 \times S^5$, and the so-called $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory on the four dimensional boundary of AdS_5. It is quite hard to find quantum field theories that are conformally invariant. In supersymmetric theories it is sometimes possible to prove exact conformal invariance and the $\mathcal{N} = 4$ $SU(N)$ SYM theory is an example of this. Another important duality that we will consider is the one between string theory on spacetimes which asymptote $AdS_3 \times S^3 \times T^4$ and the two dimensional conformal field theory living on AdS_3, called the D1-D5 CFT. The latter is very useful in order to study black hole microstates.

According to the holographic dictionary each state of the CFT is linked to a particular geometry on the gravity side. The gravity theory becomes classical in the limit in which the central charge of the CFT is large which is equivalent to the large N limit discussed above. In this limit, "heavy" states, whose conformal dimension grows as the central charge, should be described by non-trivial classical geometries that approximate the Anti-de-Sitter (AdS) solution at large distances. The purpose of this thesis will be to construct some examples of these geometries for the heavy states of the maximally supersymmetric $SU(N)$ gauge theory. The starting point in order to do this are the LLM (Lin-Lunin-Maldacena) geometries, namely a particular class of solutions for the supergravity theory which tend asymptotically to $AdS_5 \times S^5$. These solutions are dual to the class of operators of the CFT we are interested in, namely operators preserving 1/2 of the supersymmetries of the theory, and they are all uniquely determined once a particular region has been fixed on the two dimensional LLM plane. For example, in this picture the $AdS_5 \times S^5$ geometry is the LLM solution related to a disk on the LLM plane, which is dual to the vacuum state of the CFT according to the AdS/CFT correspondence. It is natural to think the geometry dual to the "light" states of the CFT (namely, those with a conformal dimension much smaller than the central charge) as the LLM solution related to a small deformation of the disk on the LLM plane, which will be therefore linear in the parameter that quantifies the deformation and which we will call ϵ. As we have already said, the main purpose of this work is to find the geometries dual to the heavy states. These states can be constructed by putting together a large number of light states and, in the LLM picture, the problem of finding the dual geometry is equivalent to finding the correct figure on the LLM plane, which will be a non-linear correction in the parameter ϵ to that related to the light states.

This work is organized as follows. In the first five chapters we will give an overview of the background material upon which the rest of the thesis rests. We begin by reviewing the basic concepts of supersymmetry in chapter 1, we give a very basic introduction to string theory in chapter 2 and supergravity seen as the low-energy limit of string theory in chapter 3. Then in chapter 4 we will motivate the AdS/CFT conjecture using the arguments of the first three chapters and in chapter 5 we will conclude the introduction describing some special operators of the CFT which allow the correspondence to be applied regardless of the energy scale; always in this chapter we will also give the basic recipe for applying the holographic dictionary, i.e. the dictionary which allows us to relate operators in the field theory with the corresponding fields in the theory of (super)gravity. Throughout

this introduction to the holography we will always refer to the correspondence between the maximally supersymmetric $SU(N)$ Yang-Mills theory and type IIB supergravity on $AdS_5 \times S^5$ since this is the protagonist of the present work. Nevertheless, in Chapter 6 we will use the duality between the D1-D5 CFT and type IIB supergravity on $AdS_3 \times S^3 \times T^4$ as an example to construct geometries dual to heavy states and to the use the holographic dictionary to check the state/geometry map. In particular we will consider some special geometries on the gravity side and we will use the holographic dictionary in order to describe the dual states on the CFT side. It is important to emphasize that the material presented in these introductory chapters is insufficient for a complete understanding of the topics covered, but cites the relevant literature and serves as review for the problem that we are going to study. In chapter 7 we will describe the LLM solutions in general as the class of geometries of type IIB supergravity which preserves half of the supersymmetries and tend asymptotically to $AdS_5 \times S^5$. Then in chapter 8 we will explicitly write the LLM solution obtained by slightly deforming $AdS_5 \times S^5$ and we will see how this geometry is related to what we will call "light" states of the CFT, i.e. the states with a "small" conformal dimension. We redevive in detail and confirm the analysis at linear order in ϵ. In particular, we write explicitly the linear order solution (never written explicitly in literature). Finally, in chapter 9 we will generalise what we saw in chapter 8 by considering the deformations of $AdS_5 \times S^5$ up to the second pertubative order in the parameter that quantifies the deformation. We will explicitly write the solution and we will try to say something about the exact solution, which could in any case be the subject of any future research. These geometries are dual to what we will call "heavy" states of the CFT, i.e. the states with a "large" conformal dimension. The second order analysis covered in this last chapter is new. We find that the "naive" LLM profile is not the one that maps to the simplest heavy state (i.e. the multi-trace made by many equal single-trace components). We find the proper profile at the second order.
Finally, the last chapter is devoted to a summary and discussion.

1

An Introduction to Supersymmetry

Supersymmetry (SUSY) is a space-time symmetry mapping particles and fields of integer spin (bosons) into particles and fields of half integer spin (fermions), so that its representations are (super)multiplets containing both bosons and fermions. It was found that the ultraviolet divergences of supersymmetric theories are less severe than in the standard model due to the cancelation between bosons and fermions in loop diagrams. For this reason, the supersymmetric extension of the standard model and, of course gravity, was developed. The supersymmetric extensions of gravity are called supergravity theories (SUGRA).

An extended review of supersymmetry is far beyond the purposes of this work. Here we will show only some basic aspects that will be useful in dealing with supergravity. We refer to a complete reference such as [3], [4] and [5] for details.

1.1 The Super-Poincaré Algebra

Supersymmetric theories have a new conserved charge that is a left-handed Weyl spinor Q^I_α, together with its right-handed counterpart $\bar{Q}^I_{\dot{\alpha}}$. This is known as the *supercharge*. Here $I = 1, \cdots, \mathcal{N}$, so it is possible to have multiple supercharges, a situation known as *extended supersymmetry* ($\mathcal{N} > 1$). From an algebraic point of view there is no limit to $\mathcal{N}$ but, as we will see later, increasing $\mathcal{N}$ the theory must contain particles of increasing spin and no consistent, interacting quantum field theory can be constructed with fields that have spin greater than 2. At the heart of the supersymmetry algebra is the anti-commutation relation

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ}, \tag{1.1}$$

where $\sigma^\mu \equiv (\mathbb{I}, \sigma^i)$ and later we also use $\bar{\sigma}^\mu \equiv (\mathbb{I}, -\sigma^i)$. It is no surprise that a spinor should have an anti-commutator. But the structure of this relation is interesting: it tells us that the supercharges should be viewed as the square-root of spacetime translations. In theories with local supersymmetry (i.e. where the spinorial infinitesimal parameter of the supersymmetry transformation depends on x^μ), the anti-commutator is an infinitesimal translation whose parameter depends on x^μ. This is nothing but a theory invariant under general coordinate transformation, namely a theory of gravity. The upshot is that theories with local supersymmetry automatically incorporate gravity. The only two non-trivial commutators with Poincaré generators are

$$[J^{\mu\nu}, Q^I_\alpha] = (\sigma^{\mu\nu})_\alpha{}^\beta Q^I_\beta, \quad [J^{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \tag{1.2}$$

that follows from the fact that $\sigma^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$ and $\bar{\sigma}^{\mu\nu} \equiv \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$ are respectively the representation $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz generator $J^{\mu\nu}$. Then we have

$$[P_\mu, Q^I_\alpha] = 0, \quad [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \tag{1.3}$$

and

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ}, \quad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*, \tag{1.4}$$

where $Z^{IJ} = -Z^{JI}$ is the central charge of the algebra (i.e. it commutes with all generators) and $\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}}$ is the spinorial symplectic metric (note that for $\mathcal{N} = 1$, the anti-symmetry of Z implies $Z = 0$).
This, then, is the supersymmetry algebra: it comprises of the well known algebra of the Poincaré group, together with the algebra of the supercharges (1.1)-(1.4). Since the latter involves non-trivial relationships with Poincaré generators, this algebra is an extension of the Poincaré algebra, and is called *super-Poincaré algebra*. So supersymmetry is a space-time symmetry.

1.2 Representations of the Superalgebra in D = 4

The representetions of the superalgebra can be obtained by acting with supersymmetries on the single-particle Poicaré representations. Now, as a particle is an irreducible representation of the Poincaré algebra, we call superparticle an irreducible representation of the supersymmetry algebra. Since the Poincarè algebra is a subalgebra of the supersymmetry algebra, it follows that any irreducible representation of the supersymmetry algebra is a representation of the Poincaré algebra, which in general will be reducible. This means that a superparticle corresponds to a collection of particles, the latter being related by the action of the supersymmetry generators Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ and having spins differing by units of half. Being a multiplet of different particles, a superparticle is often called *supermultiplet*. Since a Casimir of the superalgebra is the invariant mass $P^2 = P_\mu P^\mu$, particles in the same supermultiplet have the same mass. Therefore we have to study the massive case separately from the mass-less one. We will also work in $D = 3 + 1$ dimensions and we'll say something on higher dimensions later.

1.2.1 Massless Supermultiplets

To study mass-less representations we choose a Lorentz frame in which the momentum take the form $P^\mu = (E, 0, 0, E), E > 0$. The superalgebra relation (1.1) then reduces to

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = \begin{bmatrix} 4E & 0 \\ 0 & 0 \end{bmatrix} \delta^{IJ}. \tag{1.5}$$

The relations for $\alpha = \dot{\alpha} = 2$ and $I = J$, together with $\bar{Q}_{\dot{\alpha}} = Q^\dagger_\alpha$ and the positivity condition, implies

$$Q^I_2 \left|\text{state}\right\rangle = \bar{Q}^I_2 \left|\text{state}\right\rangle = 0, \qquad Z^{IJ} = 0, \tag{1.6}$$

where $Z^{IJ} = 0$ follow from (1.4). Therefore we can act on Poincaré particles only with Q^I_1 and $\bar{Q}^I_1$. From (1.2) follows that Q^I_1 lowers helicity by $\frac{1}{2}$ and $\bar{Q}^I_1$ raises helicity by $\frac{1}{2}$. All the states in the representation may be obtained by starting from the highest helicity state $|h_{max}\rangle$ and applying products of Q^I_1 operators. The total number of states in a multiplet will then be $2^{\mathcal{N}}$. We shall only be interested in CPT invariant theories, such as quantum field theories and string theories, for which the particle spectrum must be symmetric under a sign change in helicity. If the particle spectrum obtained is not already CPT self-conjugate, then we shall take instead the direct sum with its CPT conjugate. The table below contain the multiplets $h_{max} = 1, \frac{1}{2}, 2, \frac{3}{2}$ for some $\mathcal{N}$.

$\mathcal{N}$	$h_{max} = 1$	$h_{max} = \frac{1}{2}$	$h_{max} = 2$	$h_{max} = \frac{3}{2}$
8	none	none	$[2, \frac{3}{2}, 1, \frac{1}{2}, 0]$	none
6	none	none	$[2, \frac{3}{2}, 1, \frac{1}{2}, 0]$	$[\frac{3}{2}, 1, \frac{1}{2}, 0]$
5	none	none	$[2, \frac{3}{2}, 1, \frac{1}{2}, 0]$	$[\frac{3}{2}, 1, \frac{1}{2}, 0]$
4	$[1, \frac{1}{2}, 0]$	none	$[2, \frac{3}{2}, 1, \frac{1}{2}, 0]$	$[\frac{3}{2}, 1, \frac{1}{2}, 0]$
3	$[1, \frac{1}{2}, 0]$	none	$[2, \frac{3}{2}, 1, \frac{1}{2}]$	$[\frac{3}{2}, 1, \frac{1}{2}, 0]$
2	$[1, \frac{1}{2}, 0]$	$[\frac{1}{2}, 0]$	$[2, \frac{3}{2}, 1]$	$[\frac{3}{2}, 1, \frac{1}{2}]$
1	$[1, \frac{1}{2}]$	$[\frac{1}{2}, 0]$	$[2, \frac{3}{2}]$	$[\frac{3}{2}, 1]$

Table 1.1: Multiplets in $D = 4$.

We have not included the multiplicity of each state and CPT, for a more complete version look for example [3]. Each supermultiplet contains an equal number of bosonic and fermionic d.o.f.; this is a general result of supersymmetry. Supermultiplets with the vector boson $h = 1$ are called *gauge (or vector) multiplets*, while the supermultiplets with the graviton $h = 2$ are called *supergravity multiplets*. Finally, the multiplets with only

matter particles ($h = 0, h = \frac{1}{2}$) are called *matter (or chiral) multiplets*.
All theories with $\mathcal{N} > 4$ are supergravity theories because in this case is not possible to avoid gravity since there do not exist representations with helicity smaller than $\frac{3}{2}$. Finally, it is interesting to note that $\mathcal{N} = 8$ supergravity allows only one possible representation with highest helicity smaller than $\frac{5}{2}$ and that for higher $\mathcal{N}$ one cannot avoid states with helicity $\frac{5}{2}$ or higher. Therefore, $\mathcal{N} = 8$ is an upper bound on the number of supersymmetry generators in $D = 4$, as far as interacting local field theories are concerned. We will see that in $D = 11$ the upper bound is $\mathcal{N} = 1$ and in $D = 10$ is $\mathcal{N} = 2$. So the maximum allowed number of supersymmetry generators for non gravitational theories is 16 (which is indeed $\mathcal{N} = 4$ in four dimensions) and 32 for theories with gravity (which is $\mathcal{N} = 8$ in four dimensions). When we talk about supersymmetry in higher dimension, we will see that this is true in general and not only in $D = 4$.

1.2.2 Massive Supermultiplets and BPS Bounds

The logical steps one should follow for massive representations are similar to previous ones. There is however one important difference. Let us consider a state with mass m in its rest frame $P_\mu = (m, 0, 0, 0)$. Equation (1.1) is now

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2m\delta_{\alpha\dot{\alpha}}\delta^{IJ} \tag{1.7}$$

and no supersymmetric generators are trivially realized. This means that, generically, massive representations are longer than mass-less ones. For $\mathcal{N} = 1$ the central charge is zero. We define annihilation and creation operators satisfying the usual oscillator algebra

$$a_{1,2} \equiv \frac{1}{\sqrt{2m}} Q_{1,2}, \qquad a^\dagger_{1,2} \equiv \frac{1}{\sqrt{2m}} \bar{Q}_{1,2}, \tag{1.8}$$

where $a^\dagger_1$ lowers the spin by half unit while $a^\dagger_2$ raises it. As before all the states in the representation may be obtained by starting from the state $|m, j_0\rangle$ annihilated by both a_1 and a_2 and act with the creation operators to construct the corresponding massive representations. In this case there are only two multiplets which contain massive particles of spin lesser than 1: the matter multiplet $j_0 = 0 \to (-\frac{1}{2}, 0, 0', \frac{1}{2})$ and the gauge multiplet $j_0 = \frac{1}{2} \to (-1, \mathbf{2} \times -\frac{1}{2}, \mathbf{2} \times 0, \mathbf{2} \times \frac{1}{2}, 1)$.
For $\mathcal{N} > 1$ the algebra contain non-trivial central charges. A change of basis in the space of supersymmetry generators turns out to be useful for the following analysis. Since the central charge $\mathcal{N} \times \mathcal{N}$ matrix Z^{IJ} is antisymmetric, with a $U(N)$ rotation one can put it in the standard block-diagonal form. To do so, we split the label I into two labels: $I = (\hat{I}, \bar{I})$, where $\hat{I} = 1, 2$ and $\bar{I} = 1, \cdots, r$. Here $\mathcal{N} = 2r$ for $\mathcal{N}$ even (and we append a

further single label when $\mathcal{N}$ is odd). We then have

$$Z^{IJ} = \begin{bmatrix} 0 & Z_1 & & & & & & & \\ -Z_1 & 0 & & & & & & & \\ & & 0 & Z_2 & & & & & \\ & & -Z_2 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & Z_r & & \\ & & & & & -Z_r & 0 & & \\ & & & & & & & k & \end{bmatrix}, \tag{1.9}$$

where $k = 0$ for $\mathcal{N}$ odd and is absent for $\mathcal{N}$ even. The $Z_{\bar{I}}$ are the central charges. We can now defines the following linear combination of the supercharges

$$\mathcal{Q}^{\bar{I}}_{\alpha\pm} \equiv \frac{1}{2}(Q^{1\bar{I}}_{\alpha} \pm (\sigma^0)_{\alpha\dot{\alpha}} \bar{Q}^{2\bar{I}}_{\dot{\alpha}}), \tag{1.10}$$

which satisfy the oscillator algebra

$$\{\mathcal{Q}^{\bar{I}}_{\alpha\pm}, \bar{\mathcal{Q}}^{\bar{J}}_{\dot{\alpha}\pm}\} = \delta^{\bar{I}\bar{J}} \delta_{\alpha\dot{\alpha}} (m \pm Z_{\bar{I}}). \tag{1.11}$$

Due to the positivity of the scalar product with respect to which $\bar{Q}_{\dot{\alpha}} = Q^{\dagger}_{\alpha}$, we get

$$m \geq |Z_{\bar{I}}|, \tag{1.12}$$

that is called *BPS bound*. Whenever one of the values $|Z_{\bar{I}}|$ equals m, the BPS bound is (partially) saturated and the supercharges $\mathcal{Q}^{\bar{I}}_{\alpha_-}$ ($\mathcal{Q}^{\bar{I}}_{\alpha_+}$) must annihilate the state if $Z_{\bar{I}} > 0$ ($Z_{\bar{I}} < 0$). The supersymmetry representation then suffers *multiplet shortening*, and is usually referred to as BPS. More precisely, if we have $m = |Z_{\bar{I}}|$ only for $\bar{I} = 1, \cdots, r_0$, the corresponding representation is said to be $\frac{1}{2^{(r-r_0)+1}} BPS$. In other words, the states that are invariant under half of the supersymmetry algebra are half-BPS states. If $r_0 = r$ we have an *ultra-short multiplet*, if $0 < r_0 < r$ a *short multiplet* and if $r_0 = 0$ a *long multiplet*. The construction of the representations proceeds as before using the oscillators $\mathcal{Q}^{\bar{I}}_{\alpha\pm}$, see [3] for details. The existence of short multiplets, whose mass is fixed to an upper bound, turns out to be a wonderfully powerful tool in the study of quantum field theories with extended supersymmetry. The basic idea is that one can usually solve quantum field theories at weak coupling. As one moves into the strong coupling realm, the short multiplets are special because their mass is not affected by radiative corrections. The existance of BPS solutions is one of the main reasons that makes the study of supergravity important.

1.3 Supersymmetry in Other Dimensions

There are essentially two parameters characterizing a supersymmetric theory: the dimension of the spacetime D, and the number of supersymmetries $\mathcal{N}$. We have seen that in $D = 4$, $\mathcal{N} = 1$ is the minimal supersymmetry (4 supercharges), while $\mathcal{N} = 8$ is the maximal supersymmetry (32 supercharges). We now want to extend the analysis of supersymmetric theories to higher dimensional spacetime, and the reason is that, as we will see, the supergravity representing to the low-energy limit of string theory lives in $D = 11$ or in $D = 10$. Furthermore supergravity in $D = 4$ naturally arises as dimensional reduction of higher dimensional theories.

The extension is trivial if one considers only bosonic fields; in order to deal also with fermions, one should first study spinor representations in dimensions greater than four. This means studying the representations of Clifford algebra in higher dimensions

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \tag{1.13}$$

with $\mu, \nu = 0, \cdots, D-1$. If $J_{\mu\nu}$ is the Lorentz generator, the Dirac spinor representation is defined in terms of the standard Clifford matrices

$$U_D(J_{\mu\nu}) = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{1.14}$$

and its complex dimension is given by $2^{[D/2]}$. For D even the Dirac spinor representation is always reducible because in that case there exists a chirality matrix $\bar{\gamma}$, with square $\bar{\gamma}^2 = \mathbb{I}$, witch anti-commutes with all γ_μ

$$\{\bar{\gamma}, \gamma_\mu\} = 0 \quad \Rightarrow \quad [\bar{\gamma}, U_D(J_{\mu\nu})] = 0. \tag{1.15}$$

As a result, the Dirac spinor is the direct sum of two Weyl spinors of chirality ± 1 $U_D = U_R \oplus U_L$. The reality condition is

$$\psi = \psi^c, \tag{1.16}$$

where

$$\psi^c \equiv C\gamma_0\psi^*, \tag{1.17}$$

here C is a matrix such that $C\gamma_\mu C^{-1} = -(\gamma_\mu)^T$. It can be shown that one can impose such a condition only in dimensions $D = 0, 1, 2, 3, 4(\text{mod } 8)$. In dimensions $D = 0, 4(\text{mod } 8)$, a Majorana spinor is equivalent to a Weyl spinor, while in dimension $D = 2(\text{mod } 8)$ it is possible to impose the Majorana and Weyl conditions at the same time, resulting in Majorana-Weyl spinors. The supercharges Q_α transforms in the spinor representation

U, which could be a Dirac spinor, a Weyl spinor, a Majorana spinor or a Majorana-Weyl spinor, depending on D. Thus, α runs over the spinor indices $\alpha = 1, \cdots, dimU$. We have seen that for massless representations we can choose $P^\mu = (E, 0, \cdots, 0, E)$ and we have

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ} \begin{bmatrix} 4E & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.18}$$

Half of the supercharges effectively vanish $Q^I_\alpha = 0$ for $\alpha = \frac{1}{2}dimU + 1, \cdots, dimU$. Half of the remaining supercharges may be viewed as lowering operators, while the other half may be viewed as raising operators. Thus, the total number of raising operators is $\frac{1}{4}\mathcal{N} dimU$. Each operator raising helicity by $\frac{1}{2}$, and total helicity ranging at most from -2 to $+2$, we should have at most 8 raising operators and this produces an important bound

$$\mathcal{N} \cdot dimU \leq 32. \tag{1.19}$$

In other words, the maximum number of supercharges is always 32. The largest dimension D for which the bound may be satisfied is $D = 11$ and $\mathcal{N} = 1$, for which there are precisely 32 Majorana supercharges. In $D = 10$, the bound is saturated for $\mathcal{N} = 2$ and 16-dimensional Majorana-Weyl spinors. Many of the lower dimensional theories may be constructed by Kaluza-Klein compactification on a circle or on a torus of the $D = 11$ theory as we will see.

2

An Introduction to String Theory

String theory is an ambitious project. It purports to be an all-encompassing theory of the universe, unifying the forces of nature, including gravity, in a single quantum mechanical framework. The premise of string theory is that the fundamental objects are not point-like particles but extended one dimensional strings. From this slightly unconventional beginning, the laws of physics emerge. However, they come with baggage. String theory gives rise to a host of other ingredients, most strikingly extra spatial dimensions of the universe beyond the three that we have observed. The quantization of the strings vibration modes corresponds to different particles of various masses and spins. The particles spectrum contain also a massless spin-2 particle, the graviton. In this sense, string theory is a theory of quantum gravity. The main purpose of this chapter is to introduce the main aspects of the theory that will be relevant in the discussion of supergravity. In fact it turns out that supergravity is the low-energy limit of string theory. Standard references for the topic are, for example, [6], [7] and [8].

2.1 Bosonic Strings

The action for a relativistic point particle with mass m which moves in the Minkowski space $\mathbb{R}^{1,D-1}$, is proportional to the lenght of its worldline

$$S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}, \tag{2.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1)$ is the Minkowski metric and $X^\mu(\tau)$ are the coordinates of the worldline parameterized by τ that the particle sweeps out. This action is invariant under an arbitrary reparameterization of the worldline

$$\tilde{\tau} = \tilde{\tau}(\tau) \tag{2.2}$$

and under global Poincaré trasformation

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{2.3}$$

as one can check using the definition property of the Lorentz transformations: $\Lambda^T \eta \Lambda = \eta$. It is pretty straightforward to generalize the action of the point particle for a string: while a point particle sweeps out a worldline, a string sweeps out a 2-dimensional surface, called the *worldsheet* of the string. We want to write the string action in term of the worldsheet area; to this purpose we parameterized the surface with a time-like coordinate τ and a space-like coordinate σ that we put into a single object $\sigma^a \equiv (\sigma, \tau)$, $a = 1, 2$ for later convenience. The metric γ_{ab} on the worldsheet is the pull-back of the Minkowski metric

$$(X^*\eta)_{ab} \equiv \gamma_{ab} = \frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b}\eta_{\mu\nu}. \tag{2.4}$$

So the action proportional to the worldsheet area is

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det(\gamma)} = -T \int d^2\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})} \tag{2.5}$$

and it is called the *Nambu-Goto action*. The proportionality coefficient T is the string tension, meaning the mass per unit length, which is related to the Regge slope parameter α' via

$$T = \frac{1}{2\pi\alpha'}, \qquad \alpha' = l_s^2, \tag{2.6}$$

where l_s is the string length. The square root in the Nambu-Goto action makes the quantization of the string more difficult. We can write an equivalent string action without the square root at the expense of introducing a dynamical field g_{ab} on the worldsheet. This is called *Polyankov action*

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det(g)} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{2.7}$$

To justify the equivalence between the two actions we write down the equations of motion for X^μ

$$\partial_a(\sqrt{-g} g^{ab} \partial_b X^\mu) = 0, \tag{2.8}$$

which coincides with the equations of motion obtained from the Nambu-Goto action with the field g_{ab} instaed of γ_{ab}. The equations for the dynamical metric on the worldsheet are

$$g_{ab} = f(\sigma) \partial_a X \partial_b X, \tag{2.9}$$

where f is left arbitrary by the equations of motion for g_{ab} (because of Weyl symmetry). We see that g_{ab} differ from the pull-back metric (2.4) only by the conformal factor $f(\sigma)$. However, this doesn't matter because, rather remarkably, $f(\sigma)$ drops out of the equation of motion (2.8). This is because the $\sqrt{-g}$ term scales as f, while the inverse metric g^{ab} scales as f^{-1}. We therefore see that Nambu-Goto and Polyankov actions result in the same equations of motion for X^μ. In fact, we can see more directly that the two actions coincide by integrating away g_{ab} using its equations of motion.
The fact that the conformal factor $f(\sigma)$ didn't actually affect the equations of motion for X^μ reflects the existence of an extra symmetry which the Polyankov action enjoys. In particular the symmetries of the action are the following:

- Global Poincaré invariance

$$X^\mu \rightarrow \Lambda^\mu{}_\nu X^\nu + c^\mu. \tag{2.10}$$

- Reparameterization invariance

$$\sigma^a \rightarrow \tilde{\sigma}^a \quad \Rightarrow \quad X^\mu(\sigma) = \tilde{X}^\mu(\sigma),\;\; \tilde{g}_{ab} = \frac{\partial\sigma^c}{\partial\tilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}^b}g_{cd}(\sigma). \tag{2.11}$$

- Weyl invariance (conformal invariance)

$$g_{ab}(\sigma) \rightarrow \Omega^2(\sigma)g_{ab}(\sigma) = e^{2\omega(\sigma)}g_{ab}. \tag{2.12}$$

  This is a gauge symmetry of the string, as seen by the fact that the parameter Ω depends on the worldsheet coordinates σ. The property of Weyl invariance is special to two dimensions, for only there does the scaling factor coming from the determinant $\sqrt{-g}$ cancel that coming from the inverse metric. If we wish to keep Weyl invariance then we are strictly limited in the kind of interactions that can be added to the action.

We can simplify the equations for X^μ (2.8) by fixing a particular gauge. Firstly, we can use reparameterization invariance to fix two of the three metric independent components. We will choose to make the metric locally conformally flat, meaning

$$g_{ab} = e^{2\phi}\eta_{ab}, \tag{2.13}$$

where $\phi = \phi(\sigma)$ is some function on the worldsheet. Choosing a metric of that form is known as *conformal gauge*. Now we can use Weyl invariance to remove the last independent component of the metric and set $\phi = 0$ such that

$$g_{ab} = \eta_{ab}. \tag{2.14}$$

We end up with the flat metric on the worldsheet in Minkowski coordinates. With this gauge fixing the Polyankov action (2.7) becomes the theory of D free scalar fields

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_a X \partial^a X, \tag{2.15}$$

and the equations of motion for X^μ reduce to the free wave equation

$$\partial_a \partial^a X^\mu = 0. \tag{2.16}$$

The equations for g_{ab}, fixing $g_{ab} = \eta_{ab}$, instead become

$$T_{ab} = 0, \tag{2.17}$$

where we have defined the energy-stress tensor

$$T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{ab}}. \tag{2.18}$$

2.1.1 Closed Strings

For a closed string we take σ to be periodic, with range

$$\sigma \in [0, 2\pi), \tag{2.19}$$

and we also require

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau). \tag{2.20}$$

The equations of motion (2.16) are easily solved. We introduce lightcone coordinates on the worldsheet

$$\sigma^\pm \equiv \tau \pm \sigma, \tag{2.21}$$

in terms of which the equations of motions simply read

$$\partial_+ \partial_- X^\mu = 0. \tag{2.22}$$

The most general solution is

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{2.23}$$

for arbitrary functions X_L^μ and X_R^μ. These describe left-moving and right-moving waves respectively. The most general periodic solution can be expanded in Fourier modes

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}, \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}, \end{aligned} \tag{2.24}$$

where the variables x^μ and p^μ are the position and momentum of the center of mass of the string. Reality of X^μ requires that the coefficients of the Fourier modes, α_n^μ and $\tilde{\alpha}_n^\mu$, obey

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^*, \quad \tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^*. \tag{2.25}$$

Finally, the constraints (2.17), impose

$$L_n = \tilde{L}_n = 0, \tag{2.26}$$

where

$$L_n \equiv \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \quad \tilde{L}_n \equiv \frac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m. \tag{2.27}$$

2.1.2 Open Strings

The spatial coordinate of an open string is parameterized by

$$\sigma \in [0,\pi]. \tag{2.28}$$

The dynamics of an open string must therefore still be described by the Polyakov action. But this must now be supplemented by something else: boundary conditions to tell us how the end points move. Let's consider the string evolving from some initial configuration at $\tau = \tau_i$ to some final configuration at $\tau = \tau_f$

$$\delta S = -\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \partial_\alpha X \partial^\alpha \delta X = \frac{1}{2\pi\alpha'}\int d^2\sigma (\partial^\alpha\partial_\alpha X)\delta X + \text{ total derivative,} \tag{2.29}$$

where the boundary contribution is given by the total derivative term

$$\frac{1}{2\pi\alpha'}\left[\int_0^\pi d\sigma \dot{X}\cdot\delta X\right]_{\tau=\tau_i}^{\tau=\tau_f} - \frac{1}{2\pi\alpha'}\left[\int_{\tau_i}^{\tau_f} d\tau X'\cdot\delta X\right]_{\sigma=0}^{\sigma=\pi}, \tag{2.30}$$

where $\dot{X} \equiv \partial_t X$, $X' \equiv \partial_\sigma X$. The first term vanishes by requiring that $\delta X^\mu = 0$ at $\tau = \tau_i$ and τ_f as usual in the variational approach. To eliminate also the second term we have to impose the condition

$$\partial_\sigma X^\mu \delta X_\mu = 0, \quad \text{at } \sigma = 0, \pi. \tag{2.31}$$

There are two different types of boundary conditions that we can impose to satisfy this:

- Neumann boundary conditions

  $$\partial_\sigma X^\mu = 0, \quad \text{at } \sigma = 0, \pi. \tag{2.32}$$

  Because there is no restriction on X , this condition allows the end of the string to move freely.

- Dirichlet boundary conditions

  $$\delta X^\mu = 0, \quad \text{at } \sigma = 0, \pi. \tag{2.33}$$

  This means that the end points of the string lie at some constant position, $X^\mu = c^\mu$, in space.

Let's consider Dirichlet boundary conditions for some coordinates and Neumann for the others. This means that at both end points of the string, we have

$$\partial_\sigma X^i = 0, \text{ for i} = 0, \cdots, p; \qquad X^I = c^I, \text{ for I} = p+1, \cdots, D-1. \tag{2.34}$$

This fixes the end-points of the string to lie in a $(p+1)$-dimensional hypersurface in spacetime such that the $SO(1, D-1)$ Lorentz group is broken to

$$SO(1, D-1) \to SO(1, p) \times SO(D-p-1), \tag{2.35}$$

that is Lorentz invariance in the flat hypersurface and Lorentz invariance in the directions transverse to the membrane. This hypersurface is called a D-*brane* or, when we want to specify its dimension, a Dp-brane. Here D stands for Dirichlet, while p is the number of spatial dimensions of the brane. So, in this language, a D0-brane is a particle; a D1-brane is itself a string; a D2-brane a membrane and so on. The brane sits at specific positions c^I in the transverse space. It turns out that the D-brane hypersurface should be thought of as a new, dynamical object in its own right: string theory is not just a theory of strings, it also contains higher dimensional branes. Strings that have Neumann boundary conditions in all directions, are free to move throughout spacetime or, in other words, the space is completely covered by branes.

2.1.3 A Nod to the Quantization

We will not discuss the quantization procedure of the Polyakov action, standard references are [6] and [7]. It turns out that a consistent quantum theory of strings is possible only if the dimension of spacetime is $D = 26$. The quantization of the vibration modes found in the previous sections corresponds to different particles of various masses and spin. The masses are integer multiples of $\frac{1}{l_s}$ and at distances much greater than the string length, only the mass-less modes are relevant. Finally, the spectrum contains only bosons and, for this reason, this type of strings are called *bosonic strings*.
The ground state is a tachyon while the first excited states correspond to mass-less particles and their respective fields for the closed string are:

- $g_{\mu\nu}(X)$, a massless spin two field, which we interpret as the metric[1].
- $B_{\mu\nu}(X)$, a 2-form called the *Kalb-Ramond field*.
- $\Phi(X)$, a scalar field called the *dilaton*.

while for the open string we have:

- Excitations polarized along the brane are described by a spin 1 gauge field A_a (with $a = 0, ..., p$) living in the Dp-brane's $(p+1)$-dimensional worldvolume. We will see later that this $U(1)$ gauge theory plays a major role in the AdS/CFT duality.
- Excitations polarized perpendicular to the brane are described by scalar fields ϕ^I (with $I = p+1, ..., D-1$). They can be interpreted as fluctuations of the brane in the transverse directions, this gives us a hint that the D-brane is a dynamical object.

These mass-less fields are common to all string theories, also to superstring theories which we will discuss now.

2.2 A Nod to the Superstring Theories

Superstring theories solve the two main problems of the bosonic strings: the absence of the fermions in the spectrum and the presence of a vacuum state with negative energy: the *tachyon*. The main difference from the bosonic theory is the introduction of supersymmetry on the worldsheet. While the bosonic string theory is unique, there are a number of discrete choices that one can make when adding fermions. The most important one is

[1] There is an argument by Feynman [9] that shows that any theory of interacting massless spin two particle must be General Relativity.

whether to add fermions in both left-moving and right-moving sectors (obtaining *Type II* superstring), or allow them to move in only one direction (obtaining *Heterotic* strings). Strings without an orientation are called *Type I* superstring. However, later developments have shown that they are all parts of the same framework, which goes by the name of *M-theory*. Here we will discuss and use only Type II superstring.
We introduce D Majorana spinors $\psi^\mu = (\psi^\mu_a)$ (where $\mu = 0, ..., D-1$ is the spacetime index and $a = \pm$ is a worldsheet spinor index), with action

$$S_\psi = \frac{i}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} \bar{\psi}_\mu \rho^a \partial_a \psi^\mu, \tag{2.36}$$

where ρ^a satisfy the 2-dimensional Clifford algebra. In the conformal gauge $\gamma_{ab} = \eta_{ab}$ and using light coordinates $\sigma^\pm = \tau \pm \sigma$, the fermionic equations of motion read

$$\begin{aligned} \partial_+ \psi_- = 0 \quad &\Rightarrow \quad \psi_- = \psi_-(\sigma^-), \\ \partial_- \psi_+ = 0 \quad &\Rightarrow \quad \psi_+ = \psi_+(\sigma^+). \end{aligned} \tag{2.37}$$

Let's now combine the Polyakov action (2.7) with the spinor action and introduce the gravitino χ^α, the supersymmetric partner of g^{ab}, in such a way that the resulting action is supersymmetric. The resulting action possesses reparameterization and conformal invariance. These symmetries can be used to fix some degrees of freedom: a useful choice is the so-called superconformal gauge, in which $g^{ab} = \eta^{ab}$ and $\chi^a = 0$. With this gauge choice, the action for Type II superstring reads [10]

$$S_{II} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} [\partial_a X^\mu \partial^a X^\mu - i\bar{\psi}_\mu \rho^\alpha \partial_\alpha \psi^\mu]. \tag{2.38}$$

The quantization of the theory proceeds analogously with the bosonic string case. One can project out of the spectrum the tachyonic state that is present in the NS sector. This can be done with the GSO projection, which keeps just the states constructed applying an odd number of fermionic creation operators to a vacuum state and projects out the others. This operation removes the tachyonic state from the Fock space, as it has an even fermionic number. It turn out that the GSO projection has to be applied also to the R sector: in this case, whether to keep the states with even or odd fermionic number is a matter of choice, and this choice gives rise to two different theories. Consistency requires that the dimension of spacetime must be $D = 10$. The mass-less spectrum can be classified in 4 sectors according to the different possible boundary conditions of the fermions:

- NS-NS sector: the field content is identical to the bosonic string. It consists in the dilaton Φ, the Kalb-Ramond 2-form $B_{\mu\nu}$ and the graviton $g_{\mu\nu}$.

- The NS-R sector contains two fermionic fields: the spin-$\frac{1}{2}$ dilatino and the spin-$\frac{3}{2}$ gravitino (supersymmetric partners of the dilaton and of the graviton respectively).
- The R-NS sector contatins the same spectrum of the NS-R sector.
- R-R sector: it contains bosonic fields, but its spectrum depends on the way one makes the GSO projection. Two different theories arise: *Type IIA* and *Type IIB* superstring theories. The former contains a 1-form and a 3-form; the latter a 0-form, a 2-form and a self dual 4-form.

2.3 Toroidal Compactification

Even before the advent of string theory, the possibility of extra dimensions was discussed. A few years after Einstein wrote down his theory of general relativity Kaluza attempted to unify gravitation with electromagnetism by assuming that we live in a five-dimensional universe. By considering an effective 4D theory where one keeps only the lowest harmonics in the extra dimensions he managed to obtain the four-dimensional field equations of both gravity and electromagnetism from a five-dimensional theory of pure gravity. He also assumed that the extra coordinate was curled up as a circle, explaining why this coordinate had never been observed in experiment. The same mechanism can now also be used for ten-dimensional string theories, in order to try to obtain the four-dimensional world as we observe it and make contact with experiment. The point is that all of the dimensions need not to be infinitely extended: some of them can be compact. Consider, for instance, a 5-dimensional spacetime in which 4 directions are flat (with coordinates x^μ, $\mu = 0, ..., 3$) while the fifth direction is a circle of radius R (whose coordinate y is periodic: $y = y + 2\pi R$). Consider now a mass-less scalar field $\phi(x, y)$; we can decompose it as

$$\phi(x, y) = \sum_n \phi_n(x) e^{\frac{iny}{R}}, \tag{2.39}$$

where the integer-valued n labels the quantized momenta in the compact direction. The equation of motion $\partial_M \partial^M \phi(x, y) = 0$ (where $M = 0, ..., 4$ is the index of the 5-dimensional spacetime) gives

$$(\partial_\mu \partial^\mu - \frac{n^2}{R^2})\phi_n(x) = 0, \quad \forall n. \tag{2.40}$$

Thus, a single field in higher dimensions becomes an infinite tower of massive fields in the non-compact world, with mass m_n given by $m_n = \frac{|n|}{R}$. At energies much lower then

$\frac{1}{R}$, only the $n = 0$ mode can be excited, and at this scale we remain with only one scalar field in the non-compact dimension $\phi_0(x)$.
We can decompose the metric $\tilde{g}_{MN}$ of the 5-dimensional space into:

$$g_{\mu\nu}, \quad g_{\mu y}, \quad g_{yy}, \tag{2.41}$$

that is, in a metric on the non-compact dimensions, a vector gauge field and a scalar matter field. In order to implement this we can parameterize the metric as

$$ds^2 = \tilde{g}_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + g_{yy}(dy + A_\mu dx^\mu)^2. \tag{2.42}$$

This form still allows a reparameterizations ${x'}^\mu(x^\nu)$ and also the reparameterizations

$$y' = y + \lambda(x). \tag{2.43}$$

Under the latter

$$A'_\mu = A_\mu - \partial_\mu \lambda(x), \tag{2.44}$$

so gauge transformations arise as part of the higher-dimensional coordinate group. This is the *Kaluza-Klein* mechanism. The action of pure gravity written in terms of the new fields reads [7]

$$S = \frac{1}{2k_5^2} \int d^5x \sqrt{-\tilde{g}} R_5 = \frac{2\pi R}{2k^2} \int d^4x \sqrt{-g} e^\sigma (R_4 - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \sigma \partial^\mu \sigma), \tag{2.45}$$

where $F = dA$ and R_4 is the Ricci scalar of $g_{\mu\nu}$.
Let's now consider the Kaluza-Klein reduction from the prespepctive of a (bosonic) string. We want to study a string moving in the background $\mathbb{R}^{1,D-2} \times S^1$. One effect of the compactification is that the momentum along the circle direction p_y is quantized in integer units

$$p_y = \frac{n}{R}, \quad n \in \mathbb{Z}, \tag{2.46}$$

this is not specific to strings and it follows from the requirement that the standard wave function $e^{ip_y y}$ be single valued on S^1. Another consequence of the compactification, that is peculiar to string theory, is that the boundary conditions for the string coordinates become

$$X^y(\sigma + 2\pi) - X^y(\sigma) = 2\pi m R, \quad m \in \mathbb{Z}. \tag{2.47}$$

The integer m is the *winding number* of the string: it is the number of times the string wraps the circle. The presence of a quantized momentum and winding number contribute to the mass of the string: beside the $\frac{1}{l_s}$ oscillator contributions, it receives the correction [6]

$$\delta M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{l_s^4}. \tag{2.48}$$

3 Supergravity

Supergravity theories are supersymmetric extensions of general relativity and have a natural embedding in superstring theories, as supergravity corresponds to their low energy limit. The low-energy string effective action describes the low-energy dynamics of a given string theory: the low energy limit is equivalent to the limit $\alpha' \to 0$, because at large distances the string length can be ignored and a theory of particles is recovered. Moreover, at low energies only the mass-less modes are relevant and their dynamics is described by a theory of the corresponding mass-less fields. The low energy theory can be obtained expanding in powers of α' the action for the massless string spectrum and keeping only the lowest terms. Historically, however, supergravity and superstring theories were discovered independently. Before the advent of strings as a theory of quantum gravity, in fact, there was an attempt to control loop divergences in gravity by making the theory supersymmetric. The greater the number of supersymmetries, the better was the control of divergences. Since no consistent interacting quantum field theory can be constructed with fields that have spin greater than 2, in four dimensions the maximal number of supersymmetries is $\mathcal{N} = 8$. We can also construct a $D = 11$ $\mathcal{N} = 1$ theory by taking the low energy limit of M-theory. Then we can construct a $D = 10$ $\mathcal{N} = 2$ theory via the process of dimensional reduction explained above. Among the various supergravity theories, 11-dimensional supergravity occupies a distinguished position; eleven is the maximal space-time dimension in which a supergravity theory can be constructed for the reason above. In this chapter we will introduce $D = 11$ supergravity, as well as Type IIA and Type IIB 10-dimenional supergravity.

3.1 Basic Features of Supergravity

In this section we introduce some useful tools to deal with supergravity. The first is the language of differential p-forms and the second is the tetrad formalism needed to put spinors on curved spaces.

3.1.1 Differential p-forms

A *differential p-form* is a type $(0, p)$ tensor completely antisymmetric. Using the coordinate differentials dx^μ, we can construct differential p-forms for $p = 1, \cdots, D$ as

$$\omega^{(D)} = \frac{1}{p!}\omega_{\mu_1 \cdots \mu_p}(x) dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{3.1}$$

The product $\wedge$ between two p-forms is called *wedge product* and is defined as antisymmetric

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu. \tag{3.2}$$

A p-form $\omega^{(p)}$ and a q-form $\omega^{(q)}$ can be multiplied to give a $(p+q)$-form if $p+q \leq D$. The product vanishes if $p + q > D$ for the antisymmetry and it satisfies

$$\omega^{(p)} \wedge \omega^{(q)} = (-)^{pq} \omega^{(q)} \wedge \omega^{(p)}, \tag{3.3}$$

where the minus sign is related to the fact that we have to swap the differentials. The *exterior derivative* is a map between a p-form and a $(p + 1)$-form:

$$d\omega^{(p)} = \frac{1}{p!}\partial_\mu \omega_{\mu_1 \cdots \mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{3.4}$$

that satisfy the Leibniz rule

$$d(\omega^{(p)} \wedge \omega^{(q)}) = d\omega^{(p)} \wedge \omega^{(q)} + (-)^p \omega^{(p)} \wedge d\omega^{(q)}, \tag{3.5}$$

where the minus sign is related to the fact that dx^μ must surpass p differentials. A p-form that satisfies $d\omega^{(p)} = 0$ is called *closed*. A p-form that can be expressed as $\omega^{(p)} = d\omega^{(p-1)}$ is called *exact*. We also have

$$dd\omega^{(p)} = 0 \tag{3.6}$$

because partial derivatives commute. In a D-dimensional manifold, the number of independent parameters of a p-form is $\binom{D}{p}$. Since p-forms and q-forms have the same number of components when $p + q = D$, it is possible to define a map between them. This map is called *Hodge duality* $\star$

$$\star : \Lambda^{(p)}(\mathcal{M}) \to \Lambda^{(q)}(\mathcal{M}), \quad \Omega^{(q)} = \star\omega^{(p)} \tag{3.7}$$

and it's defined as

$$\Omega^{(q)}_{b_1 \cdots b_q} = \frac{1}{p!}\sqrt{-g}\epsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p} \omega_{a_1 \cdots a_p}. \tag{3.8}$$

This operator has an important involutive property; for a lorentzian metric

$$\star(\star\omega^{(p)}) = -(-)^{pq}\omega^{(p)}. \tag{3.9}$$

For even dimension $D = 2m$, it is possible to impose the constraint of self-duality (or anti-self-duality) on forms of degree m

$$\omega^{(m)} = \pm(\star\omega^{(m)}). \tag{3.10}$$

This condition is consistent only if duality is a strict involution, i.e.

$$-(-)^{m^2} = +1. \tag{3.11}$$

A self-dual $F^{(5)}$ is possible in $D = 10$ Lorentzian signature, and it indeed appears in Type IIB supergravity. Finally, for general p-forms we have

$$\lambda^{(p)} \wedge \star\omega^{(p)} = \omega^{(p)} \wedge \star\lambda^{(p)}. \tag{3.12}$$

The equation of motion in any field theory are most conveniently packaged in the action integral. In a gravitational theory this requires integration over the curved spacetime manifold. We thus need a procedure for integration that is invariant under coordinates transformations. The volume form is the key to this procedure.
On a D-dimensional manifold, one may choose *any* D-form $\omega^{(D)}$ as a volume form and define the integral

$$I = \int \omega^{(D)} = \frac{1}{D!}\int \omega_{\mu_1\cdots\mu_D}(x)dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D}. \tag{3.13}$$

So, with the forms we can define integral volume in curved spacetime without invoking the metric explicitly. When the physical theory contains forms field, we can use them to define the integral volume. It's easy to show that the volume forms are invariant under a general coordinate transformation. Since the the wedge product between a p-form and its dual is a D-form, we can always define the integral as

$$\int \star\omega^{(p)} \wedge \omega^{(p)} = \frac{1}{p!}\int d^Dx\sqrt{-g}\omega^{\mu_1\cdots\mu_p}\omega_{\mu_1\cdots\mu_p} \tag{3.14}$$

for any p-form $\omega^{(p)}$. In this language the Maxwell equations for the 1-form vector potential $A^{(1)}$ with field strength $F^{(2)} = dA^{(1)}$ are

$$dF = 0, \quad d\star F = 0, \tag{3.15}$$

where the Bianchi's identity $dF = 0$ follow from the fact that field strength is an exact form. In dealing with supergravity we will generalize these equations of motion to generic p-forms.

3.1.2 Tetrad Formalism: The Spin Connection

For practical reasons, when dealing with spinors in curved spacetime it is useful to introduce a new basis on the manifold's tangent space. The starting point is the Clifford algebra in flat space, $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$. In curved spacetime we have

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x). \tag{3.16}$$

Because the right-hand side depends on x, the object γ^μ on the left-hand side also depend on x as we already have indicated. We can expand $\gamma^\mu(x)$ in terms of the constant Dirac matrices γ^m of flat space as follows

$$\gamma^\mu(x) = \gamma^m {e_m}^\mu(x). \tag{3.17}$$

The matrices ${e_m}^\mu(x)$ are called the (inverse) *vielbein* fields[1].
The substitution of (3.17) in (3.16) shows that the metric is the product of two vielbeins

$$\eta^{mn} {e_m}^\mu {e_n}^\nu = g^{\mu\nu}. \tag{3.18}$$

Defining ${e_\mu}^m$ as the (matrix) inverse we also have

$$g_{\mu\nu} = {e_\mu}^m {e_\nu}^n \eta_{mn}. \tag{3.19}$$

This is the defining property of the vielbeins. Given a local Lorentz transformation, we can construct another solution

$${e'^a}_\mu(x) = (\Lambda^{-1}(x))^a{}_b {e(x)^b}_\mu. \tag{3.20}$$

All choice of frame fields related by local Lorentz transformations are viewed as equivalent. So we require that the geometrical quantities derived from it must be used in a way that is covariant with respect to this transformation. Coordinate indices transforms as a covariant

[1] In the German literature they were called Vierbein fields where vier = four in German, and bein = leg. In the English literature this became *tetrads* (tessara = four in Greek), or *frame fields* sometimes. Gell-Mann coined the word *vielbeins* because viel = many in German.

vector under diffeomorphism

$$e'^{a}{}_{\mu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} e^{a}_{\rho}(x). \tag{3.21}$$

We also have

$$e^{a}_{\mu} e^{\mu}_{b} = \delta^{a}_{b}, \qquad e^{\mu}_{a} e^{a}_{\nu} = \delta^{\mu}_{\nu}. \tag{3.22}$$

So geometrically the frame fields e^{μ}_{n} form an orthonormal set of vectors in the tangent space of the manifold at each point. Any contravariant and covariant field has a unique expansion in the new basis

$$\begin{aligned} V^{\mu}(x) &= V^{a}(x) e^{\mu}_{a}(x), \qquad V^{a}(x) = V^{\mu}(x) e^{a}_{\mu}(x); \\ \omega_{\mu}(x) &= \omega_{a}(x) e^{a}_{\mu}(x), \qquad \omega_{a}(x) = \omega_{\mu}(x) e^{\mu}_{a}(x). \end{aligned} \tag{3.23}$$

The $V^{a}(x)$ and $\omega_{a}(x)$ transform as scalar fields under coordinate transformations, and as a vector under Lorentz transformation. We can use the frame fields to define a new basis in the tangent space

$$E_{a} = e^{\mu}_{a} \partial_{\mu}. \tag{3.24}$$

While the new local Lorentz basis of 1-forms is

$$e^{a} = e^{a}_{\mu} dx^{\mu}, \tag{3.25}$$

that is the dual basis of the previous one: $(E_{a}, e^{b}) = \delta^{b}_{a}$. For 2-forms, basis consists of the wedge products $e^{a} \wedge e^{b}$, and so on. In a field theory containing only bosonic fields, which are always vectors or tensors, the use of local frames is unnecessary. Local frames are a necessity to treat the coupling of fermion fields to gravity, because spinors are defined by their special transformation properties under Lorentz transformations.
We can define a covariant derivative for the vectors in the frame bases in the same way we construct the one for the vectors in the coordinate frames. We first observe that given a 1-form e^{a}, we have

$$de^{a} = \frac{1}{2}(\partial_{\mu} e^{a}_{\nu} - \partial_{\nu} e^{a}_{\mu}) dx^{\mu} \wedge dx^{\nu}. \tag{3.26}$$

The antisymmetric components don't transform as a $(0, 2)$ tensor under local Lorentz transformation

$$de'^{a} = d((\Lambda^{-1})^{a}{}_{b} e^{b}) = (\Lambda^{-1})^{a}{}_{b} de^{b} + d(\Lambda^{-1})^{a}{}_{b} \wedge e^{b}. \tag{3.27}$$

The second term spoils the vector transformation property. To cancel it we add the contribution involving the spin connection. We have

$$\begin{aligned} D_\mu V^n &= e^n{}_\nu D_\mu V^\nu = e^n{}_\nu \partial_\mu V^\nu + e^n{}_\nu \Gamma^\nu_{\mu\lambda} V^\lambda = \\ &= e^n{}_\nu \partial_\mu (e^\nu{}_a V^a) + e^n{}_\nu \Gamma^\nu_{\mu\lambda} e^\lambda{}_a V^a = \\ &= \partial_\mu V^n + e^n{}_\nu (\partial_\mu e^\nu{}_a + \Gamma^\nu_{\mu\lambda} e^\lambda{}_a) V^a = \partial_\mu V^n + \omega_\mu{}^n{}_a V^a \end{aligned} \tag{3.28}$$

where

$$\omega_\mu{}^a{}_b \equiv e_\nu{}^a (\partial_\mu e^\nu{}_b + \Gamma^\nu_{\mu\lambda} e^\lambda{}_b) \tag{3.29}$$

is called the *spin connection*. So, for a contravariant and a covariant vector, the local Lorentz covariant derivative is

$$\begin{aligned} D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b, \\ D_\mu V_a &= \partial_\mu V_a - V_b \omega_\mu{}^b{}_a = \partial_\mu V_a + \omega_{\mu_a}{}^b V_b. \end{aligned} \tag{3.30}$$

from which we get $\omega_\mu{}^b{}_a = -\omega_{\mu_a}{}^b$. These relations can be generalized for a type (q, p) tensor. In order to determine the spin connection one has to impose the *tetrad postulate*: the covariant derivative of the vierbein field vanishes, $D_\mu e^a_\nu = 0$. In fact

$$\omega_\mu{}^a{}_b e_\nu{}^b = e_\sigma{}^a e^\lambda{}_b e_\nu{}^b \Gamma^\sigma_{\mu\lambda} - e^\lambda{}_b e_\nu{}^b \partial_\mu e_\lambda{}^a = e_\sigma{}^a \Gamma^\sigma_{\mu\nu} - \partial_\mu e_\nu{}^a. \tag{3.31}$$

Rearranging terms, we have the tetrad postulate

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - e_\sigma{}^a \Gamma^\sigma_{\mu\nu} + \omega_\mu{}^a{}_b e_\nu{}^b = 0. \tag{3.32}$$

We can define the *torsion* 2-form:

$$de^a + \omega^a{}_b \wedge e^b \equiv T^a. \tag{3.33}$$

In most application of gravity the torsion vanishes, and one deal with a torsion-free, metric-preserving connection

$$T^a = 0. \tag{3.34}$$

From (3.32),(3.33) this is equivalent to the usual symmetry $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ of Christoffel symbols. In a gravitational theory, spinors must be described through their local frame components. The local Lorentz transformation rule

$$\Psi'(x) = e^{-\frac{1}{4}\lambda^{ab}\gamma_{ab}} \Psi(x) \tag{3.35}$$

determines the covariant derivative

$$D_\mu \Psi(x) = (\partial_\mu + \frac{1}{4}\omega_{\mu ab}(x)\gamma^{ab})\Psi(x), \tag{3.36}$$

where $\omega_{\mu ab} = \eta_{ac}\omega_\mu{}^c{}_b$.

3.2 Supergravity in D = 11

This theory is not only more fundamental, but also much simpler than other supergravity theories in lower dimension, because its field content is very simple. This is the reason why it is often useful to work with this theory and then, if necessary, to obtain physical results via dimensional reduction. Supergravity in $D = 11$ is the low energy limit of M-Theory; its bosonic fields are:

- The eleven-dimensional metric G_{MN}; on shell this is a symmetric trace-less tensor with 44 d.o.f.
- The 3-form $A^{(3)} = A_{MNP}dx^M \wedge dx^N \wedge dx^P$ (84 d.o.f.), with field strength $F^{(4)} = dA^{(3)}$.

The fermionic content is given by the Majorana gravitino ψ^α_M (128 d.o.f.), with $M, N, P = 0, \cdots 10$. Eleven-dimensional supergravity is a maximal supergravity theory and so the gravity supermultiplet is the only multiplet.
The bosonic part of the action is given by (the fermionic part is fixed by supersymmetry) [8]

$$S_{11} = \frac{1}{2k_{11}^2}\int d^{11}x\sqrt{-G}(R - \frac{1}{2}|F^{(4)}|^2) - \frac{1}{12k_{11}^2}\int A^{(3)} \wedge F^{(4)} \wedge F^{(4)}, \tag{3.37}$$

where $k_{11} \equiv \sqrt{8\pi G_N^{(11)}}$ is the gravitational coupling constant in eleven dimensions. Here and in the following we will avoid writing down the fermionic part of the action. This is because we are interested in supersymmetric solutions with zero fermion fields[2], which of course have vanishing action for the fermionic part. The first term contains the Einstein-Hilbert action and the kinetic term for A_3. The second one is called *Chern-Simons* term, and is required by supersymmetry; note that it does not contain the metric: it is a topological term. The eleven-dimensional gravitational coupling constant k_{11} is related to

[2]With supersymmetryc solutions we mean solutions that are invariant under supersymmetry transformations (BPS). By setting the fermions and their supersymmetry transformations to zero we obtain pure bosonic solutions that are of course invariant under supersymmetry since the bosons transform into fermions.

the Planck length via $k_{11}^2 \sim G_N^{(11)} \sim l_P^9$. This theory has no free dimensionless parameters: there is only one scale, l_P.

3.3 Supergravity in D = 10

3.3.1 Type IIA

Type IIA supergravity can be obtained from eleven-dimensional supergravity by compactifying a coordinate, say $y \equiv x^{10}$, on a circle of radius R (which is a new length scale of the theory). The eleven dimensional metric can be written in terms of a ten-dimensional metric $g_{\mu\nu}$, a 1-form $C^{(1)}$ and a scalar σ (or, equivalently, the dilaton $\Phi \equiv \frac{3}{2}\sigma$) as

$$ds_{11}^2 = ds_{10}^2 + e^{2\sigma}(dy + C_\mu^{(1)} dx^\mu)^2, \tag{3.38}$$

where $\mu, \nu = 0, \cdots 9$. The eleven-dimensional gauge fields can be decomposed into a 2-form $B^{(2)}$ and a 3-form $C^{(3)}$ via

$$A^{(3)} = B^{(2)} \wedge dy + C^{(3)}. \tag{3.39}$$

So we have the correct bosonic fields content of Type IIA theory: the NS-NS sector (Φ,$B^{(2)}, g_{\mu\nu}$) and the R-R sector ($C^{(1)}, C^{(3)}$). The action for Type IIA supergravity can be obtained using (3.37), (3.38) and (3.39) [8]

$$\begin{aligned} S_{IIA} = & \frac{1}{2k_{10}^2} \int d^{10}x \sqrt{-g_{10}} (e^\sigma R_{10} + e^\sigma \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} e^{3\sigma} |F^{(2)}|^2) + \\ & - \frac{1}{4k_{10}^2} \int d^{10}x \sqrt{-g_{10}} (e^{-\sigma} |H^{(3)}|^2 + e^\sigma |\tilde{F}^{(4)}|^2) - \frac{1}{4k_{10}^2} \int B^{(2)} \wedge F^{(4)} \wedge F^{(4)}, \end{aligned} \tag{3.40}$$

where we have introduced the field strengths $F^{(p+1)} = dC^{(p)}, H^{(3)} = dB^{(2)}$, and $\tilde{F}^{(4)} = dC^{(3)} - C^{(1)} \wedge F^{(3)}$. The dilaton Φ is related to the string theory coupling constant by

$$g_s = e^{\Phi_\infty} \tag{3.41}$$

where Φ_∞ is the value of the dilaton at spatial infinity. We can observe that in the strong coupling limit we have $\sigma \to \infty$, i.e. the radius of the 11-th direction y becomes large, which means that we can describe Type IIA theory as a 11-dimensional theory. We say that the strong coupling limit of Type IIA is M-theory.

Written in this frame, the action has an Einstein-Hilbert term that is not written in the canonical form $\sqrt{-g_E} R_E$. To get it in its canonical form we shall move in the so-called

Einstein frame via

$$(g_E)_{\mu\nu} = e^{\frac{\Phi}{6}}(g_{10})_{\mu\nu}. \tag{3.42}$$

This is the frame one shall use when deriving physical results. Another useful frame, the *string frame*, is given by

$$(g_E)_{\mu\nu} = e^{-\frac{\Phi}{2}}(g_S)_{\mu\nu}. \tag{3.43}$$

With this choice the Einstein-Hilbert term reads $\sqrt{-g_S}e^{-2\Phi}$; this is the frame one obtains when one derives the action as the low energy limit of Type IIA superstring theory. In this frame the action reads

$$S_{IIA} = S_{NS-NS} + S_{R-R} + S_{CS}. \tag{3.44}$$

The explicit expressions are:

$$S_{NS-NS} = \frac{1}{2k_{10}^2}\int d^{10}x\sqrt{-g_S}e^{-2\Phi}(R_S + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H^{(3)}|^2), \tag{3.45}$$

$$S_{R-R} = -\frac{1}{4k_{10}^2}\int d^{10}x\sqrt{-g_S}(|F^{(2)}|^2 + |\tilde{F}^{(4)}|^2), \tag{3.46}$$

$$S_{CS} = -\frac{1}{4k_{10}^2}\int B^{(2)} \wedge F^{(4)} \wedge F^{(4)}. \tag{3.47}$$

3.3.2 Type IIB: T-Duality

We have seen that Type IIA supergravity can be obtained by dimensional reducing the eleven-dimensional theory. There is another ten-dimensional supergravity theory that is the low energy limit of Type IIB superstring: Type IIB supergravity. This theory cannot be derived via compactification, but it is related to Type IIA supergravity thanks to a duality between the fields of the two theories: the *T-duality*. As we have seen, if we wrap a IIA string on a circle of radius R it receives a mass contribution in units of $\frac{R}{l_s^2}$ from the winding number and in units of $\frac{1}{R}$ from the momentum modes. We can do the same for a Type IIB string wrapping a circle of radius $\tilde{R}$. It turns out that if $\tilde{R} = \frac{l_s^2}{R}$ the two theories not only have exactly the same spectra (momentum modes map to winding modes and vice versa) but they are also equivalent at the interacting level. To T-dualize the bosonic fields of Type IIA supergravity into those of Type IIB, it is convenient to rewrite the fields as:

$$\begin{aligned} ds^2 &= g_{yy}(dy + A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu \\ B^{(2)} &= B_{\mu y}dx^\mu \wedge (dy + A_\mu dx^\mu) + \hat{B}^{(2)} \\ C^{(p)} &= C_y^{(p-1)} \wedge (dy + A_\mu dx^\mu) + \hat{C}^{(p)} \end{aligned} \tag{3.48}$$

The fields of the corresponding Type IIB supergravity are:

$$
\begin{aligned}
ds'^2 &= g_{yy}^{-1}(dy + B_{\mu y}dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu \\
e^{2\Phi'} &= g_{yy}^{-1}e^{2\Phi} \\
B'^{(2)} &= A_\mu dx^\mu \wedge dy + \hat{B}^{(2)} \\
C'^{(p)} &= \hat{C}^{(p-1)} \wedge (dy + B_{\mu y}dx^\mu) + C_y^{(p)}
\end{aligned}
\tag{3.49}
$$

The NS-NS sector has the same fields in Type IIA and Type IIB supergravity; the R-R sector is again made of p-forms but, in Type IIB, p takes only even values ($p = 0, 2, 4$). Thus the NS-NS term in the action S_{IIA} is valid also for Type IIB supergravity. We have

$$
S_{IIB} = S_{NS-NS} + S_R + S_{CS} \tag{3.50}
$$

where, in the string frame:

$$
S_{NS-NS} = \frac{1}{2k_{10}^2}\int d^{10}x\sqrt{-g_S}e^{-2\Phi}(R_S + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H^{(3)}|^2) \tag{3.51}
$$

$$
S_{R-R} = -\frac{1}{4k_{10}^2}\int d^{10}x\sqrt{-g_S}(|F^{(1)}|^2 + |\tilde{F}^{(3)}|^2 + \frac{1}{2}|\tilde{F}^{(5)}|^2) \tag{3.52}
$$

$$
S_{CS} = -\frac{1}{4k_{10}^2}\int C^{(4)} \wedge H^{(3)} \wedge F^{(3)} \tag{3.53}
$$

Here we have introduced the fields strengths $\tilde{F}^{(3)} = F^{(3)} - C^{(0)} \wedge H^{(3)}$ and $\tilde{F}^{(5)} = F^{(5)} - \frac{1}{2}C^{(2)} \wedge H^{(3)} + \frac{1}{2}B^{(2)} \wedge F^{(3)}$.
Matching of degrees of freedoms requires $\tilde{F}^{(5)}$ to be self dual

$$
\star\tilde{F}^{(5)} = \tilde{F}^{(5)}, \tag{3.54}
$$

this is an additional condition that must be imposed in addition to the equations of motion because there is no straightforward way to incorporate in the action this self-duality condition on a middle rank (i.e. $\frac{D}{2}$-form) field strength. In fact the kinetic term of this field strength is proportional to

$$
\int \tilde{F}^{(5)} \wedge \star\tilde{F}^{(5)} = \int \tilde{F}^{(5)} \wedge \tilde{F}^{(5)} = -\int \tilde{F}^{(5)} \wedge \tilde{F}^{(5)} \quad \Rightarrow \quad \int \tilde{F}^{(5)} \wedge \star\tilde{F}^{(5)} = 0,
$$

where we used (3.3) with $p = q = 5$. So the naive kinetic term of a self-dual $F^{(5)}$ vanishes. The introduction of a Lagrange multiplier field to implement the self-duality condition does not help, because the Lagrange multiplier field itself ends up reintroducing the components it was intended to eliminate. There are several different ways of dealing with the problem of the selfdual field. The original approach is the one we use: we don't construct an action, but only the field equations and the supersymmetry transformations. The equations are highly overconstrained, so one obtains many consistency checks.

3.4 S-Duality

In the strong coupling limit $R_{11} \to \infty$ and the theory decompactifies: the strong coupling limit of Type IIA is M-theory. Obviously, the same argument does not apply to Type IIB theory as it cannot be obtained through dimensional reduction. It turns out that its strong coupling limit is still a Type IIB theory, as it follows from S-duality. S-duality is a duality under which the coupling constant changes non-trivially, and thus it relates different Type IIB theories with different values of the coupling. In its simplest form, it maps the content of one theory with coupling constant g_s, into a dual theory of coupling constant $\frac{1}{g_s}$

$$g_s' = \frac{1}{g_s}. \tag{3.55}$$

From this relation we can relate two Type IIB theories, one with small coupling and one with big coupling. We thus see that the strong coupling limit of Type IIB is again a Type IIB theory. So this duality is important to study the strong coupling limit of Type IIB theory and to generate solutions (one can S-dualize a solution, obtaining another one). The full set of transformations on the type IIB fields is:

$$\begin{aligned} \Phi' &= -\Phi \\ g'_{\mu\nu} &= e^{-\Phi} g_{\mu\nu} \\ B'^{(2)} &= C^{(2)} \\ C'^{(2)} &= -B^{(2)} \end{aligned} \tag{3.56}$$

The other fields ($C^{(0)}$ and $C^{(4)}$) remain unchanged. Changing the sign of the dilaton has the effect of inverting the coupling constant. This duality reflects the symmetry of the theory under the action of the group $SL(2,\mathbb{Z})$ on the axio-dilaton field defined as $\tau \equiv C_0 + e^{-i\Phi}$. In particular the transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$ act as $\tau \to \frac{a\tau+b}{c\tau+d}$ and following the proportionality of the dilaton to the string coupling constant, this is again a weak-strong coupling duality.

3.5 Branes in Supergravity

The fields of supergravity are the massless modes of the string. But strings are not the only fundamental objects of string theory: there are also multidimensional objects, called *branes*, that couple with the p-forms of supergravity and play the role of electric and magnetic charges. Let's review Maxwell theory: the 1-form $A^{(1)}$ couples to a point-particle (which is a 0-dimensional object) with worldline $x^\mu(\tau)$ and charge q through the interaction lagrangian

$$\mathcal{L}_{int} = q\int d\tau A_\mu \frac{dx^\mu}{d\tau} = q\int_\gamma A^{(1)}, \tag{3.57}$$

where γ is the path of the particle. The electric charge of a particle can be computed integrating the Hodge dual of the field strength $\tilde{F}^{(2)} = \star F^{(2)}$ over a 2-sphere surrounding the charge

$$q_e = \int_{S^2} \star F^{(2)}, \qquad F^{(2)} = dA^{(1)}. \tag{3.58}$$

One can also introduce magnetic charges that are monopole sources for the magnetic field. They can be defined as

$$q_m = \int_{S^2} F^{(2)}. \tag{3.59}$$

In supergravity we have p-forms, thus we have to generalize this discussion to multidimensional objects. The interaction lagrangian becomes

$$\mathcal{L}_{int} = q\int_{\gamma_p} A^{(p)}, \tag{3.60}$$

where γ_p is a p-dimensional worldvolume of a $(p-1)$-dimensional object: a $(p-1)$-brane. The analog for the charges can be obtained computing the field strength of $A^{(p)}$, $F^{(p+1)} = dA^{(p)}$ and its Hodge dual $\tilde{F}^{(D-p-1)} = \star F^{(p+1)}$: thus, each p-form couples electrically to a $(p-1)$-brane (with electric charge Q_e) and magnetically to a $(D-p-3)$-brane (with magnetic charge Q_m). The charges can be computed as

$$Q_e = \int_{S^{D-p-1}} \tilde{F}^{D-p-1}, \qquad Q_m = \int_{S^{p+1}} F^{(p+1)}. \tag{3.61}$$

It is natural to include these multidimensional objects in supergravity theories: there must be sources the p-forms couple to. In other words, just as the presence of an electric charge generates the vector potential 1-form $A^{(1)}$, the presence of a p-brane generates a potential p-form.
Solutions to supergravity with non-trivial $A^{(p+1)}$ charge are referred to as p-branes, after the space-dimension of their source. For example in $D = 11$ supergravity the possible

branes are very restricted because the only form of the theory is the 3-form $A^{(3)}$. So we have a 2-brane, denoted $M2$, and its magnetic dual $M5$.
The branes in Type IIA/B theory are further distinguished as follows.

Theory	IIA			IIB		
Forms	$B^{(2)}$	$C^{(1)}$	$C^{(3)}$	$B^{(2)}$	$C^{(2)}$	$C^{(4)}$
Electric	$F1$	$D0$	$D2$	$F1$	$D1$	$D3$
Magnetic	$NS5$	$D6$	$D4$	$NS5$	$D5$	$D3$

Table 3.1: Coupling of branes to p-form potentials in type IIA and IIB Supergravity.

As we have seen, in the presence of a compact direction, there is a 1-form gauge field in the dimensionaly reduced theory. Its electric source is a momentum wave P and the magnetic dual is KK monopole. When the form to which it couples is in the R-R sector, the brane is referred to as a D-brane. On the other hand, the 1-brane that couples to the NS-NS form $B^{(2)}$ is nothing but the fundamental string, denoted $F1$, whose magnetic dual is $NS5$.
The type IIA objects in table (3.1) can be obtained by reducing M-theory charges (M2,M5,KKm and P) along the 11-th direction y:

$$\begin{aligned} &KKm \overset{\perp}{\leftarrow} KKm \overset{\parallel}{\rightarrow} D6 && NS5 \overset{\perp}{\leftarrow} M5 \overset{\parallel}{\rightarrow} D4 \\ &D2 \overset{\perp}{\leftarrow} M2 \overset{\parallel}{\rightarrow} F1 && P \overset{\perp}{\leftarrow} P \overset{\parallel}{\rightarrow} D0 \end{aligned} \tag{3.62}$$

where the arrows denote whether the M-Theory objects point in the M-Theory direction y upon which we reduce ($\|$) or not ($\perp$).
We can also see from eq.s (3.48) and (3.49) how T-duality exchanges the branes. The exchange of the NS-NS fields $B_{\mu y}$ and $g_{\mu y}$ under T-duality corresponds to the transformation of the string winding number (F1) with momentum (P) along the string in the T-duality direction. From the transformation of the R-R fields $C^{(p)}$ we see that the dimension of Dp-branes changes under T-duality depending on whether the transformation is performed on a circle parallel ($\|$) or perpendicular ($\perp$) to the brane worldvolume. In summary:

$$F1 \overset{\parallel}{\leftrightarrow} P \quad KKm \overset{\perp}{\leftrightarrow} NS5 \quad D(p+1) \overset{\perp}{\leftarrow} Dp \overset{\parallel}{\rightarrow} D(p-1) \tag{3.63}$$

Using the T-duality rules and the relation between type IIB and type IIA one can derive the brane content of type IIB given in table (3.1). Finally, from (3.56) it is easy to see how the branes change under S-duality:

$$F1 \leftrightarrow D1 \quad NS5 \leftrightarrow D5 \quad D3 \leftrightarrow D3 \tag{3.64}$$

while KKm and P are unaffected.

3.6 Brane Solutions

Brane solutions are extensively studied solutions of the supergravity field equations, they have a non-perturbative character and, as we saw earlier, they arise as electric and magnetic excitations of the $(p+1)$-form gauge fields that appear in supergravity theories. They are classified as elementary or solitonic, according to whether they are singular or non-singular solutions of the supergravity field equations. A special class of brane solutions are *BPS-brane* solutions; these are supersymmetric solutions, characterised by the saturation of a BPS bound which equates their mass density to the p-form charge(s) they carry. The BPS property 'shields' the brane solutions against quantum corrections and thus, allows the extrapolation of results obtained in the classical limit, to the quantum level of string theory. The field equations of 11-dimensional supergravity admit two BPS brane solutions: an elementary membrane solution $M2$ and a solitonic five-brane solution $M5$, which arise as the electric and magnetic excitations of the 3-form gauge potential respectively.
If we have only closed string we have a maximal supersymmetry (i.e. 32 supercharges). Each brane carries open strings which require boundary conditions relating left and right modes and thus reducing by half the supersymmetries. We say that each brane is realized as a $\frac{1}{2}$BPS solution in supergravity. The geometry of these solutions will be important, and we describe it now. A p-brane has a $(p+1)$-dimensional flat hypersurface, with Poincaré invariance group $SO(1,p) \ltimes \mathbb{R}^{p+1}$. The transverse space is then of dimension $D-p-1$ and solutions may always be found with maximal rotational symmetry $SO(D-p-1)$ in this transverse space. Thus, p-branes in supergravity may be thought of as solutions with symmetry groups:

$$\begin{aligned}
D = 11 \quad &\rightarrow \quad SO(1,p) \ltimes \mathbb{R}^{p+1} \times SO(10-p) \\
D = 10 \quad &\rightarrow \quad SO(1,p) \ltimes \mathbb{R}^{p+1} \times SO(9-p)
\end{aligned}$$

There are two different methods to derive solutions:

- The first one consists in solving the equations of motion of the supergravity theory. As it happens for Einstein equations, this is difficult in general. However, the presence of symmetries in the brane configuration and supersymmetry simplify the task. In this contest, BPS solutions are obtained imposing constrains on the supersymmetry transformations of the fields. If we require that the configuration of the fields is supersymmetric, the fields should be invariant under a supersymmetry transformation δ_ϵ. Bosonic fields transform into fermionic ones and, when the latter are set to zero, bosonic fields are invariant. Consistency requires that also the supersymmetric variation of the fermions vanishes, leading to the BPS equations. These are typically first order equations and thus simpler than the equations of motion, that are second order.

- The second method starts from some simple neutral solution, and derives other solutions by means of boosts, T-duality and S-duality. One can add charges to the starting solution, to get a BPS solution, making boosts along a compact direction (the charge of the Kaluza-Klein gauge boson is the momentum in the direction of the compact dimension). The resulting metric is still a supergravity solution because the supergravity action is Lorentz invariant; but yet it is another solution because the boost direction is compact: the boost is not a globally defined change of coordinates and we are constructing a different solution. We will see later that, in this contest, a BPS solution can be obtained imposing the extremality condition (i.e. taking the so called BPS-limit $M \to 0$ and $\beta \to \infty$, where M is the mass and β the rapidity of the boost). The charge added by the boost is always momentum, but it can be transformed into all other possible charges by appropriate chains of S and T duality.

Now we are going to use both methods to find some solutions following [11],[12].

3.7 Solutions Generation: Some Examples

3.7.1 Direct Method

The first solution we are looking for is that corresponding to a M2 brane in 11-dimensional supergravity. We know that such solution should exist, because the 3-form $A^{(3)}$ naturally couples to this type of brane. We want this brane to extend over the directions x^i with $i = 1, 2$ and to be perpendicular to the directions x^a with $a = 3, \cdots, 10$. In the presence of a membrane, the initial Poincaré invariance in eleven dimensions is reduced to $P_3 \times SO(8)$ invariance i.e. Poincaré invariance in the flat world-volume of the membrane and rotational invariance in the trasverse to the membrane directions. So let us start with the following ansatz

$$\begin{cases} ds^2 = Z(r)(-dt^2 + dx^i dx^i) + Y(r) dx^a dx^a \\ A^{(3)} = X(r) dt \wedge dx^1 \wedge dx^2 \\ \psi^\alpha_\mu = 0, \end{cases} \tag{3.65}$$

where we used the fact that the 3-form gauge field couples electrically to the world-volume of the membrane. Here we have assumed that the solution would depend only on the three functions X, Y and Z, and $SO(8)$ invariance requires that these functions depend only on the radial coordinate $r = \sqrt{(x^a)^2}$, where we have understood a sum over a from 3 to 10. We will now see that a solution of this form is really allowed in 11 dimensional supergravity. First we want this solution to be invariant under supersymmetry transformations (BPS condition); we will not derive their exact form, but only state that they are the following:

$$\begin{aligned} \delta e_\mu^A &= \bar{\epsilon}\gamma^A\psi_\mu \\ \delta A_{\mu\nu\rho} &= -3\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} \\ \delta\psi_\mu &= D_\mu\epsilon + \frac{1}{288}({\gamma_\mu}^{\nu\rho\sigma\tau}F_{\nu\rho\sigma\tau} - 8\gamma^{\nu\rho\sigma}F_{\mu\nu\rho\sigma})\epsilon \end{aligned} \tag{3.66}$$

where, for simplicity, we have understood all spinor indices, and the local Lorentz frame index A runs from 0 to 10. A γ with more than one index must be intended as the antisymmetric product of γ matrices. The field strength $F = dA$ has only few non-trivial components, that are

$$F_{a12t} = \partial_a X(r). \tag{3.67}$$

The fact that our solution has a vanishing gravitino, implies that δe_μ^A and $\delta A_{\mu\nu\rho}$ automatically vanish. Thus we must only check that also the variation of the gravitino is zero. We remember that the covariant derivative of a spinor is defined in terms of the spin connection

$$D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}{\omega_\mu}^{AB}\gamma_{AB}\epsilon. \tag{3.68}$$

Our goal is to derive the spin connection, and we do this using the torsion-free condition (3.34). From the definig property of vielbein fields (3.19) and from our ansatz (3.65), we have

$$e^t = \sqrt{Z(r)}dt, \quad e^i = \sqrt{Z(r)}dx^i, \quad e^a = \sqrt{Y(r)}dx^a. \tag{3.69}$$

From the torsion-free condition, remembering that $\omega_{AB} = -\omega_{BA}$ and $\omega_{AB} = \eta_{AC}\omega^C{}_B$, we have

$$\begin{aligned}
&de^t + \omega^t{}_i \wedge e^i = \partial_a\sqrt{Z(r)}dx^a \wedge dt + \omega^t{}_a \wedge \sqrt{Y(r)}dx^a = 0\\
&\Rightarrow \omega_{ta} = -\frac{1}{\sqrt{Y(r)}}\partial_a\sqrt{Z(r)}dt,\\
&de^i + \omega^i{}_a \wedge e^a = \partial_a\sqrt{Z(r)}dx^a \wedge dx^i + \omega^i{}_a \wedge \sqrt{Y(r)}dx^a = 0\\
&\Rightarrow \omega_{ia} = \frac{1}{\sqrt{Y(r)}}\partial_a\sqrt{Z(r)}dx^i,\\
&de^a + \omega^a{}_b \wedge e^b = \partial_b\sqrt{Y(r)}dx^b \wedge dx^a + \omega^a{}_b \wedge \sqrt{Y(r)}dx^b = 0\\
&\Rightarrow \omega_{ab} = \frac{1}{\sqrt{Y(r)}}\partial_b\sqrt{Y(r)}dx^a - \frac{1}{\sqrt{Y(r)}}\partial_a\sqrt{Y(r)}dx^b \quad a \neq b.
\end{aligned} \tag{3.70}$$

We now have to impose that the variation of the gravitino vanishes. Let us do it explicitly for the index $\mu = 1$

$$0 = \partial_1\epsilon + \frac{1}{4}(\omega_1)_{AB}\gamma^{AB}\epsilon + \frac{1}{288}(4!\gamma_{\hat{1}}{}^{\hat{a}\hat{1}\hat{2}\hat{t}}F_{a12t} - 8\cdot 3!\gamma^{\hat{2}\hat{a}\hat{t}}F_{12at})\epsilon \tag{3.71}$$

where hatted indices of the gamma matrices should be intended as curved indices: one should express all in terms of gamma matrices with flat indices, by means of the appropriate vielbein. Since ϵ inherits the symmetries of the geometry it is indipendent of (t, x^i); thus both the first and the third term trivially vanish and we are left with

$$0 = (\frac{1}{2\sqrt{Y}}\partial_a\sqrt{Z}\gamma^{1a} - \frac{1}{6\sqrt{Y}}Z^{-1}\partial_a X\gamma^{2at})\epsilon. \tag{3.72}$$

Multiplying this expression with γ^{1a} and using the Clifford algebra we arrive at

$$\frac{1}{3Z}\partial_a X\gamma^{012}\epsilon = \partial_a\sqrt{Z}\epsilon. \tag{3.73}$$

This is a sort of projection equation for the spinor ϵ: in fact we have that $(\gamma^{012})^2 = \mathbb{I}$. Thus it must be $\gamma^{012}\epsilon = \pm\epsilon$: these two possibilities are both possible, and correspond to a brane and its anti-brane. Here we choose the $+$ sign, and the equation reduces to

$$X(r) = Z(r)^{\frac{3}{2}}. \tag{3.74}$$

Solving the same equation for $\mu = a$ one gets a link between the functions Y and X, in particular we have

$$0 = \partial_a \epsilon + (\frac{1}{4\sqrt{Y}} \partial_a \sqrt{Y} \gamma^{ab} - \frac{1}{6\sqrt{Y}} Z^{-1} \partial_a X \gamma^{012}) \epsilon. \tag{3.75}$$

The second term is again a projector for the spinor ϵ and taking again the $+$ sign we obtain

$$\partial_a \epsilon + \partial_a \sqrt{Y} \epsilon = -\frac{1}{3} Z^{-1} \partial_a Z^{\frac{3}{4}} \epsilon. \tag{3.76}$$

The well-defined solution of this equation, using (3.74), leads to

$$X(r) = Y(r)^{-3}. \tag{3.77}$$

This is all what we can say just using the supersymmetry. In order to go further, we cannot avoid solving an equation of motion, which we choose to be the equation of motion of the form A. It is simply the generalisation of Maxwell's equations in 4-dimensional electrodynamics, i.e.

$$d \star F = 0. \tag{3.78}$$

The Hodge dual of F is

$$(\star F)_{\mu_1 \cdots \mu_7} = \sqrt{-g} \epsilon_{\mu_1 \cdots \mu_7}{}^{\mu_8 \cdots \mu_{11}} F_{\mu_8 \cdots \mu_{11}}. \tag{3.79}$$

Thus we get $\star F = X^{-2} \partial_a X dx^{a_1} \wedge \cdots \wedge dx^{a_7}$, with $a_1 \cdots a_7 \neq a$. The equation of motion is then equivalent to the Laplace equation for X^{-1}

$$\partial^a \partial_a X^{-1} = 0. \tag{3.80}$$

The solution of this equation is an harmonic function in 8 dimensions. We thus write $X^{-1} = 1 + \frac{Q}{r^6}$, where the adding constant is fixed requiring that the metric is at at infinity. The constant Q is precisely the electric charge corresponding to the $M2$ brane. We rewrite here the complete solution we have found:

$\frac{1}{2}$BPS M2 solution

$$\begin{cases} ds^2 = X(r)^{\frac{2}{3}} (-dt^2 + dx^i dx^i) + X(r)^{-\frac{1}{3}} dx^a dx^a \\ A^{(3)} = X(r) dt \wedge dx^1 \wedge dx^2 \\ \psi_\mu^\alpha = 0 \end{cases} , \quad X(r) = \left(1 + \frac{Q}{r^6}\right)^{-1} \tag{3.81}$$

The fact that the metric and the $A^{(3)}$ field depend on one single function $X(r)$ is due to the presence of supersymmetry. However, for an asymptotically flat geometry the tt component of the metric is related to the mass of the object generating that solution. Having a single function that determines the metric and the 3-form means that there is a precise relation between the mass and the charge of our solution as always happens in BPS solutions. The recipe in order to extract the ADM mass from an asymptotically flat geometry in arbitrary dimensions is given for example in [13]. In the asymptotic regime our solution become

$$ds^2 \approx \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} = \frac{1}{r^6}\left[-\frac{2}{3}Q(-dt^2 + dx^i dx^i) + \frac{1}{3}Q dx^a dx^a\right] \tag{3.82}$$

and following [13] the mass M is related to h_{tt} by $(G = 1)$

$$h_{tt} \approx \frac{16\pi M}{A_7 r^6} \quad \Rightarrow \quad M = \frac{\pi^3 Q}{72}, \tag{3.83}$$

where $A_7 = \frac{\pi^4}{3}$ is the area of a unit 7-sphere. Once we have this $M2$ solution, it is quite simple to derive suitable solutions of Type IIA supergravity via dimensional reduction. There are two ways to do so, compactifying a coordinate x^i or a coordinate x^a. Let us first choose $y = x^1$; we have

$$ds^2_{11} = ds^2_{10} + e^{2\sigma}(dy + C^{(1)}_\mu dx^\mu)^2 \quad \mu \neq 1, \quad A^{(3)} = B^{(2)} \wedge dy + C^{(3)}, \qquad \sigma = \frac{2}{3}\Phi \tag{3.83}$$

So, we find the following solution (where we have made a slight change of notation $X(r)^{-1} \equiv Z(r)$):

$\frac{1}{2}$<u>BPS F1 solution</u>

$$\begin{cases} ds^2 = Z(r)^{-1}(-dt^2 + dx^2 dx^2) + dx^a dx^a \\ e^{\Phi} = Z(r)^{-\frac{1}{2}} \\ B^{(2)} = -Z(r)^{-1} dt \wedge dx^2 \\ C^{(p)} = 0 \ \ (p = 1, 3) \end{cases} \tag{3.84}$$

Here we have already turned to the string frame, following (3.42),(3.43) $ds^2 = e^{\sigma} ds^2_{10}$. This solution corresponds to a $F1$ fundamental string parallel to the x^2 direction because $B^{(2)}_{tx^2} \neq 0$. Obviously this is a Type IIA solution, because it was obtained via dimensional reduction. Obtaining this solution was straightforward, in that our $M2$ brane was parallel to the x^1 direction; therefore the brane was invariant under translations along x^1. If we now want to do the same for a direction perpendicular to the $M2$ brane, say x^3, we get some difficulties, because we have one single $M2$ brane located at $x^3 = 0$. The problem can be solved noting that the Laplace equation is linear, and so we can safely take a superposition of branes at different locations as a correct supersymmetric solution. Physically speaking, this is allowed because of the equality of mass and charge, that balances the attracting gravitational force and the repulsive gauge force between parallel branes. Suppose that we make a superposition of many branes, each one at position $x_i^3 = y_i$ and with charge Q; then the X function becomes

$$X(r)^{-1} = 1 + Q \sum_i \frac{1}{|\vec{x} - \vec{x}_i|^6}. \tag{3.85}$$

Defining $r' = \sum_{a=4}^{10} (x^a)^2$, and letting the branes be continuously distributed along x^3, we have

$$X(r)^{-1} = 1 + Q \int_{-\infty}^{\infty} \frac{dy}{[r'^2 + (x^3 - y)^2]^3} = 1 + \frac{Q'}{r'^5}, \tag{3.86}$$

where Q' is proportional to Q (it is not important the right proportionality coefficient). Thus the solution corresponding to an infinite superposition of $M2$ branes is formally identical to the previous one with X a harmonic function in a 7-dimensional transverse space. We can now safely make a dimensional reduction along the x^3 direction. The result is a solution corresponding to a $D2$ brane parallel to the directions x^1 and x^2 which, when expressed in string frame, reads:

$\frac{1}{2}$BPS D2 solution

$$\begin{cases} ds^2 = Z(r)^{\frac{1}{2}} \left[-dt^2 + (dx^1)^2 + (dx^2)^2\right] + Z(r)^{\frac{1}{2}} (dx^a)^2. \\ e^{\Phi} = Z(r)^{\frac{1}{4}} \\ B^{(2)} = 0 = C^{(1)} \\ C^{(3)} = Z(r)^{-1} dt \wedge dx^1 \wedge dx^2 \end{cases} , \quad Z(r) = 1 + \frac{Q}{r^5}, \tag{3.87}$$

Here r is the radial direction in the 7 dimensional space orthogonal to the brane. Starting from these solutions for the $F1$ and the $D2$ brane, we can use T and S dualities in order to find other supergravity solutions in 10 dimensions. For example we can find the $D3$ solution making a T duality along, say, x^3 of (3.87) obtaining:

$\frac{1}{2}$BPS D3 solution

$$\begin{cases} ds^2 = Z(r)^{-\frac{1}{2}} \left[-dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + Z(r)^{\frac{1}{2}} (dx^a)^2. \\ e^{\Phi} = Z(r)^{-\frac{1}{4}} \\ C^{(4)} = Z(r)^{-1} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{cases} , \tag{3.88}$$

where now $a = 4, ..., 10$. We will return to this solution in the next chapter when we discuss our application of the AdS/CFT correspondence.

3.8 Indirected Method

As anticipated, there is another method that can be used to derive the same supergravity solutions. One starts from a well known, neutral solution, and applies symmetries and dualities to construct the desired brane solutions. Let us now see an example. Let's consider a 10-dimensional spacetime, with topology $\mathbb{R}^{1,4} \times S^1 \times T^4$. Let's denote with (t, x^i) the coordinates on the non-compact directions, with y the coordinate on the circle and with z^a the coordinates on the 4-dimensional torus. The starting point is the Schwarzshild-Tangherlini metric in the $\mathbb{R}^{1,4}$ directions, tensored trivially with $S \times T^4$

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3 + dy^2 + \sum_a (dz^a)^2, \tag{3.89}$$

where $G = 1$ and we have used polar coordinates in the 4 non-compact spatial direction

$$x^1 = r \sin\theta \cos\phi \quad x^2 = r \sin\theta \sin\phi \quad x^3 = r \cos\theta \cos\psi \quad x^4 = r \cos\theta \sin\psi \tag{3.90}$$

with $\theta \in [0, \frac{\pi}{2}]$ and $\phi, \psi \in [0, 2\pi]$. This solution is the generalisation of the Schwarzschild solution, thus it is for sure a solution of the Einstein equations in vacuum, and then a supergravity solution if all gauge fields vanish. It can also be seen both as a Type IIA or IIB solution, because all the gauge fields are trivial.

3.8.1 The 1-charge Geometry

In our solution we have a mass M but no charge. To obtain a BPS solution we can add a charge by performing a boost along the direction y of the circle

$$y' = y \cosh\alpha + t \sinh\alpha \quad t' = t \cosh\alpha + y \sinh\alpha, \tag{3.91}$$

where $\alpha \in (-\infty, \infty)$ is the rapidity. Renaming $y' \equiv y$, $t' \equiv t$ the metric becomes

$$\begin{aligned} ds^2 = &\left(1 + \frac{2M \sinh^2\alpha}{r^2}\right) dy^2 + \left(-1 + \frac{2M \cosh^2\alpha}{r^2}\right) dt^2 + \\ &+ 2\cosh\alpha \sinh\alpha \frac{2M}{r^2} dydt + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3 + dy^2 + (dz^a)^2. \end{aligned} \tag{3.93}$$

This solution corresponds to a wave carrying momentum P_y. Let's now apply a T-duality, then result will be a solution of Type IIB supergravity describing a fundamental string wrapping the circle, $F1_y$. We rewrite the metric in the form (3.48)

$$ds^2 = \left(1 + \frac{2M\sinh^2\alpha}{r^2}\right)\left[dy + \frac{\cosh\alpha\sinh\alpha 2M/r^2}{1+2M\sinh^2\alpha/r^2}dt\right]^2 + \left(1 + \frac{2M\sinh^2\alpha}{r^2}\right)^{-1}\left(-1+\frac{2M}{r^2}\right)dt^2 + \left(1-\frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3 + dy^2 + (dz^a)^2$$

and applying the correspondence of eq. (3.49) with $B_{\mu y} = 0$, we obtain:

$$\begin{cases} ds^2 = S_\alpha^{-1}\left[dy^2 + \left(-1+\frac{2M}{r^2}\right)dt^2\right] + \left(1-\frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3 + dy^2 + (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ B^{(2)} = \frac{2M}{r^2}\cosh\alpha\sinh\alpha S_\alpha^{-1} dt\wedge dy \end{cases} \tag{3.94}$$

where $S_\alpha \equiv \left(1+\frac{2M\sinh^2\alpha}{r^2}\right)$. We define the charge of this solution by taking the BPS condition, i.e. we take the limit

$$M \to 0, \quad \alpha\to\infty \quad \text{such that} \quad Me^{2\alpha} = 2Q, \tag{3.95}$$

where Q is the charge of F1. In this limit $S_\alpha \to 1 + \frac{Q}{r^2}$ and the solution in the string frame becomes:

$\frac{1}{2}$BPS $F1_y$ solution

$$\begin{cases} ds^2 = Z(r)^{-1}(-dt^2+dy^2) + dr^2 + r^2 d\Omega_3 + (dz^a)^2 \\ e^{2\Phi} = Z(r)^{-\frac{1}{2}} \\ B^{(2)} = -Z(r)^{-1}dt\wedge dy \end{cases}, \quad Z(r) = 1+\frac{Q}{r^2} \tag{3.96}$$

As we expected this is exactly the $F1_y$ solution (3.84) we found with direct method but in polar coordinates.

3.8.2 The 2-charge Geometry

To add a second charge we can proceed as before and perform a second boost, with rapidity β. Note however that a boost acts trivially on a BPS solution, thus we must go back to the non-extremal solution (3.89). In this method, the BPS limit has to be taken only at the end of the chain of boost and dualities. The result describes a string $F1_y$ wrapped in the y direction carrying momentum P_y

$$\begin{cases} ds^2 = S_\alpha^{-1} S_\beta (dy + \frac{2M\cosh\alpha\sinh\alpha/r^2}{1+2M\sinh\beta/r^2} dt)^2 + S_\alpha^{-1} S_\beta^{-1} (-1 + \frac{2M}{r^2}) dt^2 + \\ +(1 - \frac{2M}{r^2})^{-1} dr^2 + r^2 d\Omega_3 + dy^2 + (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ B^{(2)} = \frac{2M}{r^2}\cosh\alpha\sinh\alpha S_\alpha^{-1} dt\wedge dy \end{cases} \tag{3.97}$$

where S_β is defined with the same structure of S_α. The BPS limit gives:
$\frac{1}{4}$BPS $F1_y P_y$ solution

$$\begin{cases} ds^2 = Z_1(r)\left[-dt^2 + dy^2 + K(r)(dt+dy)^2\right] + dr^2 + r^2 d\Omega_3 + (dz^a)^2 \\ e^{2\Phi} = Z(r)^{-\frac{1}{2}} \\ B^{(2)} = -Z(r)^{-1} dt\wedge dy \end{cases} \tag{3.98}$$

with $Z_1(r) = 1 + \frac{Q_1}{r^2}$ and $K(r) = Z_P - 1 = \frac{Q_P}{r^2}$.
We can obtain the $\frac{1}{4}$-BPS solution in an other duality frame: an interesting one is when the charges are $D1_y D5_{yT4}$. Starting from the $F1_y P_y$, the $D1D5$ frame can be reached performing a chain of dualities. They are schematically:

$$(F1_y P_y) \xrightarrow{S} (D1_y P_y) \xrightarrow{T \text{ along } T^4} (D5_{yT^4} P_y) \xrightarrow{S} (NS5_{yT^4} P_y)$$

$$\xrightarrow{T \text{ along } y} (NS5_{yT^4} F1_y) \xrightarrow{T \text{ along } z_1 \text{ and } S} (D5_{yT^4} D1_y)$$

Note that the $D5$-brane has been constructed out of the fundamental string $F1$ and thus its charge Q_5' is related to the charge Q_1 (and to the boost parameter α). Analogously the charge of $D1$ Q_1' derives from Q_P. The explicit D1-D5 solution can be obtained by applying the duality rules explained above. We skip the details of the calculation, and only give the final result:
$\frac{1}{4}$BPS $D1_y D5_{yT_4}$ solution

$$\begin{cases} ds^2 = Z_1(r)^{-\frac{1}{2}} Z_5(r)^{-\frac{1}{2}} (dy^2 - dt^2) + Z_1(r)^{\frac{1}{2}} Z_5(r)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_3) + Z_1(r)^{-\frac{1}{2}} Z_5(r)^{\frac{1}{2}} (dz^a)^2 \\ e^{2\Phi} = Z_1(r)^{-1} Z_5(r) \\ C^{(2)} = -Q_5' \sin^2\theta d\phi\wedge d\psi + (1 - Z_5(r)^{-1}) dt\wedge dy \end{cases} \tag{3.99}$$

with $Z_1(r) = 1 + \frac{Q_1'}{r^2}$ and $Z_5(r) = 1 + \frac{Q_5'}{r^2}$.
The geometries we have generated are guaranteed to be solutions of the supergravity equations of motion (BPS or not) carrying the allowed charges of string theory. Their

microscopic meaning, however, has to be clarified. Note that, due to the singularities of the various harmonic functions, all the solutions generated until now are singular at $r = 0$. Singularities are not necessarily a reason to discard a solution, as far as they correspond to allowed microscopic sources: think for example at the well-known singularity of the Coulomb potential of a point-like charge. Can the singularity of the F1-P solution in (3.94) be linked to a fundamental string? The answer is no: that solution should represent a string wrapped along the y circle and carrying momentum through a wave travelling along y. However, since fundamental strings have no physical longitudinal vibration modes, the momentum must be carried by vibrations transverse to y. This should make the string bend away from its central axis and the singularity cannot be confined at $r = 0$. The allowed microscopic solution for a vibrating string carrying momentum can be constructed by specifying a transverse displacement profile $F(t-y)$ and it is singular along this profile. See [11], [12] for more details. Since the D1-D5 solution (3.95) has been obtained via a chain of dualities from an unphysical one, one expects that even that solution does not describe an allowed microscopic configuration of string theory. The fact that in (3.95) the tt component of the metric vanishes at $r = 0$, suggests that that solution might represent an extremal black hole with a horizon at $r = 0$. Even this interpretation is not completely correct, since one can check that the area of the horizon vanishes for the metric in (3.95). That solution thus represents a "degenerate" black hole, also knows as small black hole, with a singular horizon of vanishing area.

In conclusion, we will refer to the solutions constructed in this chapter as "naive" solutions: they carry the expected global charges, but they do not described the actual microscopic configurations of string theory. The main goal of this thesis is to construct solutions carrying D3 charge with a precise microscopic interpretation.

4

The AdS/CFT Conjecture

The Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence, is one of the most important recent ideas that arose in theoretical physics, providing new ways of performing calculations where more conventional methods are intractable. The correspondence, roughly speaking, states the equivalence between a string theory containing gravity living in a certain geometry, and a gauge theory living on the boundary of that geometry. Here 'equivalence' means that there is a one-to-one correspondence between all aspects of the theories including the global symmetries, observables, and correlation functions. The theories are thus considered to be dual descriptions of each other; this notion of duality is an interesting one because it turns out that the regimes within which it is possible to perform calculations easily do not coincide on the two sides of the correspondence. In other words, when one theory is strongly coupled (and, thus, it is difficult to treat) the dual one is weakly coupled, and vice versa. In this chapter we will introduce the conjecture starting with a brief introduction of the two sides: CFT and AdS geometry. A CFT is a field theory with the conformal symmetry, which is a specific type of symmetry that describes how the theory remains the same under transformations that preserve angles but not lengths. On the other hand the AdS geometry is the maximally symmetrical solution of the Einstein equations with a negative curvature. As we'll see AdS geometry has the same isometry as conformal group and this will be very important in the context of the AdS/CFT correspondence. After this general introduction we'll describe the basic properties of a particular CFT, namely the $\mathcal{N} = 4$ CFT, which will be the one we will focus on the most in the future. After that we'll motivate the correspondence starting with the "large N limit" which was historically the first link to be found between $SU(N)$ gauge theories and string theories and then we'll give a stronger motivation through what is called the "open/closed duality". Over time, numerous dualities have been found between different theories. For our purposes, we will focus uniquely on the original duality due to Maldacena [14] between the 10-dimensional Type IIB superstring theory on the product space $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with gauge group $SU(N)$, living on the 4-dimensional boundary of AdS_5.

4.1 A Brief Introduction to CFT

4.1.1 The Conformal Transformations

Conformal Field Theories (CFTs) are the theories invariant under transformations of the conformal group. The conformal group is the group of local transformations $x \to x'$ which leaves the metric invariant up to an arbitrary scale factor

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \tag{4.1}$$

Therefore geometrically a conformal transformation is locally equivalent to a (pseudo) rotation and a dilatation (i.e. transformation that preserve angles). The conformal group has the Poincaré group as a subgroup, since the latter corresponds to the special case $\Omega(x) = 1$. It also include, for example, dilatation $x^\mu \to \lambda x^\mu$ and inversion $x^\mu \to x^\mu / x^2$. If we follow an inversion by a translation by b and a second inversion, we arrive at the special conformal transformation

$$x^\mu \to \frac{x^\mu + x^2 b^\mu}{1 + 2b \cdot x + b^2 x^2}, \tag{4.2}$$

which, in contrast to the inversion, can be expanded around the identity.
Under an infinitesimal transformation $x^\mu \to x^\mu + \xi^\mu(x)$ the metric, at first order in ξ, changes as follows

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu). \tag{4.3}$$

The requirement that the transformation be conformal implies that

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = f(x) g_{\mu\nu}. \tag{4.4}$$

Taking the trace of this equation gives $f(x) = \frac{2}{D}(\partial \cdot \xi)$ and we have

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D}(\partial \cdot \xi) g_{\mu\nu}. \tag{4.5}$$

One may show that in $d > 2$ the most general solution to this Killing equation is

$$\xi_\mu = a_\mu + \omega_\mu{}^\nu x_\nu + \lambda x_\mu - 2(b \cdot x) x_\mu + x^2 b_\mu, \tag{4.6}$$

where λ is a constant, a_μ and b_μ are constant vectors and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ a constant antisymmetric matrix. They parametrize infinitesimal traslations, Lorentz transformations, rescalings and special conformal transformations of x^μ. So we have a total of

$d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+1)(d+2)}{2}$ parameters.

4.1.2 The Conformal Algebra

The conformal transformations are generated by $P_\mu = \partial_\mu$, $J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$, $D = x \cdot \partial$, $K_\mu = -2x_\mu x \cdot \partial + x^2 \partial_\mu$ whose algebra is easily worked out. The non-vanishing commutators are:

$$[D, P_\mu] = -P_\mu \tag{4.7}$$

$$[D, K_\mu] = K_\mu \tag{4.8}$$

$$[P_\mu, K_\nu] = -2g_{\mu\nu} D + 2J_{\mu\nu} \tag{4.9}$$

$$[J_{\mu\nu}, P_\rho] = -g_{\mu\rho} P_\nu + g_{\nu\rho} P_\mu \tag{4.10}$$

$$[J_{\mu\nu}, K_\rho] = -g_{\mu\rho} K_\nu + g_{\nu\rho} K_\mu \tag{4.11}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = -g_{\mu\rho} J_{\nu\sigma} - g_{\nu\sigma} J_{\mu\rho} + g_{\mu\sigma} J_{\nu\rho} + g_{\nu\rho} J_{\mu\sigma} \tag{4.12}$$

So K_μ, P_μ are Lorentz vectors and D a Lorentz scalar. Furthermore (4.7),(4.8) shows that P_μ and K_μ are raising and lowering operators respectively for the dilatation operator D. Finally one can also interpret D as reading off the length dimension of the other operators since P_μ, K_μ and $J_{\mu\nu}$ have length dimensions $-1, +1, 0$ respectively.
If one defines $M_{\mu\nu} = J_{\mu\nu}$, $M_{d(d+1)} = -D$, $M_{\mu d} = \frac{1}{2}(P_\mu - K_\mu)$ and $M_{\mu(d+1)} = -\frac{1}{2}(P_\mu + K_\mu)$ the above commutation relations can be combined into the following single relation

$$[M_{ab}, M_{cd}] = -\eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}, \tag{4.13}$$

where $a, b, c, d = 0, 1, \cdots, d+1$ and $\eta_{ab} = \mathrm{diag}(-1, +1, \cdots, +1, -1)$ is the invariant metric of $SO(d, 2)$. This establishes the isomorphism of the conformal algebra of d-dimensional Minkowski space with $so(d, 2)$, the Lie algebra of $SO(d, 2)$[1] ($O(d, 2)$ if we include inversion).

Transformation	Infinitesimal	Finite	Generator
Translation	$x^\mu + a^\mu$	$x^\mu + a^\mu$	$P_\mu = \partial_\mu$
Lorentz	$x^\mu + \omega^{\mu\nu} x_\nu$	$\Lambda^\mu{}_\nu x^\nu$	$J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$
Dilatation	$x^\mu + \lambda x^\mu$	λx^μ	$D = x \cdot \partial$
Special	$x^\mu + b^\mu x^2 - 2(b \cdot x) x_\mu$	$\frac{x^\mu - x^2 b^\mu}{1 - 2(b\cdot x) + b^2 x^2}$	$K_\mu = -2x_\mu x \cdot \partial + x^2 \partial_\mu$

Table 4.1: Summary of conformal transformations

[1] Here we assume $d > 2$. In $d = 2$ the conformal algebra is infinite dimensional.

4.1.3 General Features of CFT

Under a Poincaré transformation a field operator $\Psi_A(x)$ transform as

$$\Psi_A(x) \to U(\Lambda)_{AB}\Psi_B(\Lambda^{-1}(x-a)), \tag{4.14}$$

where A is the representation index. For a conformal transformation we have to specify also how the field operator transform under a scale transformation. Under a dilatation $x \to \lambda x$ we have

$$\Psi_A^{\Delta} \to \lambda^{-\Delta}\Psi_A^{\Delta}(\lambda^{-1}x), \tag{4.15}$$

where Δ is the *scaling dimension* (or conformal dimension) of the operator. It is an eigenvalue of the dilatation operator

$$[D, \Psi_A^{\Delta}] = -\Delta\Psi_A^{\Delta}. \tag{4.16}$$

We mentioned previously that P_μ and K_μ act as raising and lowering operators for the dilatation operator D. We can see this by considering an operator Ψ^{Δ} of conformal dimension Δ and finding the conformal dimension of $[P_\mu, \Psi^{\Delta}]$. Using (4.7),(4.16) we have

$$[D, [P_\mu, \Psi^{\Delta}]] = -(\Delta+1)[P_\mu, \Psi^{\Delta}], \tag{4.17}$$

showing that $[P_\mu, \Psi^{\Delta}]$ has dimension $\Delta + 1$ as claimed. An analogous proof shows that $[K_\mu, \Psi^{\Delta}]$ has dimension $\Delta-1$. For a representation to be unitary the conformal dimension must have a lower bound (for scalar fields $\Delta \geq (d-2)/2$ which is the dimension of a free scalar field), an thus there must be an operator in the representation of lowest dimension (i.e. that is annihilated by K_μ). The lowest-dimensional operators are called *primary operators*; they are defined by the condition

$$[K_\mu, \mathcal{O}]_\pm = 0. \tag{4.18}$$

All the other fields are obtained by the action of P_μ on primary fields and they are called *descendant fields*. The primary operators define the full representation of the algebra which is classified by the dimension Δ and by the spin j of these operators (i.e. the Casimir of the conformal group).

One of the basic properties of conformal field theories is the one-to-one correspondence between local operators $\mathcal{O}$ and state $|\mathcal{O}\rangle$ in the radial quantization theory. If we consider the conformal theory in the Euclidean space the conformal group is $SO(d+1,1)$, and since $\mathbb{R}^d$ is conformally equivalent to S^d (after adding a point at infinity) the field on $\mathbb{R}^d$ is isomorphic to the theory on S^d. In radial quantization the time coordinate is chosen to be the radial direction in S^d, with the origin corresponding to past infinity, so that the field theory lives on $\mathbb{R} \times S^{d-1}$. An operator $\mathcal{O}$ can then be mapped to the state

$$|\mathcal{O}\rangle = \lim_{x \to 0} \mathcal{O}(x) |0\rangle . \tag{4.19}$$

All states in the theory can be created by operators which act locally in a small neighborhood of the origin.That is to say that the entire Hilbert space of a CFT can be thought of as living at a single point. The inverse mapping of states to operators proceeds by taking a state which is a functional of field values on some ball around the origin and using conformal invariance to shrink the ball to zero size.
Classically, a field theory is conformally invariant if there are no dimensionful couplings constant in the action (e.g. mass terms); this is intuitive, since a dimensionful coupling constant sets a scale, thereby breaking scale invariance. Upon quantization however, conformal invariance may be broken due to the renormalization process, which introduces dimensionful constants into the theory. A necessary condition for a theory to be conformally invariant quantum mechanically is the vanishing of the renormalization group beta functions

$$\beta \equiv \mu \frac{\partial g}{\partial \mu}, \tag{4.20}$$

where g is a coupling of the theory and μ the renormalisation scale.

4.2 $\mathcal{N} = 4$ Super Yang-Mills Theory

As we have seen in Section 1.2.1 non gravitational theories with $\mathcal{N} = 4$ are maximally supersymmetric. So in $\mathcal{N} = 4$ SYM the gauge multiplet is the only possible multiplet. It is given by

$$(A_\mu, \lambda^a_\alpha, X^i) \tag{4.21}$$

where A_μ is a spin-1 gauge field (μ is the index which transform in the $(\frac{1}{2}, \frac{1}{2})$ representation of $SO(3,1)$), λ^a_α ($a = 1, ..., 4$) are four complex Weyl spinors (in the $\bar{\mathbf{4}}$ of $SU(4)_R$), and X^i ($i = 1, ..., 6$) are six real scalars (in the $\mathbf{6}$ of $SU(4)_R$). Under the global R-symmetry group $SU(4)_R \simeq SO(6)_R$ these transform as a singlet, a vector, and a rank-2 antisymmetric tensor respectively. All fields transform in the adjoint representation of the $SU(N)$ gauge group. The lagrangian for the so-called $\mathcal{N} = 4$ super Yang-Mills theory is given by

$$\begin{aligned}\mathcal{L} =&\mathrm{Tr}(-\frac{1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\sigma_a\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu X^i D^\mu X^i \\ &+ g_{YM}\sum_{a,b,i} C_i^{ab}\lambda_a[X^i, \lambda_b] + g_{YM}\sum_{a,b,i}\bar{C}_{iab}\bar{\lambda}^a[X^i, \bar{\lambda}^b] + \frac{g_{YM}^2}{2}\sum_{i,j}[X^i, X^j]^2),\end{aligned} \tag{4.22}$$

where g_{YM} is the coupling constant, θ_I is the so-called instanton angle, $F_{\mu\nu}$ is the dual field-strength of the gauge field, D_μ is the usual covariant derivative, $\tilde{F}$ is the Hodge dual of F, and C_i^{ab} are the structure constants of $SU(4)_R$. The trace is over the gauge indices and is to ensure gauge invariance of the action. This theory is classically conformally invariant since $[g_{YM}] = [\theta_I] = 0$. More strikingly, upon quantisation one finds that the theory is UV finite; since no renormalisation scale is needed, the β-function vanishes to all orders and thus the theory remains conformally invariant at the quantum level.
The combination of conformal symmetry $SO(2,4) \simeq SU(2,2)$, $\mathcal{N} = 4$ supersymmetry and R-symmetry are part of a larger symmetry group, that is the superconformal group $SU(2,2|4)$. Superconformal algebra in addition to generators of the conformal group and supercharges, contains also the so-called *conformal supercharges* S_α^I with their complex conjuagte $\bar{S}_{\dot{\alpha}}^I$. These are required to close the superconformal algebra $[K, Q] \sim S$. In addition the theory exhibits a further $SL(2,\mathbb{Z})$ symmetry, i.e. it is invariant under S-duality. This can be seen by using the two coupling constants of the theory g_{YM}, θ_I to define

$$\tau \equiv \frac{\theta_I}{2\pi} + \frac{4\pi i}{g_{YM}^2}. \tag{4.23}$$

The action of the theory is invariant under $\tau \to \frac{a\tau+b}{c\tau+d}$ with $ad-bc = 1$ and $a, b, c, d \in \mathbb{Z}$. As always this symmetry is very useful for studying large-coupled theory known the perturbative regime and as the other things will feature later in the AdS/CFT correspondence.

4.3 Anti-de Sitter Space

4.3.1 Definition of Anti-de Sitter Space

A *maximally symmetric* space of d-dimensions has the maximum number of Killing vectors, namely $\frac{d(d+1)}{2}$ (d translations and $\frac{d(d-1)}{2}$ rotations). The Riemann curvature tensor for this spaces can be written as

$$R_{\mu\nu\rho\sigma} = C(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{4.24}$$

for some constant C, and thus one finds by contracting that $R_{\mu\nu} = (d-1)Cg_{\mu\nu}$ and $R = d(d-1)C$, i.e. maximally symmetric spaces have constant curvature scalars.

We define the *anti-de Sitter* (AdS) space as a space of Lorentzian signature and constant negative curvature. In a similar fashion to other constant curvature spaces, AdS space may be defined as an embedding in a higher-dimensional space. If we consider a flat embedding space $\mathbb{R}^{2,d-1}$ with coordinates X_a $(a = 0, ..., d)$ and metric

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2, \tag{4.25}$$

then we may define AdS_d as the hyperboloid

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = R^2, \tag{4.26}$$

where R is known as the *AdS radius. Euclidean AdS* may be defined in an analogous way, but embedded in $\mathbb{R}^{1,d}$ and with the defining equation

$$X_0^2 - X_d^2 - \sum_{i=1}^{d-1} X_i^2 = R^2. \tag{4.27}$$

It is obvious from the defining equations that the isometry group of lorentzian AdS_d is $O(2, d-1)$ (or $O(1, d)$ for the Euclidean case); so for example the isometry group of AdS_5 is $O(2, 4)$. Since the dimension of $O(2, d-1)$ is $\frac{d(d+1)}{2}$ we see that AdS_d is indeed a maximally symmetric space. By eliminating the final coordinate via $(X^d)^2 = R^2 + \eta_{\mu\nu} X^\mu X^\nu$, where $\mu = (0, 1, ..., d-1)$ and $\eta_{\mu\nu}$ is the d-dimensional Minkowski metric, we may provide a set of coordinates for AdS_d and write the metric as

$$ds^2 = \left(\eta_{\mu\nu} - \frac{\eta_{\mu\lambda}\eta_{\nu\rho} X^\lambda X^\rho}{X \cdot X + R^2} \right) dX^\mu dX^\nu. \tag{4.28}$$

Calculating the Riemann tensor with this metric we get $C = -\frac{1}{R^2}$, therefore AdS_d has constant negative curvature scalar.

4.3.2 Coordinate Systems on AdS

Let us for convenience now set $R = 1$. We may introduce a set of coordinates on AdS_d by writing:

$$\begin{aligned} X_0 &= \tilde{r} \cos t \\ X_d &= \tilde{r} \sin t \\ X_i &= r x_i \end{aligned} \tag{4.29}$$

where $\sum_{i=1}^{d-1} x_i^2 = 1$, and the other coordinates range over $\tilde{r}, r > 0$ and $t \in [0, 2\pi)$. The defining equation (4.26) then clearly implies $\tilde{r}^2 - r^2 = 1$. Using (4.25) we thus find

$$ds^2 = -d\tilde{r}^2 - \tilde{r}^2 dt^2 + dr^2 + r^2 d\Omega_{d-2}^2. \tag{4.30}$$

Using the constraint $\tilde{r}^2 - r^2 = 1$ we find that $d\tilde{r}^2 = \frac{r^2}{\tilde{r}^2} dr^2$ and thus simple algebra gives the metric

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{d-2}^2. \tag{4.31}$$

We have thus eliminated $\tilde{r}$ and now have a set of d coordinates for AdS_d. We see that t acts as a time coordinate, yet from it's definition in (4.29) this coordinate appears to be periodic. To avoid the existance of closed timelike curves and causal inconsistencies, we thus unwrap the time coordinate (technically, we move to the universal cover) and simply define the space AdS_d by equation (4.31) (which is, after all, a solution to the Einstein field equations with negative cosmological constant) for $t \in \mathbb{R}$. Note that the this metric has the same form as the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2, \tag{4.32}$$

but here $f(r) = 1 + r^2 > 0$, and thus we see that the anti-de Sitter space does not have an event horizon.

We now make a further coordinate transformation in (4.31) given by $r = \sinh \rho$ for $\rho > 0$. Using $1 + r^2 = \cosh^2 \rho$ we easily find

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-2}^2. \tag{4.33}$$

These are known as *global coordinates* (so-called because they cover the entire AdS space) and are the coordinates that we'll use in the future. Finally, we can make instead a different coordinates substitution in (4.31) given by $r = \tan \beta$ for $\beta \in [0, \pi/2)$. Using $1 + r^2 = \sec^2 \beta$ we find the metric

$$ds^2 = \frac{1}{\cos^2 \beta}(-dt^2 + d\beta^2 + \sin^2 \beta d\Omega_{d-2}^2) = \frac{1}{\cos^2 \beta}(-dt^2 + d\Omega_{d-1}^2). \tag{4.34}$$

These are known as *conformal coordinates*, so called because AdS_d is conformally equivalent to the cylinder $\mathbb{R} \times S^{d-1}$.

4.3.3 The Conformal Boundary of AdS

In the conformal coordinates metric (4.34) the coordinate β clearly plays the role of a latitude; strangely, however, we saw from its definition that it ranges over the values $\beta \in [0, \frac{\pi}{2})$ rather than the usual $[0, \pi]$. So the spatial part really only covers the northern hemisphere and not the full sphere. Thus, after a conformal transformation and neglecting t, we have a hemisphere of S^{d-1} with boundary at the equator that of course is topologically equivalent to the ball B^{d-1}. Since $\partial(B^{d-1}) = S^{d-2}$ which we commonly associate with $\mathbb{R}^{d-2}$ with spatial infinity identified as a single point, we arrive at the important result (taking the time coordinate into account)

$$\partial(AdS_d) = \mathbb{R}^{1,d-2}, \tag{4.35}$$

or, if one includes the point at infinity

$$\partial(AdS_d) = \mathbb{R}_t \times S^{d-2}. \tag{4.36}$$

So AdS_d is bounded by Minkowski space $\mathbb{R}^{1,d-2}$. This result is of crucial importance in the AdS/CFT correspondence since it is at the heart of its holographic nature.

4.3.4 Poincaré Coordinates

We now introduce one further set of coordinates for AdS_d which are particularly useful in the AdS/CFT corrispondence. We will here restore the radius R. We introduce the coordinates $y > 0$ and $(t, \vec{x}) \in \mathbb{R}^{d-1}$ via:

$$\begin{aligned} X_0 &= \frac{1}{2y}[1 + y^2(R^2 + \vec{x}^2 - t^2)] \\ X_d &= Ryt \\ X_{d-1} &= \frac{1}{2y}[1 - y^2(R^2 - \vec{x}^2 + t^2)] \\ X_i &= Ryx_i \end{aligned} \tag{4.37}$$

where $(i = 1, ..., d-2)$ and $\vec{x}^2 = \sum_{i=1}^{d-2} x_i^2$. These coordinates satisfy (4.27) and give the metric

$$ds^2 = \frac{R^2}{y^2}dy^2 + \frac{y^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{4.38}$$

where $x^\mu = (t, \vec{x})$. Making the coordinate substitution $u = \frac{R^2}{y}$ we find

$$ds^2 = \frac{R^2}{u^2}(du^2 + \eta_{\mu\nu}dx^\mu dx^\nu), \tag{4.39}$$

which is the metric in *Poincaré coordinates*. So we see that AdS_d is conformally equivalent to Minkowski space $\mathbb{R}^{1,d-1}$. The slices of constant u are copies of Minkowski space $\mathbb{R}^{1,d-2}$, in particular the conformal boundary is given by the slice $u = 0$ (i.e. $y \to \infty$). Finally, in these coordinates one can also see that the dilatations

$$\begin{aligned} u &\to \lambda u, \\ x^\mu &\to \lambda x^\mu, \end{aligned} \tag{4.40}$$

for any $\lambda \in \mathbb{R}$ form an isometry of AdS space.

4.4 The Large N Limit

QCD is a gauge theory based on the $SU(3)$ group, where 3 is the number of colors. While the gauge theory description is very useful for studying the high-energy behavior of the strong interactions, it is very difficult to use it to study low-energy regime. The difficulty stems from the lack of a small, dimensionless parameter which we can use as the basis for a perturbative expansion. Soon after the advent of QCD, 't Hooft pointed out that gauge theories based on the group $SU(N)$ simplify in the limit $N \to \infty$ (despite the large number of degrees of freedom), and have a perturbation expansion in terms of the parameters $1/N$. First, we need to understand how to scale the coupling g_{YM} as we take $N \to \infty$. The confinement and the mass gap all occur at the strong coupling scale Λ_{QCD}, so it is natural to scale g_{YM} so that Λ_{QCD} remains constant in the large N limit. The beta function equation for pure $SU(N)$ YM theory is

$$\mu\frac{dg_{YM}}{d\mu} = -\frac{11}{3}N\frac{g_{YM}^3}{16\pi^2} + o(g_{YM}^5), \tag{4.41}$$

where μ is the renormalisation scale. So the leading terms are of the same order for large N if we take $N \to \infty$ while keeping $\lambda \equiv g_{YM}^2 N$ fixed (one can show that the higher terms are also of the same order in this limit). We thus have the *t'Hooft limit*

$$N \to \infty, \quad \lambda \equiv g_{YM}^2 N \text{ fixed}. \tag{4.42}$$

This ensures that the physical scale Λ_{QCD} also remains fixed and this limit can also be applied to theories with $\beta = 0$, like the $\mathcal{N} = 4$ SYM theory introduced before. Let's see more closely at the Feynman diagrams that arise from the Yang-Mills action

$$S_{YM} = -\frac{1}{2g_{YM}^2}\int d^4x \mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) = -\frac{N}{2\lambda}\int d^4x \mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{4.43}$$

Each gluon field is an $N \times N$ matrix

$$\left(A_\mu\right)^i{}_j \qquad i, j = 1, ..., N \tag{4.44}$$

and the propagator has the index structure

$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y)\rangle = \Delta_{\mu\nu}(x-y)\left(\delta^i_l\delta^k_j - \frac{1}{N}\delta^i_j\delta^k_l\right), \tag{4.45}$$

where $\Delta_{\mu\nu}(x)$ is the usual photon propagator for a single gauge field and the $1/N$ term arises because we're working with trace-less $SU(N)$ gauge fields. At leading order in $1/N$ we have

$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y)\rangle = \Delta_{\mu\nu}(x-y)\delta^i_l\delta^k_j. \tag{4.46}$$

The fact that the gauge field has two indices i, j suggests that we can represent it as two lines in a Feynman diagram rather than one. Each line comes with an arrow, and the arrows point in the opposite ways. This reflects the fact that the upper and lower lines are associated to complex conjugate representations ($[(A_\mu)^i{}_j]^\dagger = (A^*_\mu)_j{}^i$).

Figure 4.1

The propagator scales as λ/N as can be read off from the action (4.43). We have also the cubic and the quartic coupling vertex; each vertex come with a factor N/λ.
The general scaling will be

$$\text{diagram} \sim \left(\frac{\lambda}{N}\right)^{\#\text{propagators}}\left(\frac{N}{\lambda}\right)^{\#\text{vertices}} N^{\#\text{index contractions}} \tag{4.47}$$

where the index contractions come from the loops in the diagram. It turns out that, among all the possible Feynman diagrams, a subset dominate in the large N limit. The dominant diagrams are those which can be drawn flat on a plane in the double line notation. These are referred to as *planar diagrams*. The key idea is that the planar diagrams can all be drawn on the surface of a sphere. In contrast, the non-planar diagrams must be drawn on higher genus surfaces, for example a torus. In general we have

$$\begin{aligned} E &\equiv \#\text{ of edges} = \#\text{ of propagators} \\ F &\equiv \#\text{ of faces} = \#\text{ of index loops} \\ V &\equiv \#\text{ of vertices} \end{aligned}$$

and from (4.47) a given diagram scales as

$$\text{diagram} \sim N^{F+V-E}\lambda^{E-V}. \tag{4.48}$$

The following combination determines the topology of the Riemann surface Σ

$$\chi(\Sigma) = F + V - E, \tag{4.49}$$

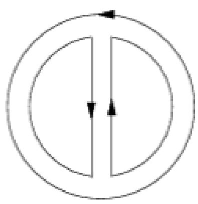
where χ is called *Euler character* and it only depends on the topology of Σ. It is related to the number of handles H of the Riemann surface, also called the *genus*, by

$$\chi(\Sigma) = 2 - 2H. \tag{4.50}$$

The sphere has $H = 0$, the torus has $H = 1$ and so on. In this way, the large N expansion is a sum of Feynman diagrams, weighted by their topology

$$\text{diagram} \sim N^{\chi}\lambda^{E-V}. \tag{4.51}$$

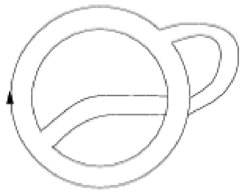
As an example of a planar diagram we can consider the vacuum bubbles



$$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N^3 \sim \lambda N^2$$

Figure 4.2: Vacuum bubbles: an example of a planar diagram

which using (4.47) scales as λN^2. Instead a non-planar diagram is



$$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N \sim \lambda$$

Figure 4.3: A non planar diagram cannot be drawn on a surface with $H = 0$

which scales as λ and therefore in the large N limit doesn't contribute; more precisely the relationship between the amplitudes is $A_{\text{non-planar}}/A_{\text{planar}} = N^{-2}$. This is true in general for all non planar diagrams.
The fact that in this limit the perturbation theory is based on the topology of the Riemann surface on which the Feynman diagram rests was the first hint of some sort of link between field theories and string theory. In fact also in string theory the perturbation theory is based on the Riemann surface topology. In particular the sum over Riemann surfaces is weighted by the string coupling constant; by analogy we have

$$g_s = \frac{1}{N}. \tag{4.52}$$

We have thus seen that, in the t'Hooft limit of non-abelian gauge theories, perturbative string theory seems to provide a dual description of the guage theory's perturbation expansion. This is just an idea: nothing tells us which gauge theory is associated with which string theory. As we will see, the AdS/CFT correspondence realizes this idea.

4.5 The Open/Closed String Duality

Now that we have introduced the fundamental elements, let us motivate the conjecture that we will enunciate in the next section. One of the strongest motivations for believing the AdS/CFT correspondence (and the original one, due to Maldacena [14]) is to consider it as a realization of the open/closed string duality. We have seen that superstring theories contain multidimensional objects: D-branes. On one hand, these objects are considered to be dynamical hyperplanes upon which the endpoints of open strings are fixed (but are free to move parallel to the brane). On the other hand, D-branes are massive objects and therefore can be considered as sources for closed strings; one can then consider closed string propagating in such a background. That these points of view are equivalent is of great importance, since by considering a particular physical set-up from each in turn, we shall see that (in certain limits) there are two decoupled theories in both interpretations; by recognising a common theory present, we are then led to identify the two theories as equivalent or dual descriptions, which is exactly the AdS/CFT correspondence mentioned above. We now discuss the important subject of how gauge theories arise on the worldvolumes of D-branes. Then we consider the set-up from the open and closed strings points of view in turn.

4.5.1 Gauge Theories on the Worldvolumes of D-branes

Let's first consider D-branes from the open string prospective. As we have seen in Section 2.1.3, the quantization of the theory gives an open string spectrum that can be identified with fluctuations of the brane. For a single D-brane, the massless spectrum consists in scalar field ϕ_i describing fluctuations of the brane in the transverse direction and a $U(1)$ gauge field A_μ that lives on the brane. If we consider a stack of N coincident branes then we must further label the string states by indices which denote which brane the endpoints lie on. Open strings that have both endpoints on the same brane form $U(1)$ gauge fields as before, so that we have an overall gauge group $U(1)^N$; we will denote the gauge fields with $(A_\mu)^a{}_a$, where the upper (lower) index labels the brane on which the string starts (ends). We can also have strings that have endpoints on different branes $(A_\mu)^a{}_b$ (with $a \neq b$) that are mass-less gauge fields if the branes are coincident. In this case the resulting theory is a non-Abelian gauge theory with gauge group $U(N)$. The $U(N)$ gauge group is equivalent to $U(1) \times SU(N)$; the diagonal $U(1)$ degree of freedom describes the motion of the branes' center of mass (i.e. rigid motion of the entire system of branes); we are not interested in this trivial type of motion and we will focus only on the $SU(N)$ gauge group.
D-brane breaks one half of the 32 supersymmetries of the $D = 10$ $\mathcal{N} = 2$ superstring theory and so, in particular, for a stack of N D3-branes, the brane dynamics is described by $D = 4$ $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$.

4.5.2 The Open String Point of View

From the open string point of view, the action describing the physical set-up has the form

$$S = S_{\text{bulk}} + S_{\text{branes}} + S_{\text{bulk-branes}}, \tag{4.53}$$

where S_{bulk} is the ten-dimensional supergravity action, S_{brane} is the brane action and $S_{\text{bulk-branes}}$ describes the interaction between the branes and the bulk theory that scales with Newton's constant $\sqrt{G_N} \sim g_s \alpha'^2$. In the low energy limit $\alpha' \to 0$ and we thus see that the interaction term drops out. We remain with two decoupled theories

$$(\text{brane theory}) \oplus (\text{bulk theory}) \tag{4.54}$$

The particular brane and bulk theories depend on the D-brane. If we consider as before a stack of N D3-branes we have

$$(D = 4\,\mathcal{N} = 4\,\text{SYM}) \oplus (\text{Type IIB SUGRA}) \tag{4.55}$$

If we introduce the t'Hooft coupling $\lambda \equiv g_{YM}^2 N$, for $\lambda \ll 1$ we recover the weakly coupled Yang-Mills theory, where the perturbative expansion is reliable.

4.5.3 The Closed String Point of View

Now we consider the same system from a different point of view. D-branes are massive charged objects which act as a source for the various supergravity fields. The D3-brane supergravity solution as derived in chapter 3 is

$$ds^2 = H^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \tag{4.56}$$

where x^μ are the coordinates parallel to the brane. We also have

$$H = 1 + \frac{R^4}{r^4}, \tag{4.57}$$

where if we consider N branes the scale factor R is related to the string coupling costant by $R^4 = 4\pi g_s \alpha'^2 N$. Note that the supergravity description is valid when the curvature radius (which is set by the scale R) is large compared to the string length l_s since otherwise string effects are important and cannot be ignored. So the useful regime is given by $R \sim \sqrt{\alpha'}(g_s N)^{\frac{1}{4}} \gg l_s \sim \sqrt{\alpha'}$ and thus we require $\lambda \equiv g_s N \gg 1$. We thus see that this is the opposite regime to the one in which the gauge theory description is useful.
In the limit $r \gg R$ the solution (4.54) becomes that of 10-dimensional flat Minkowski space. Instead, in the *near horizon limit* $r \ll R$ the metric becomes

$$ds^2 \to \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2. \tag{4.58}$$

Using (4.38) we see that this is nothing but the metric for the product geometry $AdS_5 \times S^5$, where the radius for both parts of the geometry is R.
Since (4.56) becomes flat at $r \to \infty$, the coordinate t is the proper time for an observer at infinity. In contrast, the proper time for an observer at some other point is given by $\Delta\tau = \sqrt{-g_{tt}}\Delta t$, and correspondingly the energies are related by $E = \frac{1}{\sqrt{g_{tt}}}E_\infty$. In particular, close to the brane we have (4.58) $E_\infty = \frac{rE}{R}$ and thus the energy as observed at infinity goes to zero as $r \to 0$. For an observer at infinity there are two decoupled low

energy regimes:

- 10-dimensional supergravity close to the observer, since gravity becomes free at low energies/large distances.
- Full Type IIB string theory close to the branes (i.e. in the geometry $AdS_5 \times S^5$); due to the large red-shift, everything (i.e. all strings) becomes a low energy effect close to the branes for an observer at infinity, and thus there is no need to restrict to low energy massless modes.

We thus have two decoupled theories:

$$(\text{Type IIB String theory on } AdS_5 \times S^5) \oplus (\text{Type IIB Supergravity in 10D}) \tag{4.59}$$

4.6 Statement of the Correspondence

We finally reach the celebrated AdS/CFT correspondence; by looking at (4.55) and (4.59) we see that both from the point of view of a field theory of open strings living on the brane, and from the point of view of the supergravity description, we have two decoupled theories in the low energy limit. In both case one of the decoupled systems is supergravity in flat space. So, it natural to identify the second system which appears in both descriptions:

$$(\text{Type IIB String theory on } AdS_5 \times S^5) \equiv (D = 4\, \mathcal{N} = 4\, \text{SYM}) \tag{4.60}$$

The fact that D-branes have a dual interpretation has led us to identify these two theories as dual descriptions of each other. Although not generally regarded as a proof, the decoupling argument provides strong motivation for the above correspondence.
The parameters g_s, R of the superstring theory are related to the parameters g_{YM}, N of the flat theory on the brane by

$$g_s = g_{YM}^2, \qquad R^4 = 4\pi g_s N \alpha'^2. \tag{4.61}$$

The second equation as we have seen is related to the fact that we consider the geomtry of a stack of N branes, while the first comes from the fact that the closed string coupling constant is the square of the open string coupling constant. So we have the following relations

$$\lambda \equiv g_{YM}^2 N = g_s N = \frac{R^4}{4\pi\alpha'^2} = \frac{R^4}{4\pi l_s^4}. \tag{4.62}$$

We see that the supergravity regime $R^4/l_s^4 \gg 1$ and the perturbative field theory regime $\lambda \ll 1$ are perfectly incompatible. This is the reason that this correspondence is called a

"duality". The two theories are conjectured to be exactly the same, but when one side is weakly coupled the other is strongly coupled and vice versa.
Let's now consider the holographic nature of the correspondence. We saw in Section 4.3.3 that AdS_5 has a boundary given by conformally invariant Minkowski space $\mathbb{R}^{1,3}$. So, it is in fact possible to identify the branes as being in some sense on the boundary of AdS_5, and thus the gauge theory (which lives on the branes) can be said to live on the boundary of AdS_5. This is the sense in which the correspondence is a holographic principle, since the 5-dimensional dynamics of Type IIB theory (after compactification on S^5) can be encoded in a gauge theory living on the 4-dimensional boundary.
This is the strong form of the correspondence as it is supposed to hold for all values of the coupling constant. However this strong form is difficult to check due to the need of defining the string theory on curved manifolds such as $AdS_5 \times S^5$. We can state some slightly less general forms. First we can note that it's not possible to get into the gravity regime by taking N small and g_s very large because this would give a very quantum gravity theory. So, it is always necessary, but not sufficient, to have large N in order to have a weakly coupled supergravity description. We have the *t'Hooft form* of the correspondence, by going to the t'Hooft limit

$$N \to \infty, \quad g_{YM} \to 0, \quad \lambda \text{ fixed.} \tag{4.63}$$

In the gauge theory side this corresponds to the perturbation theory topological expansion in $1/N$. On the string theory side one has a classical Type IIB string theory with small coupling $g_s = \lambda/N$. Finally we have the *weak form* of the correspondence which, after taking the t'Hooft limit, involves taking the large λ limit. This corresponds to the strong coupling (i.e. non perturbative) regime on the gauge theory, whereas on the string theory side we have a classical Type IIB supergravity, with an expansion in small α'. This final form turns out to be extremely powerful, since one may use classical gravity to perform calculations in the non-perturbative gauge theory.

4.7 The Symmetry Map

As a first check of the correspondence we can show that there is a one to one map between the symmetries of the two theories. In fact the bosonic part of $D = 4$ $\mathcal{N} = 4$ SYM is invariant under the conformal transformations in the conformal group $SO(4,2)$ and the $SU(4)_R \sim SO(6)_R$ R-symmetry group. On the other hand the isometry group of AdS_5 is $SO(4,2)$ and for S^5 it is $SO(6)$. Furthermore SYM theory has also the S-duality simmetry group $SL(2, \mathbb{Z})$, the same of Type IIB string theory. Finally, the $\mathcal{N} = 4$ theory has 16 supersymmetries (Q) + 16 conformal symmetries (S) and the D3-brane breaks precisely half of the Poincaré supersymmetries (i.e. 16 of the 32). On the other hand in $AdS_5 \times S^5$ near-horizon limit we have as usual 32 supersymmetries. We can therefore conclude that the whole supergroup $SU(2, 2|4)$ is the same for the $\mathcal{N} = 4$ field theory and the $AdS_5 \times S^5$ geometry.

$\mathcal{N} = 4$ SYM	IIB on $AdS_5 \times S^5$
Conformal group $O(4, 2)$	Isometry group $O(4, 2)$ of AdS_5
R-symmetry $SU(4)_R$	Isometry group $SO(6) \simeq SU(4)$ of S^5
Supersymmetries = 16Q + 16S	32 supersymmetries

Table 4.2: Symmetry map between the two theories

5

Chiral Operators and the Holographic Dictionary

As described in the previous chapter, the AdS/CFT correspondence is a strong/weak coupling duality. Since that correspondence is a conjecture without a formal proof, one must test it by computing physical quantities such as correlation functions. This task is generally not possible to do since we can only compute physical quantities perturbatively in λ on the field theory side and perturbatively in $\frac{1}{\sqrt{\lambda}}$ on the string theory side. It turns out that there are several properties of some supersymmetric theories (such as the $\mathcal{N}=4$ SYM theory itself) which do not depend on the coupling λ, so they can be compared to test the duality. In this chapter we'll look at some of these quantities, namely the chiral primary operators (CPOs) of the $\mathcal{N}=4$ SYM theory. These special operators form a short (or chiral) multiplet of $SU(2,2|4)$ whose dimension is "protected" from the quantum corrections. Later we will see how to associate a field in AdS to each of these kind operator, i.e. we'll give the holographic dictionary.

5.1 Chiral Operators

The operator spectrum of the $\mathcal{N}=4$ SYM theory consists of all possible gauge invariant combinations of the elementary fields. Since the theory is (super)conformal we can limit our analysis to the primaries. The descendants will be obtained by acting on the primaries with the appropriate operators.

In a superconformal algebra the special conformal transformations K_μ do not commute with the supercharges Q. Since both are symmetries, their commutator must also be a symmetry, and these are the special supersymmetry transformations S^I_α with their complex conjugates $\bar{S}^I_{\dot{\alpha}}$. The dimensions of the generators of the full superconformal algebra are the following:

$$[D]=[J_{\mu\nu}]=0,\ \ [P^\mu]=+1,\ \ [K^\mu]=-1,\ \ [Q]=+1/2,\ \ [S]=-1/2 \tag{5.1}$$

A *superconformal primary operator* generalizes the idea of a conformal primary operator given in section 4.1.3. It is defined by

$$[S, \mathcal{O}]_{\pm} = 0, \tag{5.2}$$

and it's the lowest dimension operator in a given superconformal multiplet. It is important to distinguish a superconformal primary operator from a conformal primary operator. Since K^{μ} reduce the dimension by 1 and S by 1/2, the (5.2) is a stronger condition than (4.18). Note also that since $\{S, \bar{S}\} \sim K$, the condition (5.2) implies (4.18). In addition, a *superconformal descendant operator* $\mathcal{O}$ can be written as

$$\mathcal{O} = [Q, \mathcal{O}']_{\pm}. \tag{5.3}$$

Again, since P^{μ} raises the dimension by 1 and Q by 1/2, this condition is stronger than that for the conformal descendant operators defined in section 4.1.3. We can observe that $\mathcal{O}$ can never be a primary operator since the dimensions are related by $\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}'} + 1/2$. So, a superconformal primary operator is not the Q-commutator of another operator. Since the actions of the supercharges on the canonical fields are:

$$\begin{aligned}
&[Q^A_{\alpha}, \phi^I] \sim \lambda_{\alpha B} \\
&\{Q^A_{\alpha}, \lambda_{\beta B}\} \sim (\sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu} + \epsilon_{\alpha\beta}[\phi^I, \phi^J] \\
&\{Q^A_{\alpha}, \bar{\lambda}^B_{\dot{\beta}}\} \sim (\sigma^{\mu})_{\alpha\dot{\beta}} \mathcal{D}_{\mu}\phi^I \\
&[Q^A_{\alpha}, A_{\mu}] \sim (\sigma_{\mu})_{\alpha\dot{\alpha}} \bar{\lambda}^A_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}
\end{aligned} \tag{5.4}$$

a superconformal primary operator can involve neither the gauginos λ nor the gauge field A. Moreover it can involve neither derivatives nor commutators of ϕ. As a result, superconformal primary operators are gauge invariant scalars involving only traces of ϕ's. The simplest are the *single trace operators*, which are of the form

$$\mathcal{O}^{I_1 \dots I_n} \equiv \mathrm{Tr}(\phi^{I_1} \dots \phi^{I_n}). \tag{5.5}$$

Since in (5.4) the commutators of ϕ's appear on the right side, if some of the indices are antysimmetric the field will be a descendant (because we can write him in term of commutators). Thus, only symmetric combinations of the indices $(I_1, \dots, I_n)$ will be primary operators. In the AdS/CFT correspondence one is interested in the operators whose dimension does not depend on the scale λ. These are the *chiral primary operators*, which are in short representation of the superconformal algebra (this happens if they are annihilated by some of the supercharges Q). In analogy to what happens in standard extended supersymmetry, where the mass of the short or BPS multiplets is determined by their quantum numbers, chiral primary operators are also called *BPS states*. In general, using the superconformal algebra, one can show that the simplest class of $\frac{1}{2}$BPS states is formed by the operators (5.5) for symmetric and traceless combinations of the indices $(I_1, ..., I_n)$ [15]. They form a representation of weight $(0, n, 0)$ of $SU(4)_R$ and their conformal dimension is simply

$$\Delta = n. \tag{5.6}$$

In the case $n = 2$ we have

$$\sum_I \mathrm{Tr}(\phi^I \phi^I) \to \text{Konishi multiplet}, \tag{5.7}$$

that is the lowest component of a long (unprotected) multiplet, called the Konishi multiplet, and

$$\mathrm{Tr}(\phi^I \phi^J) - \frac{1}{6}\delta_{IJ}\mathrm{tr}(\phi^I \phi^J) \to \text{supergravity multiplet}, \tag{5.8}$$

that is the CPO of a short (protected) multiplet, called supergravity multiplet.
One can find the form of all fields in such a multiplet by using the algebra (5.4) starting from this primary. Short multiplets have an important status in the AdS/CFT correspondence which we have already mentioned: they have a "protected" conformal dimension. In fact, given a CPO $\mathcal{O}_n$, we have

$$0 = [Q, \mathcal{O}_n] = [K, \mathcal{O}_n] \;\; \Rightarrow \;\; 0 = [S, \mathcal{O}_n] \sim [[K, Q], \mathcal{O}_n]. \tag{5.9}$$

So, using the superconformal algebra $[Q, S] = J + D + R$ we obtain

$$0 = [[Q, S], \mathcal{O}_n] \sim [J + D + R, \mathcal{O}_n] = (\Sigma - \Delta + R)\mathcal{O}_n, \tag{5.10}$$

where $\Sigma = 0$ is the spin of the chiral primary operator and R it's $SU(4)_R$ quantum number. So Δ can only take discrete values and therefore does not depend on the parameter λ of the theory. For this reason CPO's remain in the spectrum at $\lambda \to \infty$ and so they are dual to supergravity fields. Therefore these operators allow a reliable comparison between quantities computed in the bulk versus quantities derived in the CFT.

5.2 The Field-Operator Map

As we just explained, there must be a map between individual fields on AdS_5 and (chiral) operators in the CFT. We now construct this map. For simplicity we only consider scalar

fields on AdS but the argument can be generalised.
A basic point of the correspondence due to Witten [16] is the following statement: a scalar field ϕ_0 in AdS_5 is associated with an operator $\mathcal{O}$ in the CFT via the following boundary coupling

$$\int_{\text{boundary}} \phi_0 \mathcal{O}. \tag{5.11}$$

We can think of the source ϕ_0 as the boundary value of a five dimensional field ϕ in AdS_5. Let's now consider the equation of motion for the field ϕ in AdS_5

$$(\Box_{AdS_5} - m^2)\phi(x) = 0, \tag{5.12}$$

for some mass m on AdS_5. Using Poincaré coordinates of AdS_5 (4.37) with $R = 1$ we have

$$[u^2\partial_x^2\phi + u^5\partial_u(\frac{1}{u^3}\partial_u\phi) - m^2]\phi = 0. \tag{5.13}$$

In Fourier space $x \to ip$ this becomes

$$[-u^2p^2 + u^5\partial_u(\frac{1}{u^3}\partial_u\phi) - m^2]\phi = 0. \tag{5.14}$$

Let us see the asymptotic behavior at the boundary, $u \sim 0$. The term with momentum can be neglected, and the solutions are power-law

$$\phi \sim u^{\alpha_\pm}, \quad \alpha_\pm = 2 \pm \sqrt{4 + m^2} \tag{5.15}$$

The solution with α_- dominates as $u \to 0$, and the solution with α_+ always decays. Since the one with α_- could diverge we then impose

$$\phi(x, u)_{u=\epsilon} = \epsilon^{\alpha_-}\phi_0^{ren}(x). \tag{5.16}$$

In such a way when we send $\epsilon \to 0$ the solution in the bulk has finite limit: $\phi_0^{ren}(x)$ is a renormalized boundary condition. If we perform a rescaling of coordinates in the boundary theory, which is the AdS isometry

$$x \to \lambda x, \quad u \to \lambda u, \tag{5.17}$$

the bulk field ϕ remains invariant but $\phi_0^{ren}(x)$ has to rescale with dimension α_-. Since we identify it with the source, from the Witten ansatz (5.11) we conclude that the corresponding boundary operator $\mathcal{O}$ has dimension

$$\Delta = 4 - \alpha_- = \alpha_+ = 2 + \sqrt{4 + m^2}. \tag{5.18}$$

So we have found that a scalar field of mass m^2 in AdS_5 is associated to an operator $\mathcal{O}$ with conformal dimension Δ according to the relation

$$m^2 = \Delta(\Delta - 4). \tag{5.19}$$

Although the above discussion has been for scalars, similar correspondences exist for higher spin fields. In table 5.1 there are all the relations between the dimensions and the masses for different fields.

Field	Relation
Scalars	$m^2 = \Delta(\Delta - 4)$
Spin $1/2, 3/2$	$\|m\| = \Delta - 2$
p-form	$m^2 = (\Delta - p)(\Delta + p - 4)$
Massive spin 2	$m^2 = \Delta(\Delta - 4)$
Massless spin 2	$m^2 = 0$ iff $\Delta = 4$
Rank s symmetric traceless tensor	$m^2 = (\Delta + s - 2)(\Delta - s - 2)$

Table 5.1: The field-operator map

5.3 Mapping the Representations

Since the two theories of the correspondence have the same superconformal symmetry group $SU(2,2|4)$, not only are the individual fields related to the individual operators according to the relationship $m(\Delta)$ seen in the previous section, but so are all the entire representations of the group. Since it is not known how to compute the full spectrum of type IIB string theory on $AdS_5 \times S^5$, one considers only the supergravity spectrum obtained by compacting the theory on S^5. This was done, for example, in [17] expanding the ten dimensional fields in appropriate spherical harmonics on S^5, plugging them into the supergravity equations of motion, linearized around the $AdS_5 \times S^5$ background, and diagonalizing the equations to give equations of motion for free (massless or massive) fields on AdS_5. In doing so, each field of the 10-dimensional SUGRA theory gives rise to an infinite tower of fields one for each S^5 spherical harmonic (the Kaluza-Klein spectrum) that collectively organize into chiral multiplets of $SU(2,2|4)$. So there is a complete correspondence between the Kaluza-Klein spectrum and the single-trace short multiplets of $\mathcal{N} = 4$ SYM theory: in each case we have precisely one short multiplet of the superconformal algebra for every $\Delta \geq 2$. In particular the multiplet A'_Δ on the gravity side built on a lowest dimension scalar field in the $(0, \Delta, 0)$ representation of $SU(4)_R$ with mass $m^2 = \Delta(\Delta - 4)$, corresponds to the multiplet A_Δ on the gauge theory side built on the chiral primary operator $\mathcal{O}_\Delta$ with dimension Δ. The lowest dimension scalar field in each representation related to the CPO turns out to arise from a linear combination of spherical harmonics modes on S^5 which are components of the graviton h_a^a (expanded around the $AdS_5 \times S^5$ vacuum) and the 4-form $D^{(4)}_{abcd}$, where a, b, c, d are indices on S^5. This will be described in detail in section 8.

We conclude the section by mentioning the fact that string theory on $AdS_5 \times S^5$ is expected to have many additional states, with masses of the order of the string scale $1/l_s$. Such state would correspond (using the mass/dimension relation described above) to single trace operators in the field theory with dimensions of order $\Delta \sim (g_s N)^{1/4} \sim N^{1/4}$ for large $N, g_s N$. Presumably none of these single particle states are in short multiplets of the superconformal algebra (at least, this would be the prediction of the AdS/CFT correspondence).

The CPO's described in this section are single-trace operators whose conformal dimension does not scale with N. In the next sections we will consider multi-trace CPO's with a large number of traces, s.t. $\Delta \sim N^2$. We call this operators heavy and we'll look for their gravity description.

6

D1-D5 CFT and Holography

Before obtaining the dual gravitational description for the heavy states of the $\mathcal{N} = 4$ SYM CFT, let us consider the case of the D1-D5 CFT where the holographic dictionary between heavy states and geometries has been extensively worked out in the literature. This case will serve as a useful guide for the study of $\mathcal{N} = 4$ heavy states.
In the first section we will describe some of the main aspects of the D1-D5 theory that will be useful for our purposes (for a more exhaustive treatment see, for example, [18]). Then we will use holography in order to obtain some informations about some specific chiral operators of the theory from the corresponding gravitational description.

6.1 D1-D5 CFT and the Dual Description

In section 3.8 we have seen that starting from a 10-dimensional spacetime with topology $\mathbb{R}^{1,4} \times S^1 \times T^4$ (the S^1 direction is distinguished from the T^4 because we'll consider the S^1 to be much larger than the T^4), the bound state of D5-branes wrapping the whole compact space and D1-branes wrapping the circle S^1, in the decoupling (or near horizon) limit becomes $AdS_3 \times S^3 \times T^4$. Just as for the D3-brane system, according to the AdS/CFT correspondence there is dual description and, in particular, we expect the dual theory to be a 1+1-dimensional conformal field theory with 8 supercharges (as the D1-D5 breaks $\frac{1}{4}$ supersymmetries) living on the conformal boundary of AdS_3; this is the so called D1-D5 CFT. While the low energy field theory living on the D3 branes is simple to describe, as there is a unique maximally supersymmetric SU(N) gauge theory, the D1-D5 brane system is more complicated, and there are several methods to obtain the CFT. One method is to consider only N_5 D5-branes wrapping $T^4 \times S^1$ which give rise to a 5+1-dimensional $U(N_5)$ gauge theory with 16 supercharges. Embedded in this theory we can consider the N_1 D1 branes as istantonic solutions, that is dynamical strings wrapping S^1 that are localized in T^4. These solutions break half of the 5-brane worldvolume theory's supersymmetries. From this point of view we therefore obtain a 2-dimensional sigma model on the D1-branes worldsheet with target space the moduli space of N_1 $U(N_5)$ istantons on T^4. In general this space is complicated but one can show that in a particular configuration it reduces to

$$\frac{(T^4)^N}{S_N}, \quad N \equiv N_1 N_5, \tag{6.1}$$

where S_N is the symmetric group of degree N permuting the N copies of T^4. This is the so-called orbifold point of the CFT moduli space[1]. We will always work in this configuration where we can visualize the CFT as a collection of N strings wrapping the circle with target space T^4; the S_N identification is required as there is no physical distinction between permutations of the strings.

To summarize, we consider the correspondence between type IIB superstring theory defined on an asympototically $AdS_3 \times S^3 \times T^4$ space and the D1-D5 CFT at the orbifold point, i.e. a 1+1-dimensional sigma-model with target space given by (6.1). As always, as a first check of the correspondence, we can see if the global symmetries of the two theories are in agreement between each others. On the gravity side we have the $SO(2,2)$ isometry group of AdS_3, an $SO(4)_E \simeq SU(2)_L \times SU(2)_R$ isometry group of S^3 and another $SO(4)_I \simeq SU(2)_1 \times SU(2)_2$ isometry group of T^4 broken by the compactification. On the CFT side, the conformal algebra in 2 dimensions is infinite-dimensional, with Virasoro generators L_n, $\bar{L}_n$ ($n = -\infty, \cdots, +\infty$). The vacuum state of the theory is invariant under the subalgebra spanned by $L_0, L_{\pm 1}$ which one can identify with the AdS_3 isometry group. The CFT has also an $SO(4)$ R-symmetry group which we identify with the isometry group of S^3 and another $SO(4)$ symmetry group which we identify with the isometry group of T^4.

6.1.1 Field Content

We can parameterize the 2-dimensional worldsheet of the sigma model with a timelike coordinate τ and a spacelike coordinate σ on S^1. We find it more convenient to Wick rotate to Euclidean time and map the cylinder to a complex plane, breaking the theory into left and right-movers

$$z = e^{\tau_E + i\sigma}, \quad \bar{z} = e^{\tau_E - i\sigma}. \tag{6.2}$$

Functions of z are the "left-movers'' while functions of $\bar{z}$ are the "right-movers''.

At the free orbifold point the CFT can be visualized as a collection of N strands, i.e. maps from (τ, σ) to T^4, each one with 4 bosons and four doublets of fermions that we can organise in the previous two sectors

$$\partial X^i_{(r)}(z), \;\; \bar{\partial} X^i_{(r)}(\bar{z}), \;\; \psi^{\alpha A}_{(r)}(z), \;\; \bar{\psi}^{\dot{\alpha} A}_{(r)}(\bar{z}) \tag{6.3}$$

[1] Moduli are the CFT deformations that preserve the superconformal symmetry. While the $\mathcal{N} = 4$ SYM CFT has only one complex modulus, parametrized by g_{YM} and θ_I, the D1-D5 has a more complicated 20-dimensional space. The orbifold point correspond to a submanifold in this moduli space.

where $r = 1, \cdots N$ is a "copy" index. Since the target space is the symmetric product of N T^4s, we have N copies of a sigma model with target space T^4. Moreover indices correspond to the following representations

$$\begin{array}{ll} \alpha, \beta \text{ fundamental of } SU(2)_L, & \dot{\alpha}, \dot{\beta} \text{ fundamental of } SU(2)_R \\ A, B \text{ fundamental of } SU(2)_1, & \dot{A}, \dot{B} \text{ fundamental of } SU(2)_2 \\ i, j \text{ fundamenta of } SO(4)_I & \end{array}$$

Each of the N copies of the CFT contribute with $c_{(r)} = 4 + 2 = 6$ to the central charge (corresponding to 4 free bosons and 4 free fermions). Overall we have $c = 6N$.
The untwisted sector is composed of singly wound strand, i.e. by a collection of N independent strands with winding one. In this case we have the following periodic boundary condition for the scalars

$$\partial X^i_{(r)}(e^{i2\pi} z) = \partial X^i_{(r)}(z), \tag{6.4}$$

while for fermions we can have either Ramond (R) or Neveu-Schwartz (NS) boundary conditions, which correspond, respectively, to periodic and antiperiodic boundary conditions on the cylinder. Using complex coordinates $z, \bar{z}$ there is a -1 factor coming from the Jacobian of the transformation from (τ, σ) to $(z, \bar{z})$ that switches the periodicity. So in the R sector we have

$$\psi^{\alpha A}_{(r)}(e^{i2\pi} z) = -\psi^{\alpha A}_{(r)}(z), \tag{6.5}$$

while in the NS sector

$$\psi^{\alpha A}_{(r)}(e^{i2\pi} z) = \psi^{\alpha A}_{(r)}(z). \tag{6.6}$$

The R sector can be related to the NS sector via spectral flow (see later). If we have a global $AdS_3 \times S^3$ space, then the CFT is in the NS sector since global AdS_3 has a contractible cycle and going around S^1 at the boundary looks like a 2π rotation at a point in AdS space. So, since fermions are invariant under a 4π rotation, a 2π rotation gives a minus sign and this identifies the NS sector. However, more complicated geometries such as the one we'll see are instead dual to the CFT in the R sector since the geometries have non-trivial gauge fields that mix AdS_3 and S^3. So in order to obtain the NS sector we'll apply spectral flow after using the holographic dictionary.

6.1.2 Vacuum States and Chiral Primaries

Each state is labelled by the quantum numbers of $SU(2)_L \times SU(2)_R$ $(j^3_L, j^3_R) \equiv (j, \bar{j})$ and also by the conformal dimension $\Delta = h + \bar{h}$. The NS vacuum state $|0, 0\rangle_{NS}$ is the "real" vacuum of the theory: it is in the completely untwisted sector and the gravity dual is global AdS. For this state $j_{NS} = \bar{j}_{NS} = h_{NS} = \bar{h}_{NS} = 0$. The NS sector states can be mapped to R sector states by spectral flow transformation, that is an automorphism of the superconformal algebra acting on the charges and dimensions as

$$\begin{aligned} h_R &= h_{NS} - j_{NS} + \frac{c}{24}, \\ j_R &= j_{NS} - \frac{c}{12}. \end{aligned} \tag{6.7}$$

In the R sector there are many vacua. The NS vacuum maps under spectral flow to the R vacuum with $j_R = \bar{j}_R = -\frac{N}{2}$, denoted as $|-\frac{N}{2}, -\frac{N}{2}\rangle$, which correspond to N copy of the state $|-\frac{1}{2}, -\frac{1}{2}\rangle \equiv |-,-\rangle$. Note also that CPO states with $j_{NS} = h_{NS}$ map under spectral flow to R vacua with $h_R = \frac{c}{24}$ (and $-N/2 \leq j_R \leq N/2$).
It turns out that the region of moduli space of the CFT dual to the low-energy supergravity regime in the bulk is distant from the solvable free orbifold point. In order to compute quantities free from any radiative corrections at the free orbifold point we need to focus on chiral operators of the CFT. From the CFT algebra one obtains that the chiral primary operators satisfy

$$h_{NS} = j_{NS}. \tag{6.8}$$

In the singly twisted sector there are four CPOs and the one that we'll consider in the future is

$$S_1 = \epsilon_{AB} \sum_r \psi^{+A}_{(r)} \bar{\psi}^{+B}_{(r)}, \tag{6.9}$$

with $j_{NS} = \bar{j}_{NS} = h_{NS} = \bar{h}_{NS} = \frac{1}{2}$. One can also make heavy operators with $h_{NS} \sim \bar{h}_{NS} \sim N^2$ by taking $\sim N^2$ copies of S_1.

6.2 Holographic Dictionary

In this section we describe a specific class of geometries dual to heavy states of the D1-D5 CFT and then we motivate the holographic map looking at the asymptotic regime. In particular we consider the $\frac{1}{4}$-BPS geometries which are the simplest one for a system with two charges.
The general solution of type IIB supergravity compactified on $T^4 \times S^1$ preserving the same

supersymmetries as the D1-D5 system is [19]

$$ds^2_{(10)} = -\frac{2\alpha}{\sqrt{Z_1 Z_2}}(dv+\beta)[du+\omega] + \sqrt{Z_1 Z_2}ds^2_4 + \sqrt{\frac{Z_1}{Z_2}}d\hat{s}^2_4, \tag{6.10}$$

$$e^{2\phi} = \alpha\frac{Z_1}{Z_2}, \tag{6.11}$$

$$B = -\frac{\alpha Z_4}{Z_1 Z_2}(du+\omega)\wedge(dv+\beta), \tag{6.12}$$

$$C_0 = \frac{Z_4}{Z_2}, \tag{6.13}$$

$$C_2 = -\frac{\alpha}{Z_1}(du+\omega)\wedge(dv+\beta), \tag{6.14}$$

$$C_4 = \frac{Z_4}{Z_2}\mathrm{vol}_4 - \frac{\alpha Z_4}{Z_1 Z_2}\gamma_2\wedge(du+\omega)\wedge(dv+\beta), \tag{6.15}$$

where $\alpha \equiv \frac{Z_1 Z_2}{Z_1 Z_2 - Z_4^2}$. Here ds^2_4 is a (generically non trivial) Euclidean metric in the 4 spatial non compact directions that reduces asymptotically to flat $\mathbb{R}^4$ and $d\hat{s}^2_4$ denotes the flat metric on T^4. We have also introduced light-cone coordinates

$$u = \frac{t-y}{\sqrt{2}}, \qquad v = \frac{t+y}{\sqrt{2}}, \tag{6.16}$$

where t is the time coordinate and y is the coordinate on S^1, whose radius will be denoted by R.

The simplest 2-charge solution is the naive superposition of D1 and D5 branes, which corresponds to setting all functions to zero, except Z_1 and Z_2. In section 3.8 we have derived this naive solution by applying boost and dualities to a simple neutral seed solution. It is easy to see that this naive solution fits with the ansatz above. We have also anticipated that this solution is not dual to any proper D1-D5 microstates, but that the microstate solutions can be obtained by giving a non-trivial transverse vibration profile to the F1 string in the F1-P duality frame. From a geometrical point of view, to discuss the most general F1-P state, we should start giving 8 functions $g_A(v)$ transverse to the fundamental string in order to describe its profile; these functions can be split into four $\mathbb{R}^4$ components ($A = 1, \cdots 4$) and four T^4 components ($A = 5, \cdots, 8$). When the latter are non-vanishing, invariance under rotation in the T^4 directions is broken. However, when one applies the chain of dualities from the F1P frame to the D1D5 frame, it turns out that in the latter frame geometries that have non trivial values of the profile $g_A(v)$ for $A = 1, \cdots, 5$ preserve rotational symmetry in the T^4 directions. This class of 2-charge solutions can be written in terms of the ansatz (6.10)-(6.15) by choosing

$$ds_4^2 = dx^i dx^i, \tag{6.17}$$

$$Z_2 = 1 + \frac{Q_5}{L}\int_0^L \frac{1}{|x_i - g_i(v')|^2}dv', \quad Z_4 = -\frac{Q_5}{L}\int_0^L \frac{\dot{g}(v')}{|x_i - g_i(v')|^2}dv', \tag{6.18}$$

$$Z_1 = 1 + \frac{Q_5}{L}\int_0^L \frac{|\dot{g}_i(v')|^2 + |\dot{g}(v')|^2}{|x_i - g_i(v')|^2}dv', \quad d\gamma_2 = \star_4 dZ_2, \quad d\delta_2 = \star_4 dZ_4, \tag{6.19}$$

$$A = -\frac{Q_5}{L}\int_0^L \frac{\dot{g}_j(v')dx^j}{|x_i - g_i(v')|^2}dv', \quad dB = -\star_4 dA, \tag{6.20}$$

$$\beta = \frac{-A+B}{\sqrt{2}}, \quad \omega = \frac{-A-B}{\sqrt{2}}, \tag{6.21}$$

Here $g(v)$ is the extra component of $g_A(v) \equiv g_5(v)$ in the particular direction of T^4 necessary in order to preserve invariance under T^4 rotations. Furthermore the dot on the profiles denote a derivative with respect to v and $\star_4$ is the hodge dual with respect to the flat metric ds_4^2.

We are interested in the following profile

$$g_1(v') = a\cos\left(\frac{2\pi v'}{L}\right), \quad g_2(v') = a\sin\left(\frac{2\pi v'}{L}\right), \quad g(v') = -b\sin\left(\frac{2\pi v'}{L}\right) \tag{6.22}$$

with all other components trivial. This choice yields a geometry that can be embedded in the ansatz (6.10)-(6.15) with appropriate choice of coordinates (r, θ, ϕ, ψ) in $\mathbb{R}^4$ as follows [19]

$$ds_4^2 = (r^2 + a^2\cos^2\theta)(\frac{dr^2}{r^2+a^2} + d\theta^2) + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2, \tag{6.23}$$

$$\beta = \frac{Ra^2}{\sqrt{2}(r^2+a^2\cos^2\theta)}(\sin^2\theta d\phi - \cos^2\theta d\psi), \tag{6.24}$$

$$Z_1 = 1 + \frac{R^2}{Q_5}\frac{a^2+\frac{b^2}{2}}{r^2+a^2\cos^2\theta} + \frac{R^2a^2b^2}{2Q_5}\frac{\cos 2\phi \sin^2\theta}{(r^2+a^2\cos^2\theta)(r^2+a^2)}, \tag{6.25}$$

$$Z_2 = 1 + \frac{Q_5}{r^2+a^2\cos^2\theta}, \quad a_1 = 0, \quad \gamma_2 = -Q_5\frac{(r^2+a^2)\cos^2\theta}{r^2+a^2\cos^2\theta}d\phi\wedge d\psi, \tag{6.26}$$

$$Z_4 = Rab\frac{\cos\phi\sin\theta}{\sqrt{r^2+a^2}(r^2+a^2\cos^2\theta)}, \quad a_4 = 0, \tag{6.27}$$

$$\delta_2 = \frac{-Rab\sin\theta}{\sqrt{r^2+a^2}}[\frac{r^2+a^2}{r^2+a^2\cos^2\theta}\cos^2\theta\cos\phi d\phi\wedge d\psi + \sin\phi\frac{\cos\theta}{\sin\theta}d\theta\wedge d\psi], \tag{6.28}$$

$$\omega = \frac{Ra^2}{\sqrt{2}(r^2+a^2\cos^2\theta)}(\sin^2\theta d\phi + \cos^2\theta d\psi), \tag{6.29}$$

$$\mathcal{F} = 0. \tag{6.30}$$

As we said in the previous section, our geometries are dual to the R sector of the CFT. If one set $b = 0$ the 10-dimensional metric becomes simply that of the vacuum $AdS_3 \times S^3 \times T^4$ after the coordinate shift $\phi \to \phi + t/R$, $\psi \to \psi + y/R$ which implements the spectral flow from R to NS sector. In this case from (6.11)-(6.15) follows that the fields C_0, B_2 and C_4 are zero and the dilaton is constant

$$e^{2\phi} = \frac{Z_1}{Z_2} = \frac{R^2 a_0^2}{Q_5^2} \quad \text{with} \quad a_0^2 \equiv \frac{Q_1 Q_5}{R^2}. \tag{6.31}$$

The 2-form C_2 is also non vanishing and the associated field strength can be written as

$$F_3 = -\mathrm{vol}(AdS_3) + \mathrm{vol}(S_3), \tag{6.32}$$

where in our coordinates

$$\mathrm{vol}(AdS_3) = \frac{r}{Q_1 Q_5} dr \wedge dt \wedge dy, \qquad \mathrm{vol}(S^3) = \sin\theta\cos\theta d\theta \wedge d\phi \wedge d\psi. \tag{6.33}$$

Thus at $b = 0$, the geometry is dual to the R sector vacuum $|-,-\rangle^N$. If we take the limit $b \to 0$, at linear order in b, we have a deformation of the vacuum geometry caused by the fields B and C_4 and it turns out that the operator dual to this linear deformations is the chiral primary operator (6.9). The dual state at this slightly excited geometry is obtained by acting once with this CPO on the N copies of the vacuum in the R sector, schematically this state is

$$|0,0\rangle\,(|-,-\rangle)^{N-1}. \tag{6.34}$$

As we increase b, the geometry gets deformed further away from the vacuum $|-,-\rangle^N$ and this, intuitively, corresponds to the heavy state of the CFT obtained by acting not once, but a number of times $p \sim N^2$ with the CPO (6.9) on the N copies of the vacuum state in the R sector. To be more precise, since on the gravitational side we are working in the classical regime, the dual state is a coherent superposition of the CPO, so the schematic form of the dual state at the geometry with $b \neq 0$ is

$$\sum_{p=0}^{N} \mathcal{N}(p)(|0,0\rangle)^p(|-,-\rangle)^{N-p} \simeq (|0,0\rangle)^{\bar{p}}(|-,-\rangle)^{N-\bar{p}}, \tag{6.35}$$

where $\mathcal{N}(p)$ is a normalization factor and $\bar{p}$ is the value on which, in a good approximation, the sum over p which defines the coherent state is peaked. Superficially it seems that there is a mismatch of parameters, since the CFT state depends on the single parameter $\bar{p}$, while the geometry contains the two parameters a and b. However the analysis of the regularity of the supergravity solution shows that absence of unphysical singularities requires the constraint [19]

$$a^2 + \frac{b^2}{2} = a_0^2, \tag{6.36}$$

and thus only one parameter, which we could take to be b, can be freely varied. The simplest way to find the relationship between the microscopic and the supergravity parameters $\bar{p}$ and b, is to match the conserved quantities, like angular momenta. The holographic recipe to extract the angular momenta $(j_R, \bar{j}_R)$ and the conformal dimension $(h_R, \bar{h}_R)$ in the R sector from an asymptotically AdS geometry is given for example in [20],[21]. Without considering the T^4 part, our six dimensional metric for the profile (6.22) is

$$ds_6^2 = -2\sqrt{\frac{\alpha}{\mathcal{P}}}(dv+\beta)(du+\omega) + \sqrt{\mathcal{P}\alpha}ds^4, \tag{6.37}$$

where $\mathcal{P} \equiv Z_1 Z_2 - Z_4^2$. Using the data (6.23)-(6.30) it is possible to write it as

$$ds_6^2 = \frac{det(G^{(0)})}{det(G)} g_{\mu\nu}dx^\mu dx^\nu + G_{\alpha\beta}(dx^\alpha + A^\alpha_\mu dx^\mu)(dx^\beta + A^\alpha_\mu dx^\mu), \tag{6.38}$$

where $G^{(0)}$ is the background S^3 metric, equal to G for $b=0$. The $\frac{det(G^{(0)})}{det(G)}$ factor is needed so that $g_{\mu\nu}$ is the 3D Einstein frame metric. Here x^μ denote the AdS_3 coordinates, x^α denote the S^3 coordinates and A^α_μ are $SO(4)$ gauge fields. For this 2-charge geometry both $g_{\mu\nu}$ and $G_{\alpha\beta}$ are diagonal, we have

$$ds_3^2 = g_{\mu\nu}dx^\mu dx^\nu = -\frac{a^4 + a_0^2 r^2}{a_0^2 R_{AdS}^2}dt^2 + \frac{r^2}{R_{AdS}^2}dy^2 + \frac{R_{AdS}^2}{(r^2+a^2)^2}(r^2 + \frac{a^4}{a_0^2})dr^2 \tag{6.39}$$

with $R_{AdS}^2 = \sqrt{Q1Q5}$ and

$$A^\theta = 0, \quad A^\phi = -\frac{a^2}{a_0^2 R}dt, \quad A^\psi = -a^2\frac{r^2+a^2}{a_0^2 r^2 + a^4}\frac{1}{R}dy, \tag{6.40}$$

$$G_{\theta\theta} = \sqrt{\mathcal{P}}\Sigma, \quad G_{\phi\phi} = \frac{R_{AdS}^4}{\sqrt{\mathcal{P}}\Sigma}\sin^2\theta, \quad G_{\psi\psi} = \frac{R_{AdS}^4}{\sqrt{\mathcal{P}}\Sigma}\frac{a_0^2 r^2 + a^4}{a_0^2(r^2+a^2)}\cos^2\theta, \tag{6.41}$$

with $\Sigma \equiv r^2 + a^2\cos^2\theta$. In order to obtain the angular momenta one defines [20]

$$A^\pm \equiv A^\phi \pm A^\psi, \quad \tau \equiv \frac{t}{R}, \quad \sigma \equiv \frac{y}{R} \tag{6.42}$$

and, according to the holographic dictionary [20], in the R sector we have

$$j_R = \frac{N}{4}(A^+_\tau + A^+_\sigma) = \frac{N}{2}\frac{a^2}{a_0^2}, \quad \bar{j}_R = \frac{N}{4}(A^-_\tau - A^-_\sigma) = \frac{N}{2}\frac{a^2}{a_0^2}. \tag{6.43}$$

This gives the gravity prediction for $j_R, \bar{j}_R$, which should be compared to the CFT one, that can be easily derived from (6.35)

$$j_R = \bar{j}_R = \frac{N - \bar{p}}{2}, \tag{6.44}$$

since each $|-,-\rangle$ carries $j_R = \bar{j}_R = 1/2$. Comparing the gravity (6.43) and CFT (6.44) predictions, and using the regularity constraint (6.36), one obtains the map between $\bar{p}$ and b

$$\frac{\bar{p}}{N} = \frac{b^2}{2a_0^2}. \tag{6.45}$$

As a further check of the holographic map, one can compute h_R and $\bar{h}_R$. In order to compute the conformal dimension is convenient to define dimensionless quantities

$$\rho \equiv \frac{r}{a}, \quad \eta \equiv \frac{a}{a_0}, \tag{6.46}$$

in terms of which the 3D metric in the Einstein frame (6.39) becomes

$$\frac{ds_3^2}{R^2_{AdS}} = -\eta^2(\rho^2 + \eta^2)d\tau^2 + \eta^2\rho^2 d\sigma^2 + \frac{\rho^2 + \eta^2}{(\rho^2 + 1)^2}d\rho^2. \tag{6.47}$$

We want to define a coordinate z in terms of which, for $z \to 0$

$$\frac{ds_3^2}{R^2_{AdS}} = \frac{dz^2}{z^2} + \frac{1}{z^2}(g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu}) + o(z^2). \tag{6.48}$$

By defining z such that

$$\rho = \frac{1}{z\eta}(1 + cz^2) \tag{6.49}$$

the equation (6.48) is satisfied if $c = \frac{1}{4}(-2\eta^2 + \eta^4)$. In particular we have

$$g^{(0)}_{\mu\nu}dx^\mu dx^\nu = d\sigma^2 - d\tau^2, \quad g^{(2)}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{2}\eta^2[(2 - 3\eta^2)d\tau^2 + (-2 + \eta^2)d\sigma^2]. \tag{6.50}$$

According to the holographic dictionary the conformal dimension in the R sector [20]

$$h_R = \frac{N}{4}[g^{(2)}_{\tau\tau} + g^{(2)}_{\sigma\sigma} + \frac{1}{4}(A^+_\tau + A^+_\sigma)^2] + \frac{N}{4} \tag{6.51}$$

$$\bar{h}_R = \frac{N}{4}[g^{(2)}_{\tau\tau} + g^{(2)}_{\sigma\sigma} + \frac{1}{4}(A^-_\tau - A^-_\sigma)^2] + \frac{N}{4} \tag{6.52}$$

which gives, using (6.43) and (6.50), $h_R = \bar{h}_R = \frac{N}{4} = \frac{c}{24}$. This is the expected dimension for a Ramond sector vacuum like the one in (6.35). By spectrally flowing to the NS sector, this translates to $j_{NS} = h_{NS}$ as follows from (6.7).
So, starting from a particular geometry, using the holographic dictionary we were able to map the geometry to a CFT state by matching some of the simplest conserved charges associated with the corresponding CFT operator. This chapter is meant to be an example of how to proceed in the future. In particular, we will now return to consider the theory of gravity on $AdS_5 \times S^5$ and we'll try to analyse the dual geometries of the CFT states that interest us. We will describe these type of geometries in general in the next chapter.

7

LLM Geometries

In this chapter we'll discuss some special geometries constructed by Lin, Lunin, and Maldacena in 2004 [22], which are called *LLM geometries*. These are all the $\frac{1}{2}$-BPS excitations of $AdS_5 \times S^5$ and are associated to the $\frac{1}{2}$-BPS operators in the dual CFT following the AdS/CFT prescription. When the dimension of these operators is large, of the order of $c \sim N^2$, the dual gravitational configuration is a large deformation of $AdS_5 \times S^5$ and LLM ansatz describes this most general geometry consistent with the supersymmetries preserved by the state. Here we will briefly look at the general LLM solution following [22] and we explicitly write the $AdS_5 \times S^5$ background geometry from the general ansatz. In the next chapter we will study some of these geometries in detail by perturbing this background.

7.1 The LLM Solutions

The construction is based on the assumption that the geometries we are looking for have to preserve the same amount of symmetries that the states in the CFT do. Being interested in $\frac{1}{2}$-BPS states, we are looking for a type IIB geometry that is invariant under

$$SO(4) \times SO(4) \times \mathbb{R}. \tag{7.1}$$

BPS operators are built out of a complex combination of two of the six scalars: $Z \equiv \phi^1 + i\phi^2$. The first $SO(4)$ is the rotation symmetry acting on the remaining four scalars. Since BPS operators have the lowest dimension for their charge, they do not depend on the space directions, which we can take to form a compact S^3. The second $SO(4)$ acts on this S^3. A $\frac{1}{2}$BPS operator thus depends on time t and one of the S^5 coordinates, $\tilde{\phi}$, corresponding to the R charge carried by the operator. However, the condition that the dimension is equal to the charge implies that the operator only depends on the combination $\tilde{\phi} + t$, and this leaves an extra $U(1)$ Killing vector, which is the third factor in (7.1). Hence we can single out two 3-spheres, Ω_3 and $\tilde{\Omega}_3$, and a Killing vector, t. Assuming that only the five-form field strength is excited, the most general geometry with this symmetry is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\Omega_3^2 + e^{H-G}d\tilde{\Omega}_3^2, \tag{7.2}$$

$$F_{(5)} = F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3, \tag{7.3}$$

where $\mu, \nu = 0, ..., 3$. The complete solutions of the supersymmetric equations which impose the $\frac{1}{2}$-BPS condition have the following form [22]:

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G}d\tilde{\Omega}_3^2 \tag{7.4}$$

$$h^{-2} = 2y\cosh G \tag{7.5}$$

$$y\partial_y V_i = \epsilon_{ij}\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z \tag{7.6}$$

$$z = \frac{1}{2}\tanh G \tag{7.7}$$

$$F = dB_t \wedge (dt + V) + B_t dV + d\hat{B} \tag{7.8}$$

$$\tilde{F} = d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + \hat{d}\tilde{B} \tag{7.9}$$

$$B_t = -\frac{1}{4}y^2 e^{2G} \qquad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G} \tag{7.10}$$

$$d\hat{B} = -\frac{1}{4}y^3 \star_3 d\left(\frac{z + 1/2}{y^2}\right) \qquad \hat{d}\tilde{B} = -\frac{1}{4}y^3 \star_3 d\left(\frac{z - 1/2}{y^2}\right) \tag{7.11}$$

where $i = 1, 2$ and $\star_3$ is the flat space epsilon symbol in the three dimensions parametrized by y, x_1, x_2. The full solution is determined in terms of a single function $z = z(x_1, x_2, y)$ that obeys the linear equation

$$\partial_i\partial_i z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0. \tag{7.12}$$

This is a Laplace equation in the 6D space made by (y, x_1, x_2) plus three extra coordinates on which z does not depend. Hence a solution of this equation is uniquely determined by giving a boundary condition, which can be imposed on the plane $y = 0$. From (7.5),(7.7) we see that in order to ensure the regularity of (7.4), we must have that $h^{-2} = \frac{y}{\sqrt{\frac{1}{4} - z^2}}$ remains finite as y approaches zero. This implies $z = \pm\frac{1}{2}$ when $y = 0$ and therefore a generic LLM solution can be specified by black(white) color-coding “droplets” attributed to the regions in which z takes the values $-\frac{1}{2}(+\frac{1}{2})$ on the $\{x_1, x_2, y = 0\}$ plane.

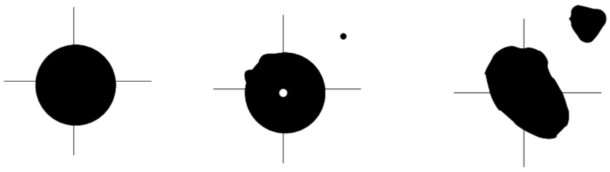


Figure 7.1: This figure is taken from [22]. In (a) we have the droplet corresponding to the $AdS_5 \times S^5$ ground state geometry as we'll see in the next section while in (b) and (c) we have droplets which correspond to more complicated geometries.

The equation (7.12) is a 6D laplacian equation for the function z/y^2 with y the radial coordinate

$$\Delta_6 \left(\frac{z}{y^2} \right) = 0. \tag{7.13}$$

Using the spherical symmetry in four of the dimensions and the fact that the values of z on the $y = 0$ plane play the role of sources for this laplacian equation, the solution can be written as an integral over the droplet $\mathcal{D}$ and, integrating by parts, over the boundary of the droplet $\partial\mathcal{D}$

$$z(x_1, x_2, y) = \frac{y^2}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0) dx_1' dx_2'}{[(x - x')^2 + y^2]^2} = -\frac{1}{2\pi} \oint_{\partial\mathcal{D}} dl\, n_i' \frac{x_i - x_i'}{[(x - x')^2 + y^2]} + \sigma, \tag{7.14}$$

where n_i is the unit normal vector to the droplet and σ is the contribution from infinity such that $\sigma = \pm\frac{1}{2}$ when $z = \pm\frac{1}{2}$ asymptotically. Using the first of (7.6) we can also write the integral form for V

$$V_i(x_1, x_2, y) = \frac{1}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0)(x_j - x_j') dx_1' dx_2'}{[(x - x')^2 + y^2]^2} = \frac{1}{2\pi} \oint_{\partial\mathcal{D}} \frac{dx_j'}{[(x - x')^2 + y^2]}, \tag{7.15}$$

and it is immediate to see that the second of (7.6) is also satisfied.

7.2 $AdS_5 \times S^5$ Geometry From the LLM Ansatz

The familiar ground state geometry $AdS_5 \times S^5$ corresponds to a circular droplet of radius $r_0 = R^2_{AdS_5} = R^2_{S^5}$ on the (x_1, x_2) plane as shown in the figure 7.1 (a). In order to show this it is convenient to introduce a function $\tilde{z} \equiv z - \frac{1}{2}$ and choose polar coordinates (r, φ) in the (x_1, x_2) plane. Integrating on the disk one obtain the following solution of (7.12) [22]

$$\tilde{z}(r, y; r_0) = \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} - \frac{1}{2}. \tag{7.16}$$

So, from (7.6) we read off that the 1-form $V = V_r dr + V_\phi d\phi$ has only the φ component

$$V_\varphi(r, y; r_0) = -\frac{1}{2}\left(\frac{r^2 + y^2 + r_0^2}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} + 1\right). \tag{7.17}$$

Performing the change of coordinates:

$$y = r_0 \sinh\rho \sin\theta \tag{7.18}$$

$$r = r_0 \cosh\rho \cos\theta \tag{7.19}$$

$$\tilde{\phi} = \phi - t \tag{7.20}$$

from (7.7) we get

$$e^G = \sqrt{\frac{1+2z}{1-2z}} = \frac{\sinh\rho}{\sin\theta}, \tag{7.21}$$

and from (7.5)

$$h^{-2} = y(e^G + e^{-G}) = r_0(\sin^2\theta + \sinh^2\rho). \tag{7.22}$$

Putting it all together in the ansatz (7.4) we get the standard $AdS_5 \times S^5$ metric

$$ds^2 = r_0(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\tilde{\Omega}_3^2), \tag{7.23}$$

with $r_0 = R^2_{AdS_5} = R^2_{S^5}$. In the future we will place $r_0 = 1$. Finally, using (7.3),(7.8)-(7.11) we get the 5-form

$$\bar{F}_{(5)} = \mathrm{Vol}(AdS_5) + \mathrm{Vol}(S^5) = \cosh\rho\sinh^3\rho dt\wedge d\rho\wedge d\Omega_3 + \cos\theta\sin^3\theta d\theta\wedge d\tilde{\phi}\wedge d\tilde{\Omega}_3. \tag{7.24}$$

This is the background LLM solution associated to the vacuum state of the CFT. All the others geometries are obtained by perturbing this background and for a droplet of finite size, the geometry asymptotically approaches $AdS_5 \times S^5$. In the next chapter we will consider small fluctuations around this background whose dual CFT states are well known.

8

LLM Excitations Around the Background

In this chapter we'll discuss some special LLM deformations of the $AdS_5 \times S^5$ background corresponding to small ripples of the circular droplet already treated in [23]. The geometries solve the supergravity equations at first order in the deformation parameter. Since all the LLM solutions asymptotically approaches $AdS_5 \times S^5$ one can use AdS/CFT methods to extract holographic data from the geometries as we did in Chapter 6 and it turns out that they all describe the CFT in a non-trivial state. In the case that we will study in this chapter of small excitations the dual description is well known; in particular these geometries are dual to $\frac{1}{2}$-BPS chiral primary operators $\mathcal{O}^k = Tr(Z^k)$ with $Z \equiv \phi_1 + i\phi_2$ described in chapter 5 whose conformal dimension k is small compared to the central charge of the CFT $c = \frac{N^2}{4}$. Since the conformal dimension of an operator is dual to the energy of the corresponding geometry, when the dimension grow to become comparable to N^2 we expect that the backreaction on the geometry is no longer negligible and we get new geometries that represent exact, fully non-linear solutions of the supergravity equations. In principle, since the LLM ansatz describes all the $\frac{1}{2}$BPS solutions, these geometries should correspond to some complicated droplet configuration, which reduces to the small ripple in the small deformation limit. The dual description of these geometries is not yet fully known and it is given by the heavy states of the CFT, i.e. by the states whose conformal dimension is of the order of the central charge. Some of these states can be constructed for example by taking p times the previous CPOs $[Tr(Z^k)]^p$ with $pk \sim N^2$. We will try to address the problem of finding the dual geometries of these heavy states in the next chapter.

In [23] the deformations that we are going to study in this chapter were be found directly by doing the LLM integrals of the previous chapter for z and V for small ripples of the circular droplet. Instead, we'll find them by using the deformations associated to the CPOs $\mathcal{O}^k$ introduced in [17] and after that we'll match the solution with the LLM ansatz by calculating the integrals.

8.1 Background Geometry and Deformations

The background geometry seen in the previous chapter is $AdS_5 \times S^5$. We'll use the following coordinates

$$d\bar{s}^2_{10} = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\theta^2 + \cos^2 \theta d\tilde{\phi}^2 + \sin^2 \theta d\tilde{\Omega}_3^2. \tag{8.1}$$

Apart from the metric, in this background only the 5-form is excited

$$\bar{F}_{(5)} = \mathrm{vol}(AdS_5) + \mathrm{vol}(S^5) = \cosh\rho \sinh^3 \rho dt \wedge d\rho \wedge d\Omega_3 + \cos\theta \sin^3 \theta d\theta \wedge d\tilde{\phi} \wedge d\tilde{\Omega}_3, \tag{8.2}$$

which is self-dual since $\bar{\star}_{10}\mathrm{vol}(AdS_5) = \mathrm{vol}(S^5)$ and $\bar{\star}_{10}\mathrm{vol}(S^5) = \mathrm{vol}(AdS^5)$.
In the following we'll use the bar to indicate quantities in this background, $\mu, \nu, ...$ indices for the AdS_5 geometry and $\alpha, \beta, ...$ indices for the S^5 geometry.
Following [17] the metric deformations of this background associated to our $\frac{1}{2}$BPS chiral primary operators are

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} = \bar{g}_{\mu\nu} + h'_{\mu\nu} - \frac{1}{3}\bar{g}_{\mu\nu} h^{\alpha}{}_{\alpha}, \tag{8.3}$$

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + h_{\alpha\beta} = \bar{g}_{\alpha\beta} + \frac{1}{5} h^{\alpha}{}_{\alpha}\bar{g}_{\alpha\beta}, \tag{8.4}$$

where the deformations are expanded in terms of scalar spherical harmonics on S^5

$$h'_{\mu\nu} = H^{(k)}_{\mu\nu}(\rho, t) Y^{(k)}(\theta, \tilde{\phi}), \tag{8.5}$$

$$h^{\alpha}{}_{\alpha} = \pi^{(k)}(\rho, t) Y^{(k)}(\theta, \tilde{\phi}). \tag{8.6}$$

Here a sum over k is understood. Similarly, the deformation of the 5-form is given by

$$C_{(4)} = \bar{C}_{(4)} + c^{AdS_5}_{(4)} + c^{S^5}_{(4)}, \qquad F_{(5)} = dC_{(4)}, \tag{8.7}$$

where

$$c^{AdS_5}_{(4)} = b^{(k)}_{(4)}(\rho, t) Y^{(k)}(\theta, \tilde{\phi}), \tag{8.8}$$

$$c^{S^5}_{(4)} = b^{(k)}(\rho, t) \star_{S^5} dY^{(k)}(\theta, \tilde{\phi}). \tag{8.9}$$

Excitation modes on AdS_5 will only depend on ρ and t requiring the $SO(4)$ symmetry of the LLM geometries. For the same reason, the spherical scalar harmonics on S^5 will only depend on θ and $\tilde{\phi}$. The extra $U(1)$ symmetry requires that the solution only depends on $\tilde{\phi} + t$, as we will see later.

8.2 The Excitation Modes

The scalar spherical harmonics on S^5 satisfy the following eigenvalue equation

$$\Box_{S^5} Y^{(k)} = \frac{1}{\mathrm{vol}(S^5)} d \star_{S^5} dY^{(k)} = -k(k+4)Y^{(k)}, \tag{8.10}$$

where $\Box_{S^5} \equiv \bar{g}^{\alpha\beta} D_\alpha D_\beta$. In the S^5 coordinates

$$ds^2_{S^5} = d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\tilde{\Omega}_3^2, \tag{8.11}$$

requiring $SO(4)$ symmetry, the highest degree general solution is given by

$$Y \equiv Y^{(k,k)}(\theta, \tilde{\phi}) \equiv Y^{(k)}(\theta, \tilde{\phi}) = \cos^k\theta e^{ik\tilde{\phi}}. \tag{8.12}$$

The 4-form excitation mode $b^{(k)}_{(4)}$ is related to the scalar mode $b^{(k)}$ by the self-duality condition $F_{(5)} = \star_{10} F_{(5)}$. It is immediate to see that this implies

$$b^{(k)}_{(4)} = - \star_{AdS_5} db^{(k)}. \tag{8.13}$$

The scalar excitation modes $\pi^{(k)}$ and $b^{(k)}$ are also linked by the equations of motion. In particular they satisfy [17]

$$\Box_{AdS_5} \begin{bmatrix} \pi^{(k)} \\ b^{(k)} \end{bmatrix} - \begin{bmatrix} k(k+4)+32 & 80k(k+4) \\ \frac{4}{5} & k(k+4) \end{bmatrix} \begin{bmatrix} \pi^{(k)} \\ b^{(k)} \end{bmatrix} = 0, \tag{8.14}$$

where $\Box_{AdS_5} \equiv \bar{g}^{\mu\nu} D_\mu D_\nu$. The matrix is diagonalisable. The eigenvalue and the corresponding eigenvector we are interested in are

$$M^2 = k(k-4), \quad \begin{bmatrix} \pi^{(k)} \\ b^{(k)} \end{bmatrix} = b^{(k)} \begin{bmatrix} -10k \\ 1 \end{bmatrix}. \tag{8.15}$$

This is the eigenvector related to our CPOs since $M^2 = \Delta(\Delta - 4)$ implies $\Delta = k$. In this one-dimensional eigenspace, we have the following conditions for the scalar modes

$$\Box_{AdS_5} \pi^{(k)} = \frac{1}{\mathrm{vol}(AdS_5)} d \star_{AdS_5} d\pi^{(k)} = k(k-4)\pi^{(k)}(\rho, t), \quad \pi^{(k)} = -10kb^{(k)}, \tag{8.16}$$

which are therefore harmonic functions on AdS_5. Using the following coordinates for AdS_5

$$ds^2_{AdS_5} = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2, \tag{8.17}$$

the previous equation is solved in a similar manner to that for the spherical harmonics. The highest degree solution is given by

$$\pi \equiv \pi^{(k,k)}(\rho,t) \equiv \pi^{(k)}(\rho,t) = \cosh^{-k}\rho e^{ikt}. \tag{8.18}$$

The last excitation mode that we have to write is the two index symmetric tensor $H^k_{\mu\nu}$ that appears in (8.5). The explicit form of $H^{(k)}_{\mu\nu}$ is not given in [17], but the only natural ansatz is

$$H_{\mu\nu} = \alpha \bar{g}_{\mu\nu}\pi + \beta D_\mu D_\nu \pi, \tag{8.19}$$

where the covariant derivatives are made with the background metric of AdS_5 (8.17). We can derive the constants α, β using the 10-dimensional equations of motion. In particular we have the following equations [17]

$$H_{\mu\mu} = \frac{16}{15}\pi, \tag{8.20}$$

$$D_\rho H_{\rho\mu} - D_\mu(\frac{8}{15}\pi + 16b) = 0. \tag{8.21}$$

By contracting (8.19) with the background AdS_5 metric $\bar{g}^{\mu\nu}$, using (8.20) one obtain the following relation

$$\frac{16}{15}\pi = 5\alpha\pi + \beta\Box_{AdS_5}\pi = 5\alpha\pi + \beta k(k-4)\pi, \tag{8.22}$$

which is the first equation for the coefficients. The second one is given by (8.21) which using our ansatz becomes

$$\alpha D_\mu \pi + \beta \Box_{AdS_5} D_\mu \pi - D_\mu(\frac{8}{15}\pi + 16b) = 0. \tag{8.23}$$

We have

$$\begin{aligned}\Box_{AdS_5} D_\mu \pi = \bar{g}_{\rho\sigma} D^\rho D^\sigma D_\mu \pi = \bar{g}_{\rho\sigma} D^\rho D_\mu D^\sigma \pi = D_\mu \Box_{AdS_5}\pi + \bar{g}_{\rho\sigma}[D^\rho, D_\mu] D^\sigma \pi \\ = D_\mu \Box_{AdS_5}\pi + \bar{g}_{\rho\sigma} R^\rho{}_\mu{}^\sigma{}_\lambda D^\lambda \pi = k(k-4) D_\mu \pi - 4 D_\mu \pi,\end{aligned}$$

where in the last equality we have used the explicit form of the Riemann tensor on AdS_5

$$R_{\mu\nu\rho\sigma} = -(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}). \tag{8.24}$$

So $D_\mu\pi$ is an eigenvector of the laplacian on AdS_5 with eigenvalue $k(k-4)-4$. Using this fact, from (8.22) and (8.23) we obtain

$$\alpha = \frac{2}{15}\frac{k+4}{k+1}, \qquad \beta = \frac{2}{5}\frac{1}{k(k+1)}, \tag{8.25}$$

and the two index symmetric tensor becomes

$$H_{\mu\nu} = \frac{2}{5k(k+1)}\left(D_\mu D_\nu \pi + \frac{k(k+4)}{3}\bar{g}_{\mu\nu}\pi\right). \tag{8.26}$$

We have therefore explicitly written down all the excitation modes. In particular, using (8.12) and (8.18), the deformations will be defined in terms of the function

$$\epsilon\pi Y \equiv \epsilon\pi^{(k)}(\rho,t)Y^{(k)}(\theta,\tilde{\phi}) = -10kb^{(k)}(\rho,t)Y^{(k)}(\theta,\tilde{\phi}) = \epsilon e^{ik\phi}\left(\frac{\cos\theta}{\cosh\rho}\right)^k. \tag{8.27}$$

Here we introduced $\phi \equiv \tilde{\phi} + t$ so that we confirm that BPS geometries depend on ϕ, but are independent of t. Furthermore, we have introduced a small factor ϵ that quantifies the deformation. Since we are interested in small deformations in the following we will always work at the first order in this parameter.

8.3 The Excited Geometry

Using (8.7)-(8.9) and (8.13) we can explicitly write the perturbed 5-form. The 4-form is

$$C_{(4)} = \bar{C}_{(4)} + b \star_{S^5} dY - Y \star_{AdS_5} db, \tag{8.28}$$

and thus the 5-form is

$$\begin{aligned} F_{(5)} = dC_{(4)} &= \bar{F}_{(5)} + db \wedge \star_{S^5} dY + bd\star_{S^5} dY - dY \wedge \star_{AdS_5} db - Yd\star_{AdS_5} db = \\ &= \bar{F}_{(5)} + db \wedge \star_{S^5} dY - bYk(k+4)\mathrm{vol}(S^5) - dY \wedge \star_{AdS_5} db - bYk(k-4)\mathrm{vol}(AdS_5). \end{aligned} \tag{8.29}$$

Similarly, using (8.3)-(8.6) with our ansatz (8.26), the perturbed metric is

$$ds^2_{10} = ds^2_{AdS_5}\left(1 + 2k\frac{k-1}{k+1}bY\right) - \frac{4}{k+1}(D_\mu D_\nu b)Y dx^\mu dx^\nu + ds^2_{S^5}(1-2kbY). \tag{8.30}$$

We have

$$
\begin{aligned}
(D_\mu D_\nu b)dx^\mu dx^\nu =&(\partial_t^2 b - \cosh\rho\sinh\rho\partial_\rho b)dt^2 + \partial_\rho^2 b\, d\rho^2 + \cosh\rho\sinh\rho\partial_\rho b\, d\Omega_3^2 + \\
&(\partial_t\partial_\rho b - \tanh\rho\partial_t b)dt d\rho,
\end{aligned}
$$

and using the explicit form of the excitation mode (8.27) the 10-dimensional perturbed metric becomes

$$
\begin{aligned}
ds_{10}^2 =& d\Omega_3^2(1+2kbY)\sinh^2\rho + d\tilde{\Omega}_3^2(1-2kbY)\sin^2\theta + d\rho^2[1-2k(-1+2\tanh^2\rho)bY] + \\
& dt^2[-\cosh^2\rho - k(-3+\cosh 2\rho)bY] + d\theta^2(1-2kbY) + d\tilde{\phi}^2(1-2kbY)\cos^2\theta + \\
& d\rho dt\, 4ki\tanh\rho bY.
\end{aligned} \tag{8.31}
$$

The self-duality condition of the 5-form at the first orderd in ϵ is

$$
F_{(5)} = \star_{10} F_{(5)} = \bar{\star}_{10} F_{(5)} + \delta\star_{10}\bar{F}_{(5)}, \tag{8.32}
$$

where $\bar{\star}_{10}$ is performed with the background geometry (8.1) and $\delta\star_{10}$ with the perturbated 10-dimensional metric. To verify this condition, since $\bar{F}_{(5)}$ is a volume form in both AdS_5 and S^5, one can compute $\delta\star_{10}$ by using only the diagonal part of the deformed metric. This metric is obtained by placing $\beta = 0$ in the ansatz (8.19) and using again (8.3)-(8.6); we have

$$
ds^2 = (1 - \frac{3}{25}\pi Y)ds^2_{AdS_5} + (1 + \frac{1}{5}\pi Y)ds^2_{S^5} \quad \leftrightarrow \quad \star_{10}, \tag{8.33}
$$

$$
\delta ds^2 = -\frac{3}{25}\pi Y ds^2_{AdS_5} + \frac{1}{5}\pi Y ds^2_{S^5} \quad \leftrightarrow \quad \delta\star_{10}, \tag{8.34}
$$

$$
d\bar{s}^2 = ds^2_{AdS_5} + ds^2_{S^5} \quad \leftrightarrow \quad \bar{\star}_{10}. \tag{8.35}
$$

Using the 10 dimensional orientation $\text{vol}_{10} = \text{vol}(AdS_5)\wedge\text{vol}(S^5)$ we have

$$
\star_{10}\,\text{vol}(AdS_5) = (1 + \frac{4}{5}\pi Y)\text{vol}(S^5), \qquad \star_{10}\text{vol}(S^5) = (1 - \frac{4}{5}\pi Y)\text{vol}(AdS_5), \tag{8.36}
$$

$$
\bar{\star}_{10}(db\wedge\star_{S^5}dY) = -\star_{AdS_5} db\wedge(\star_{S^5})^2 dY = -\star_{AdS_5} db\wedge dY, \tag{8.37}
$$

$$
\bar{\star}_{10}(dY\wedge\star_{AdS_5}db) = -\star_{S^5} dY\wedge(\star_{AdS_5})^2 db = \star_{S^5} dY\wedge db, \tag{8.38}
$$

where we have used $(\star_{AdS_5})^2 = -1$ and $(\star_{S^5})^2 = 1$. So the self-duality condition (8.32) becomes

$$
-\frac{4}{5}\pi - bk(k+4) = -bk(k-4), \tag{8.39}
$$

which confirms the relationship between the scalar modes that we have already obtained from the equations of motion $\pi^k = -10kb^k$.

8.4 Match With the LLM Ansatz

The perturbed geometry we obtained in (8.29), (8.31) must be an LLM solution. In this section we will make this explicit by comparing our solution with the LLM ansatz

$$ds^2 = -h^{-2}(dt' + V_i dx^i)^2 + h^2(dy^2 + dr^2 + r^2 d\varphi^2) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2, \tag{8.40}$$

where we are using coordinates t' and φ, instead of t and ϕ, to distinguish them from the coordinates of (8.29), (8.31). We will determine the proper coordinate transformation below. By comparing the spherical terms at the first order in the perturbation parameter we can immediately derive

$$y = \sinh\rho \sin\theta, \tag{8.41}$$

and

$$e^G = \frac{\sinh\rho}{\sin\theta}(1 - \frac{1}{5}\pi Y) = \frac{\sinh\rho}{\sin\theta}(1 + 2kbY). \tag{8.42}$$

So, using (7.7) we get

$$z(\rho,\theta,\phi) = \frac{1}{2}\frac{e^G - e^{-G}}{e^G + e^{-G}} \simeq -\frac{1}{2} + \frac{1}{1 + \frac{\sin^2\theta}{\sinh^2\rho}} + \frac{4k\sin^2\theta}{\sinh^2\rho(1 + \frac{\sin^2\theta}{\sinh^2\rho})}bY. \tag{8.43}$$

Similarly, using (7.5) we get

$$h^{-2}(\rho,\theta,\phi) = y(e^G + e^{-G}) \simeq \sin^2\theta + \sinh^2\rho - 2k(\sin^2\theta - \sinh^2\rho)bY, \tag{8.44}$$

$$h^2(\rho,\theta,\phi) \simeq \frac{1}{\sin^2\theta + \sinh^2\rho} + 2k\frac{\sin^2\theta - \sinh^2\rho}{(\sin^2\theta + \sinh^2\rho)^2}bY. \tag{8.45}$$

To complete the matching, we must write the change of coordinates from the LLM coordinates (y, r, t', φ) to those of our perturbed geometry $(\rho, \theta, t, \phi \equiv \tilde{\phi} + t)$. To this end, we make the following ansatz

$$y = \sinh\rho\sin\theta \tag{8.46}$$

$$r = \cosh\rho\cos\theta + \epsilon e^{ik\phi} f(\rho,\theta) \tag{8.47}$$

$$t' = t + \epsilon e^{ik\phi} g(\rho,\theta) \tag{8.48}$$

$$\varphi = \phi + \epsilon e^{ik\phi} l(\rho,\theta) \tag{8.49}$$

so that at the zero order we recover the change of cooordinates (7.15)-(7.17). Furthermore, since the 1-form V has only a non zero φ component at the zero order (7.14), we also make the following ansatz

$$V_\varphi = V_\varphi^{(0)} + \epsilon e^{ik\phi} v_\varphi(\rho,\theta), \qquad V_\varphi^{(0)} = \frac{2\cos^2\theta}{\cosh 2\rho - \cos 2\theta}, \tag{8.50}$$

$$V_r = \epsilon e^{ik\phi} v_r(\rho,\theta). \tag{8.51}$$

We have a total of five unknown functions to derive by performing the coordinate change and comparing the result with our perturbed geometry (8.31).
With the coordinate change $\phi = \tilde{\phi} + t$, the perturbed metric becomes

$$\begin{aligned} ds_{10}^2 =& d\Omega_3^2(1+2kbY)\sinh^2\rho + d\tilde{\Omega}_3^2(1-2kbY)\sin^2\theta + d\rho^2[1-2k(-1+2\tanh^2\rho)bY] + \\ & dt^2[-\cosh^2\rho + \cos^2\theta - k(-3+\cosh 2\rho)bY - 2k\cos^2\theta bY] + d\theta^2(1-2kbY) + \\ & d\phi^2(1-2kbY)\cos^2\theta - d\phi dt 2\cos^2\theta(1-2kbY) + d\rho dt\, 4ki\tanh\rho bY. \end{aligned} \tag{8.52}$$

By writing the LLM metric (8.40) using the previous ansatz and comparing the result with this perturbed metric, at the first order in ϵ we obtain:

$$f(\rho,\theta) = -\frac{2k}{k+1}\left(\frac{\cos\theta}{\cosh\rho}\right)^{k+1} \tag{8.53}$$

$$g(\rho,\theta) = l(\rho,\theta) = \frac{2ik}{\cosh^2\rho(1+k)}\left(\frac{\cos\theta}{\cosh\rho}\right)^{k} \tag{8.54}$$

$$v_r(\rho,\theta) = \frac{4ik^2\sec\theta\mathrm{sech}\rho}{(1+k)(\cos 2\theta - \cosh 2\rho)}\left(\frac{\cos\theta}{\cosh\rho}\right)^{k} \tag{8.55}$$

$$v_\varphi(\rho,\theta) = -\frac{4k[-1-k+\cosh 2\rho + \cos 2\theta(-1+(1+k)\cosh 2\rho)]}{(1+k)(\cos 2\theta - \cosh 2\rho)^2}\left(\frac{\cos\theta}{\cosh\rho}\right)^{k}. \tag{8.56}$$

So our perturbed geometry is an LLM solution that corresponds to a certain boundary condition on the (x_1, x_2) plane with $y = 0$. The droplet is a small perturbation of the disk that correspond to the background geometry $AdS_5 \times S^5$ as shown in figure 8.1.

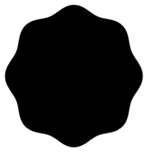


Figure 8.1: This figure is taken from [24]. This is the droplet associated to our small deformations of $AdS_5 \times S^5$. The boundary of the droplet is the wave $r = 1 + \epsilon \cos k\phi$.

To see this, we can for example calculate the 1-form V

$$V_i(x_1, x_2, y) = \frac{1}{2\pi} \oint_{\partial\mathcal{D}} \frac{dx'_i}{(x - x')^2 + y^2}, \tag{8.57}$$

integrating on the boundary of a disk of radius $r = 1 + \epsilon \cos k\phi$ and then comparing the result with our solution (8.55),(8.56). Using polar coordinates r, ϕ on the (x_1, x_2) plane we have

$$\begin{aligned} x_1 &= r\cos\phi, & x_2 &= r\sin\phi \\ x'_1 &= (1 + \epsilon\cos k\phi')\cos\phi', & x'_2 &= (1 + \epsilon\cos k\phi')\sin\phi' \end{aligned} \tag{8.58}$$

and the one-form components are

$$\begin{aligned} V_r &= \cos\phi V_1 + \sin\phi V_2, \\ V_\varphi &= -r\sin\phi V_1 + r\cos\phi V_2. \end{aligned} \tag{8.59}$$

It is convenient to define $A \equiv \frac{2r}{1+r^2+y^2}$ and $B \equiv 1 + r^2 + y^2$. The zero order terms are

$$\begin{aligned} V_r^{(0)} &= -\frac{1}{2\pi}\int_0^{2\pi} d\phi' \frac{\sin\phi'}{B - 2r\cos\phi'} = 0, \\ V_\phi^{(0)} &= \frac{r}{2\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{B - 2r\cos\phi'} = \frac{1}{2}\left(\frac{1}{\sqrt{1-A^2}} - 1\right), \end{aligned} \tag{8.60}$$

which are the ones discussed in section 7.2. The terms of order one are

$$\begin{aligned} V_r^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} d\phi' \frac{Bk\cos\phi'(-1 + A\cos\phi')\sin[k(\phi'+\phi)] - (B-2)\cos[k(\phi'+\varphi)]\sin\phi'}{B^2(-1 + A\cos\phi')^2}, \\ V_\phi^{(1)} &= \frac{r}{2\pi}\int_0^{2\pi} d\phi' \frac{Bk\sin\phi'(-1 + A\cos\phi')\sin[k(\phi'+\phi)] + (B-2)\cos[k(\phi'+\phi)]\cos\phi'}{B^2(-1 + A\cos\phi')^2}. \end{aligned} \tag{8.61}$$

These integrals can be calculated using the residue method. We pass in the complex plane by working in terms of complex exponentials. Defining $z = e^{i\phi'}, dz = izd\phi'$ the poles are

$$Az^2 - 2z + A = 0 \quad \Rightarrow \quad z_\pm = \frac{1 \pm \sqrt{1-A^2}}{A}, \tag{8.62}$$

and since $z_+ z_- = 1$ only z_- is contained in the unit circle over which we integrate. By calculating the residual in the pole z_- and multiplying it by $2\pi i$ one obtain

$$\begin{aligned} V_r^{(1)} &= \frac{4ik^2}{(1+k)B^2 A\sqrt{1-A^2}} z_-^{-k}\sin(k\phi), \\ V_\phi^{(1)} &= \frac{2kr}{(1+k)B^2(1-A^2)^{3/2}}\left[(r-A) + k(r-A^{-1})\sqrt{1-A^2}\right] z_-^{-k}\sin(k\phi), \end{aligned} \tag{8.63}$$

and using our change of coordinates (8.46)-(8.49) one obtain (8.55),(8.56) up to a normalization factor as expected (for $V_\phi^{(1)}$ one has to include also the terms of order one which come from the change of coordinates on $V_\phi^{(0)}$). Note that since we are integrating over a real profile, instead of complex exponentials we have the trigonometric factors. Since we are working at linear order, it is of course immediate to switch between complex exponentials and trigonometric functions.

In this chapter we have therefore understood which are the deformations of $AdS_5 \times S^5$ dual to the chiral primary operators $\frac{1}{2}$-BPS $\mathcal{O}^k = Tr\{Z^k\}$ with $\Delta = k \ll N^2$. In this section we have also shown that these deformations correspond to LLM solutions defined by small ripples of the disk associated to the background solution. In the next chapter we'll generalize this and we'll try to say something about the solution at higher perturbative orders in the parameter ϵ. These geometries are dual to the heavy states construed by taking p times the previous light operators $\mathcal{O} = [Tr\{Z^k\}]^p$ with $\Delta = pk \sim \epsilon N^2$.

9
Beyond the Linear Order

In the previous chapter we have studied linear deformations of $AdS_5 \times S^5$ which are dual to the light states of the $\mathcal{N} = 4$ $SU(N)$ CFT, namely the single-trace operators $\mathcal{O}_{k,p} = (\mathrm{Tr}Z^k)^p$ with $p \sim 1$, $k = 2, 3, 4, \ldots$ finite and $Z = \phi^1 + i\phi^2$ a complex combination of the adjoint scalars ϕ^I with $I = 1, \ldots, 6$. Now we would like to say something about heavy operators, namely those with a conformal dimension that grows as the central charge of the CFT: $p \sim c \sim N^2$. On the CFT side these multi-trace operators are products of the single-trace light operators and it is therefore intuitive to think of the dual geometry as a large perturbation of the background in the heavy-classical supergravity limit, defined by taking

$$c = \frac{N^2}{4} \gg 1, \quad p \gg 1, \quad \frac{p}{N^2} \text{ fixed.} \tag{9.1}$$

While the behavior of the geometry is the same as the one of the D1-D5 theory at the linear order (in fact, as we have seen in chapter 6 and in chapter 8, in this limit both geometries are small perturbations of the corresponding background) the situation is different for great value of the perturbative parameter. In the D1-D5 theory the "stringy exclusion principle" [25], put an upper bound on the possible values of p. The situation is different in the case of AdS_5 where the dual CFT is in $D = 4$ and such a limit on the possible values of p does not exist. Furthermore, by studying the asymptotic limit of the linear order geometry, it was shown in [24], that $p \sim \epsilon^2$ and consequently we can deduce that there is not an upper bound on ϵ either. From this discussion, it is natural to think that for large values of ϵ one can in principle obtain arbitrarily complicated geometries contrary to what happens with the theory on AdS_3. In the LLM picture this is due to the existence of infinitely many ways to choose the droplet that reduces to $r = 1 + \epsilon \cos(k\phi)$ at linear order in ϵ. We can also deduce that the profile on the LLM droplet must receive corrections at orders higher than the first since for $\epsilon \geq 1$ it becomes singular and thus does not provide a well-defined LLM geometry (this is hard to verify explicitly because the LLM integrals are difficult to compute in closed form for finite values of ϵ).
The problem of finding the geometry at all perturbative orders in the heavy classical limit

and determining the corresponding LLM droplet is left to future developments and one possible way to approach this problem is better explained in the conclusive chapter. Here we limit ourselves to borrow the solution up to the second perturbative order obtained in a parallel development that will not be described in this thesis, and we prove that it falls into the class of LLM solutions related to a specific droplet.

9.1 The Second Perturbative Order

As we just said in the introduction to the chapter, determining the solution at the second perturbative order is beyond the scope of this work and hence we limit ourself to describe it. This result comes from the consistent truncation of the equations of motion [26] which, combined with the supersymmetry conditions, greatly simplifies the problem of solving explicitly the equations of motion. The fundamental point is that the solution of these equations is unique and consequently it is absolutely not trivial that it is dual to our heavy states $\mathcal{O}_{k,p}$ instead that to some other more complicated heavy multi-traces. At this stage this is just a conjecture motivated by the analogy with the case on AdS_3 and by the fact that the consistent truncation eliminates all of the d.o.f. apart from those associated with the lightest CPOs $\mathcal{O}_2$. Consequently we assume that the following result is dual to the heavy states $\mathcal{O}_{2,p}$. This assumption can however be confirmed through the AdS/CFT correspondence by calculating the appropriate correlation functions as better explained in the conclusions of this work. As we have seen in the previous chapters the solution must have the $SO(4) \times SO(4) \times \mathbb{R} \times \mathbb{Z}_2$ symmetry (this symmetry must remain at all perturbative orders). The 10-dimensional metric can be written as

$$ds_{10}^2 = \Delta^{1/2} ds_5^2 + \Delta^{-1/2} T_{ij}^{-1} D\mu^i D\mu^j, \tag{9.2}$$

where

$$\Delta = T_{ij}\mu^i\mu^j, \quad D\mu^i = d\mu^i + A^{ij}\mu^j \quad (i, j = 1, \dots, 6). \tag{9.3}$$

The coordinates μ^i are a parametrization of S^5

$$\begin{aligned} &\mu^1 + i\mu^2 = \cos\theta e^{i\tilde{\phi}}, \\ &\mu^I = \sin\theta \hat{x}^I \quad \text{for} \quad I = 3, \dots, 6 \quad \text{with} \quad \hat{x}^I \hat{x}^I = 1. \end{aligned} \tag{9.4}$$

The five dimensional metric is the asymptotically AdS_5 Einstein metric given by

$$ds_5^2 = d\rho^2 - (1 + 2\epsilon^2\omega) \cosh\rho^2 d\tau^2 + \sinh^2\rho d\Omega_3^2 \quad \text{with} \quad \omega = -\frac{\cosh^4\rho}{6}, \tag{9.5}$$

while the symmetric unimodular tensor $T_{ij} = T_{ji}$, $\det(T) = 1$ is

$$T = \left(\begin{array}{cc|c} 1+\epsilon\lambda-\epsilon^2(2\mu-\frac{\lambda^2}{2}) & 0 & \multirow{2}{*}{$\mathbf{0}$} \\ 0 & 1-\epsilon\lambda-\epsilon^2(2\mu-\frac{\lambda^2}{2}) & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & (1+\epsilon^2\mu)\mathbb{I}_4 \end{array}\right), \tag{9.6}$$

with $\lambda = \cosh^{-2}\rho$ and $\mu = \frac{1}{6}\cosh^{-2}\rho$. Finally, the gauge fields $A^{ij} = -A^{ji}$ that appear in the covariant derivative are given by

$$A^{12} = (1+\epsilon^2\Phi)d\tau \quad \text{with} \quad \Phi = -\frac{1}{2}\cosh^{-2}\rho, \tag{9.7}$$

with the other components null.
As a first check of this solution we note that the background geometry $AdS_5 \times S^5$ is recovered at the zero order. Moreover, at first order this solution is the linearised metric (8.30) of the previous chapter with $k = 2$ if we apply to the latter the following diffeomorphism

$$(\xi_\mu, \xi_\alpha) = \frac{2}{3}(\nabla_\mu b^{(2)} Y^{(2)}, -b^{(2)}\nabla_\alpha Y^{(2)}), \tag{9.8}$$

under which the perturbative terms of (8.30) transform as

$$\begin{aligned} h_{\mu\nu} &= h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu, \\ h_{\alpha\beta} &= h_{\alpha\beta} + \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha. \end{aligned} \tag{9.9}$$

Note that now we have to take only the real part of the harmonic functions, i.e. we have $Y^{(2)} = \cos^2\theta\cos(2\phi)$ and $b^{(2)} = \cosh^{-2}\rho$. We can therefore conclude that this is a solution only for the lighest CPO $\mathcal{O}_{k=2}$. As we have done for the first order we can demonstrate that this geometry is an LLM solution. Note that this check is highly non-trivial since the comparison between the two metrics provides a system of 10 first-order differential equations. Since we don't know the second order correction of the LLM profile used in the previous section we do the match with a generic LLM solution by finding the appropriate change of coordinates instead of calculating the LLM integrals. By comparing the spherical terms of (9.2) with those of the LLM solution (7.4) we find the second order correction for y

$$y = \sin\theta\sinh\rho - \frac{\epsilon^2}{12}\frac{\sinh\rho}{\cosh^2\rho}\sin\theta, \tag{9.10}$$

from which we can calculate also the second order correction for e^G and h^{-2} using (7.5). The most general change of coordinates up to the second order is

$$y = \sin\theta \sinh\rho - \frac{\epsilon^2}{12}\frac{\sinh\rho}{\cosh^2\rho}\sin\theta, \tag{9.11}$$

$$r = \cos\theta\cosh\rho - \frac{\epsilon}{2}\frac{\cos\theta}{\cosh\rho}\cos 2\phi + \epsilon^2(f_{r1}(\rho,\theta)\cos 4\phi + f_{r2}(\rho,\theta)), \tag{9.12}$$

$$t = \tau + \epsilon^2 f_t(\rho,\theta)\sin 4\phi, \tag{9.13}$$

$$\varphi = \phi + \frac{\epsilon}{2}\cosh^{-2}\rho\sin 2\phi + \epsilon^2 f_\varphi(\rho,\theta)\sin 4\phi, \tag{9.14}$$

while the ansatz for the 1-form V is

$$\begin{aligned} V_\varphi =& \frac{2\cos^2\theta}{\cosh 2\rho - \cos 2\theta} - \epsilon\frac{4\cos^4\theta\tanh^2\rho}{(\cos 2\theta - \cosh 2\rho)^2}\cos 2\phi + \epsilon^2(v_{\varphi 1}(\rho,\theta)\cos 4\phi + v_{\varphi 2}(\rho,\theta)),\\ V_r =& -\epsilon\frac{2\cos\theta\operatorname{sech}^3\rho}{\cos 2\theta - \cosh 2\rho}\sin 2\phi + \epsilon^2 v_r(\rho,\theta)\sin 4\phi, \end{aligned} \tag{9.15}$$

where we have chosen the terms of the first order to match the LLM solution with (9.2) (i.e. we have applied the diffeomorphism (9.9) to the coordinates (8.46)-(8.49)) and we are using again the notation $\phi \equiv \tilde{\phi} + \tau$. By writing the LLM metric (7.4) under the previous change of coordinates and comparing the result with (9.2) at the second order we obtain

$$f_{r1}(\rho,\theta) = -\frac{1}{16}\cos\theta\operatorname{sech}^3\rho, \tag{9.16}$$

$$f_{r2}(\rho,\theta) = \frac{1}{48}\cos\theta(5 + 4\cosh 2\rho)\operatorname{sech}^3\rho, \tag{9.17}$$

$$f_t(\rho,\theta) = 0, \tag{9.18}$$

$$f_\varphi(\rho,\theta) = \frac{1}{8}\operatorname{sech}^4\rho, \tag{9.19}$$

$$v_{\varphi 1}(\rho,\theta) = -\frac{\cos^2\theta[\cosh^2\rho(-2 + \cos 2\theta + \cosh 2\rho) + 8\cos^4\theta\tanh^4\rho]}{2(\cos 2\theta - \cosh 2\rho)^3}, \tag{9.20}$$

$$v_{\varphi 2}(\rho,\theta) = \frac{\cos^2\theta[3 + \cosh 2\rho - \cos 2\theta(1 + 3\cosh 2\rho)]\tanh^2\rho}{3(\cos 2\theta - \cosh 2\rho)^3}, \tag{9.21}$$

$$v_r(\rho,\theta) = -\frac{[-2\cos\theta + (\cos\theta + \cos 3\theta)\cosh 2\rho]\operatorname{sech}^5\rho}{4(\cos 2\theta - \cosh 2\rho)^2}. \tag{9.22}$$

The existence of a well-defined solution demonstrate that our second order geometry (9.2) falls into the class of the LLM solutions and it is therefore associated to a certain droplet in the LLM plane. As we have mentioned in the introductory part of the chapter, the absence of a stringy exclusion principle for the $\mathcal{N} = 4$ $SU(N)$ theory, implies that the

LLM profile we have used in the previous chapter for the linear solution must receive a correction beyond the first order

$$r(\phi) = 1 + \epsilon \cos(2\phi) + o(\epsilon^2), \tag{9.23}$$

where here we are using $k = 2$. We may ask whether such a correction is already present at the second order. For this purpose, similarly to what we did in the previous chapter, we can calculate the one form V with (8.57) by using the usual linear profile and compare the result with (9.15) after applying the change of coordinates (9.11)-(9.14). By doing this we find a result for the components of V that do not agree with (9.15) and consequently the LLM profile must receive corrections also to this order. It is intuitive to think that this correction is proportional to $\cos 4\phi$; in particular we have found that the correct profile that reproduces the results (9.15) up to second order is

$$r(\phi) = 1 - \frac{\epsilon}{2}\cos(2\phi) + \frac{3\epsilon^2}{16}\cos(4\phi), \tag{9.24}$$

where we have also inserted the correct normalization factor in the first order term.
To conclude, in this chapter we have found the geometry related to the heavy states up to the second perturbative order and we have demonstrated that this solution falls into the class of LLM solutions by finding the corresponding droplet on the LLM plane. The second order correction of the LLM droplet (9.24) is the most important original result of this thesis since in some works such as that of Skenderis et al. [24] it was not taken into consideration. As we will explain better in the conclusive chapter, a possible future development is to solve the $\frac{1}{2}$-BPS supersymmetry constraints starting from this solution to find the exact one at all perturbative orders.

Summary and Future Developments

In this work we have studied an important aspect of the relationship between the $\mathcal{N}=4$ $SU(N)$ SYM theory and the supergravity theory on $AdS_5 \times S^5$. In particular we have used the AdS/CFT correspondence to find the geometries in the gravitational theory dual to $\frac{1}{2}$-BPS operators of the CFT. We have started by studying geometries dual to such operators with a small conformal dimension (light operators) and then we have done a step forward in studying the geometries dual to operators with a conformal dimension of the order of the central charge in the classical limit (heavy operators).

In order to find these geometries we have introduced some basic ingredients of the AdS/CFT correspondence. Initially we have described the fundamental aspects of supersymmetric field theories, then we have presented the first part of the correspondence starting from string theory and arriving at its low-energy limit, i.e. supergravity, which is the gravitational theory we worked with. In this context we have given more importance to the type IIB supergravity theory since it is the relevant one for the purposes of our application; we have described its fields and their coupling with branes and then we have provided some solutions that carry charges associated to these branes. Afterwards, we have presented the latter side of the duality, introducing the basic principles of any conformal field theory paying more attention to the theory of our interest, namely the $\mathcal{N}=4$ $SU(N)$ SYM theory. The motivation of the AdS/CFT correspondence was given following the historical developments: we first showed the link between $SU(N)$ gauge theories in the large N limit and string theories, and then we have given a more concrete justification through the argument of the open/closed string duality. After that we have described the chiral primary operators, i.e. some special operators of the CFT whose conformal dimension does not depend on the coupling. For this reason such operators can be studied quantitavely on both sides of the duality and are used to test the AdS/CFT correspondence on a practical level; for our application we have focussed on the $\frac{1}{2}$-BPS operators. As an example of the application of the correspondence we have made a short digression on the $D1$-$D5/AdS_3$ duality and we have considered a particular well-known geometry dual to a specific state of the D1-D5 CFT. Before considering the main problem we have described the LLM geometries which are all the $\frac{1}{2}$-BPS excitations of $AdS_5 \times S^5$ dual to the $\frac{1}{2}$-BPS operators of the CFT following the AdS/CFT correspondence. These geometries are all uniquely specified by black and white color coding droplets on a 2-dimensional plane and all the geometries we are interested in falls into this class. Finally, we have explained the main problem that was approached in this work. We first wanted to find the supergravity dual to the single-trace $\frac{1}{2}$-BPS light operators of the CFT

$$\mathcal{O}_k = Tr(Z^k), \quad k = 2, 3, 4, \dots \tag{10.1}$$

where $Z = \phi^1 + i\phi^2$ is a complex combination of the adjoint scalars ϕ^I with $I = 1, \dots, 6$. We have written explicitly the dual geometry as a small deformation of the background $AdS_5 \times S^5$ working at the linear order in the perturbative parameter ϵ. Subsequently we have proved that our linearised geometry falls within the class of the LLM solutions: at first order in ϵ it coincides with the LLM geometry associated with a deformed circular droplet with boundary

$$r(\phi) = 1 - \frac{\epsilon}{2}\cos(k\phi), \tag{10.2}$$

where ϕ is the angle in polar coordinates on the LLM plane and ϵ the parameter that quantifies the fluctuations around the background geometry. After that, we have considered more complex operators constructed by taking products of these light operators obtaining multi-traces heavy operators

$$\mathcal{O}_{k,p} = \left[Tr(Z^k)\right]^p, \tag{10.3}$$

with k finite and $p \sim N^2$ and we set ourselves the problem of finding the classical dual geometry (to be more precise the states that admit a classical supergravity dual are actually coherent state superpositions of the states $\mathcal{O}_{k,p}$, centered over some average value of p but with a non-vanishing spread over a finite range of p's). At this end in principle one must start with the linear solution related to the light operators defined by the limit $\epsilon \to 0$ and explicitly solve the equations of motion of the supergravity theory to obtain the exact solution at all perturbative orders. However, this approach encounters two difficulties: first solving directly the equations of motion is complicated and second, the solution is not unique, since at each order one can add an arbitrary solution of the homogeneous equations. To circumvent these problems, one can work in a consistent truncation (such that any solution of the truncated theory also solves the full equations) and further simplify the equations by using the supersymmetry constraints. This development will be carried out elsewhere and it was not be described in this thesis. We have borrowed the second order result obtained via the consistent truncation for the lightest heavy state with $k = 2$ and we have shown that this result is reproduced by an LLM geometry where the droplet (10.2) is modified at second order

$$r(\phi) = 1 - \frac{\epsilon}{2}\cos(2\phi) + \frac{3\epsilon^2}{16}\cos(4\phi). \tag{10.4}$$

This represent the most important and new result of this thesis since in some recent works such as [24] this correction was not taken into consideration. Furthermore it provides a link between the LLM droplet and multi-traces operators like the one in (10.3).
Future developments concern determining the exact geometry at all perturbative orders

by imposing supersymmetry constraints. As we were able to see in Chapter 6, when we have considered the case of the D1-D5 theory dual to the supergravity theory on AdS_3, for very small values of the fluctuation parameter ϵ the problem on AdS_5 is analogous to that on AdS_3 in the sense that both geometries result to be small perturbations of the corresponding background. However, one difference between the two theories is given by the "stringy exclusion principle" of the D1-D5 CFT [25]. According to this principle in 2D CFT there is a maximum value for the parameter p in (10.3) related to the existence of an upper limit on the $U(1)$ charge of chiral primaries

$$Q \leq \frac{c}{3} \text{ with } c \sim N. \tag{10.5}$$

The validity of the bound follows from general simmetry considerations of the CFT in $D = 2$. As we have seen a chiral primary of the conformal field theory will be a single-particle state on AdS_3 and a second chiral primary can be constructed by squaring this chiral primary. In general p-th power of the chiral primary corresponds to p particles in the same mode. An upper bound on p translates into an exclusion principle limiting the occupation numbers of bosonic BPS particle modes: for values of p of the order of the central charge the chiral primary will vanish. From the geometrical point of view, this principle means that for large values of the parameter $\epsilon \sim p$, we have pathological geometries with closed time-like curves, as is well known (see, for example [19],[20] and [21]). The situation is different in the case of AdS_5 where such a limit does not exist and consequently in the limits of large ϵ one expects to obtain arbitrarily complicated geometries. In the LLM picture this complication is related to the existence of infinitely many ways to choose the droplet that reduces to (10.2) at the linear order in ϵ. We can also deduce that the profile on the LLM droplet (10.2) must receive corrections at orders higher than the first; in fact for $\epsilon \geq 1$ the profile becomes singular and thus cannot provide a well-defined LLM geometry (however, this is hard to verify explicitly because the integrals needed to write down the LLM metric are difficult to compute in closed form for finite values of ϵ). Here we have found the correction up to the second order and in the future it will be necessary to find the exact profile at all perturbative orders.
As usual, once we have obtained the exact geometry, which we symbolically denote by $ds^2_\epsilon \longleftrightarrow |H\rangle$ where $|H\rangle$ is the dual (coherent superposition) heavy state, the correspondence must be tested by calculating protected quantities, i.e. quantities that match in the CFT and gravity sides. Such quantities are, for example, the correlation functions of some CPO O_L between two heavy states

$$\langle H(\infty)|O_L(1)|H(0)\rangle \,. \tag{10.6}$$

Such correlation functions can be calculated in both CFT and gravity theory by considering the asymptotic limit of ds^2_ϵ [24]. After that one can calculate unprotected quantities such as

$$\langle H(\infty)|O_L(z_1)O'_L(z_2)|H(0)\rangle \,, \tag{10.7}$$

where z_1 and z_2 are two different points in the CFT theory in which we evaluate the CPOs. Such quantities are unprotected since one can insert in the intermediate channel a resolution of the identity, $\sum_I |O^I_L\rangle\langle O^I_L|$, which is a sum over all possible CFT states, BPS and non-BPS. In general such correlation functions have never been calculated; since it's complicate to calculate them in the CFT one must use the AdS/CFT correspondence by solving a wave equation for the metric ds^2_ϵ and then by applying the holographic principle. As we have already mentioned in the introduction of this work, one of the major applications of the holographic principle concerns also the study of black holes. Black holes are singular solutions of Einstein's equations that have an event horizon, i.e. a one-way membrane that causally divides spacetime into the external universe and the black hole interior. The quantum theory of black holes presents many paradoxes and the AdS/CFT duality offers a useful guide towards their solution. One of the biggest problems is obviously the information paradox that originates from the process of evaporation of black holes into thermal radiation, which is a violation of the unitarity of quantum mechanics. According to the fuzzball proposal [27], the microstates of a black hole manifest themselves as regular and horizonless solutions of the supergravity theory. In our work, however, black holes were never mentioned since in the gravity theory on AdS_5 they are $\frac{1}{16}$-BPS solutions. This is a complication compared to the D1-D5 theory where black holes are $\frac{1}{4}$-BPS solutions and for this reason, despite the CFT being more complicated than our $\mathcal{N}=4$ SYM theory, it lends itself well to the study of these objects. However, since the $\mathcal{N}=4$ SYM theory has a lagrangian formulation, it would be nice to explicitly write the $\frac{1}{16}$-BPS solutions of this theory to be able to study black holes through it. A possible future development therefore concerns using our $\frac{1}{2}$-BPS solutions to obtain the $\frac{1}{16}$-BPS one by breaking the correct number of supersymmetries in a similar way to what is done in the D1-D5 theory to go from $\frac{1}{2}$-BPS to $\frac{1}{4}$-BPS solutions. Once this is done, one can use the holographic principle to describe the microstates of the black hole in terms of the dual $\frac{1}{16}$-BPS heavy operators in the dual CFT description.

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