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Moduli Space of Curves

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Introduction

The aim of this thesis is to study one of the possible construction of the moduli space $\mathcal{M}_g$ of curves of a given genus g.
To these days, there are different ways to obtain the moduli space of curves from a parameter space of curves with additional data attached, by taking its quotient by the relation that identifies those additional structures, and so on. Once we have described the construction of the moduli space of curves, we will analyze some of its topological properties that will follow directly from its construction.

The moduli space of curves was first introduced last century. The reason why has been introduced is quite easy. Since the 19^{th} century the mathematicians successfully classified all the possible compact oriented topological surface on $\mathbb{R}$. In particular they discovered that they can be described just by their genus g, which means that two compact and oriented topological surfaces are homeomorphic if and only if their genera are the same.
Going one step further we can consider compact, oriented differentiable surfaces. Again mathematicians found that every compact and oriented differentiable surface is characterized only by its genus, which means that two compact differentiable surfaces are diffeomorphic if and only if they have the same genus.
Taking another step further in the field of geometry, we would like to endow these surfaces with a complex structure. In this way they become one-dimensional complex manifold. On the contrary to the previous cases, in this case the genus is not sufficient anymore. Here comes the idea, that goes back to Riemann, of constructing a space that can parametrize all of them up to biholomorphism.

The approach we will use, it is known as the Teichmüller approach. This was the first and fully successful approach to the construction of the moduli space of curves.

The idea behind it is simple: there is a one-to-one correspondence between isometry classes of hyperbolic surfaces homeomorphic to a topological surface S_g of given genus g and the isomorphism classes of Riemann surfaces homeomorphic to S_g. The Teichmüller approach take advantage of this correspondence.

In particular we can describe the space of hyperbolic structures on a compact, oriented differentiable surface S up to homotopy. This is called the Teichmüller space of S, and we will denote it by $Teich(S)$. In particular we will prove the following theorem.

Theorem 0.0.1. *Let S be a differentiable surface of genus $g \geq 1$. Then $Teich(S)$ is a topologica space. Moreover if $g \geq 2$ $Teich(S)$ is homeomorphic to $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ and it is a topological manifold of real dimension $6g-6$.*

Obviously this alone does not describe the moduli space of curves, since it parametryze hyperbolic structures on S which are hyperbolic surfaces with a marking. In order to obtain $\mathcal{M}_g$, observe that there is a group, called the mapping class group of a surface S, that acts on the Teichmüller space. The mapping class group $Mod(S)$ of a compact oriented topological surface S is the group of isotopy classes of orientation preserving homeomorphisms of S. If S is also endowed with a differentiable structure than one can prove that the mapping class group is isomorphic to the group of isotopy classes of diffeomorphisms on S. Using this characterization of $Mod(S)$ it is easy to see that it describes an action on the Teichmüller space of S. The following result on the action of the mapping class group is central in the construction of the moduli space of curves.

Theorem 0.0.2. *The action of the mapping class group $Mod(S)$ of a compact differentiable surface S on the Teichmüller space $Teich(S)$ of S is properly discontinuous.*

The importance of this result is the fact that having a group that acts properly discontinuously on a topological manifold gives rise to an orbifold considering the quotient of the topological manifold and the group action. The orbifold $\mathcal{M}_g = Teich(S)/Mod(S)$ is the moduli space of curves. Thanks to the topology on $Teich(S)$ and the description of $\mathcal{M}_g$ as the quotient of the Teichmüller space and the action of the mapping class group, we can derive some topological properties on $\mathcal{M}_g$. For example we will give a proof of the compacteness criterion of Mumford:

Theorem 0.0.3. ***(Mumford's compactness criterion)*** *Let $g \geq 1$. For each $\varepsilon > 0$ the space $\mathcal{M}_\varepsilon(\mathbb{S}_g)$ is compact.*

The aim of this thesis is not only presenting a way to construct $\mathcal{M}_g$, but also to give the reader all the necessary preliminaries to understand the construction of the three principal notions.

In general (but not only), we will consider the case of S a compact, oriented surface without puncture or boundary component. Indeed one can extend all the results here presented in the more general case of S a topological surface with b boundary components and n puncture.

Let us give a brief presentation of the contents of each chapter of the thesis.

In the first chapter we will just give some preliminary notions that we will use in the following chapters. In particular we will recall the notion of isotopy, curves on a surface and the concept of hyperbolic surface. Then we will introduce the pants decomposition of a surface, this will be used to calculate the Teichmüller space of a surface of genus $g \geq 2$. The last two thing we will introduce are the concept of orbifold, since we will see that the moduli space of curves is an orbifold, and the notion of quasiconformal map, that we will be using in Chapter 4 to prove Theorem 4.2.3.
The reader that is already familiars with all this notions can skip to the second chapter where we will begin the discussion on the Teichmüller space of a differentiable surface.

The second chapter will introduce the concept of the Teichmüller space of a compact differentiable surface S, denoted with $Teich(S)$. In particular we will start by studying the case $g = 1$, which corresponds to consider a differentiable torus $\mathbb{T}$ and the set of homotopy classes of flat metrics on it. Then we will define the Teichmüller space of S in the case $g \geq 2$ as the set of homotopy classes of hyperbolic structures on S. After that we will equip $Teich(S)$ with a topology by making a bijection a homeomorphism. After that we will be able to create a system of coordinate on $Teich(S)$ and prove that the dimension of $Teich(S)$ over $\mathbb{R}$ is $6g-6$.

In the third chapter we will study the mapping class group of a surface $Mod(S)$, i.e. the group of isotopy classes of orientation-preserving homeomorphism of the surface S. We will begin by giving the definition and see some concrete examples of mapping class group. From these explicit examples we will extrapolate a general method, called the Alexander method, which states that an element of $Mod(S)$ is often determined by its action on a well-chosen collection of curves and arcs in S. We will then analyze some particular elements of $Mod(S)$, that are called Dehn twists, and some of their properties. We will then use them to find some set of generators for $Mod(S)$, in particular we will see two explicit examples of generators of $Mod(S)$ called the Lickorish generators and the Humpries generators.

In the last chapter we finally study the construction of the moduli space of curves of given genus $\mathcal{M}_g$. First we study the action of the mapping class group of a surface $Mod(S)$ on the Teichmüller space of the surface $Teich(S)$, in particular we will describe this action and study its characteristic. We will see the example of genus $g = 1$ and give also a description of a fundamental domain for $\mathcal{M}_1$. Then we will prove that the action of $Mod(S)$ on $Teich(S)$ is properly discontinuous for $g \geq 2$. As a corollary of this result we have that $\mathcal{M}_g$ is an orbifold. From this results we will be able to study some topological properties of

$\mathcal{M}_g$, in particular we will give a metric on $\mathcal{M}_g$ deriving from the metric on $Teich(S)$ and we will describe an exhaustion of $\mathcal{M}_g$ by compact subset, while itself is not compact. We will also be able to prove that the topological space underling $\mathcal{M}_g$ is simply connected and calculate the orbifold fundamental group of $\mathcal{M}_g$, using the properties of orbifolds obtained as a quotient of a manifold by a properly discontinuous action.
In the last section we will give a result on the rational cohomology of $\mathcal{M}_g$ for every $g \geq 2$ by constructing an isomorphism between $H^\bullet(\mathcal{M}_g, \mathbb{Q})$ and $H^\bullet(BMod(S), \mathbb{Q})$, where $BMod(S)$ is the classifying space of $Mod(S)$.

In the Appendix we will give just an idea of two other possible approaches to the construction of the moduli space of curves and observing their advantages and disadvantages. In particular we will see the idea behind the Hodge theory approach, which is again an analytic approach, and the geometric invariant theory approach, which is an algebraic approach.

Notation

Let us introduce some notations we will use in this document.

1. We will denote with $\mathfrak{S}_n$ the group of permutation of n elements.

2. Hereafter $\mathbb{H}^2$ will denote the upper half plane. Let $\mathbb{T}$ denote a differential torus.

3. With $\mathrm{PGL}(2,\mathbb{R})$ we will indicate the projective general linear group of $\mathbb{R}^2$ and with $\mathrm{PSL}(2,\mathbb{R})$ the projective special linear group of $\mathbb{R}^2$. Recall that an element $A \in \mathrm{PSL}(2,\mathbb{R})$ is called: hyperbolic if $|tr(A)| > 2$, parabolic if $|tr(A)| = 2$, and elliptic if $|tr(A)| < 2$.

4. With S we will denote a compact orientable topological surface, i.e. a locally euclidean Hausdorff space of real dimension 2.
   With $\mathbb{S}$ we will denote a differentiable surface, i.e. a Hausdorff topological orientable space S with a countable basis and a diffentiable atlas of dimension 2 on $\mathbb{R}$.
   With $\mathcal{S}$ we will denote a compact Riemann surface, i.e. a one-dimensional complex manifold, with underlying topological space S.
   With S_g we will denote a genus g topological surface, with $\mathbb{S}_g$ we will denote a genus g differentiable surface, and with $\mathcal{S}_g$ we will denote a genus g Riemann surface.
   From now on we will only write surface and the type of surface we are considering, topological, differentiable or Riemann, will be clear from the notation used.

5. For every surface S we will denote with ∂S its boundary, and $\chi(S)$ its Euler characteristic, i.e., if S has genus g, b boundary components and n punctures, $\chi(S) = 2 - 2g - (b + n)$. With $S_{g,n}$ we will indicate a surface with genus g and n punctures.

6. Let X and Y be topological spaces, with $C(X,Y)$ we will denote the set of continuous map from X to Y.
   With $Homeo(S)$ we will denote the group of homeomorphisms of S, $Homeo^+(S)$ will denote the subgroup of orientation preserving homeomorphisms and $Home_0(S)$ will denote the subgroup of the homeomorphisms isotopic to the identity of S.
   With $Diff(\mathbb{S})$ we will denote the group of diffeomorphisms of $\mathbb{S}$, $Diff^+(\mathbb{S})$ we will denote the subgroup of orientantion preserving diffeomorphisms and $Diff_0(\mathbb{S})$ will denote the subgroup of diffeomorphisms isotopic to the identity of $\mathbb{S}$. With $Isom(\mathbb{S})$ we will denote the group of isometries of $\mathbb{S}$, again $Isom^+(\mathbb{S})$ we will denote the subgroup of orientation preserverving isomentries and $Isom_0(\mathbb{S})$ will denote the subgroup of isometries isotopic to the identity of $\mathbb{S}$.

Chapter 1

Preliminaries

In this first chapter we will introduce some notions that will be useful later on. In each section we will introduce a different notion and some of its property that will be useful in the following chapters.

1.1 Isotopy

The first notion we want to introduce is the concept of isotopy. This is a particular case of homotopy. This notion will be used in different context later on. In particular we will be interested in the isotopy classes of curves on differentiable surfaces and of hyperbolic metric. We will encounter this notions in the Chapter 2.

Let X, Y be two topological spaces.

Definition 1.1.1. Let $f, g\colon X \longrightarrow Y$ be topological embeddings, i.e. they are homeomorphisms onto their images. An *isotopy* between f and g is a map $H\colon X \times I \longrightarrow Y$ such that

i. $H(x, 0) = f(x)$ for all $x \in X$,

ii. $H(x, 1) = g(x)$ for all $x \in X$,

iii. $H(-, t)\colon X \longrightarrow Y$ is an embedding for all $t \in I$.

If there exist an isotopy between f and g then f is said to be *isotopic* to g.

Remark 1.1.2. *Observe that an isotopy is a homotopy, but the two notions are not equivalent in general.*

Example 1.1.3. Let $f\colon S^1 \longrightarrow \mathbb{R}^2$ be the map defined by $f(x, y) = (2x, 2y)$ and $g\colon S^1 \longrightarrow \mathbb{R}^2$ defined by $g(x, y) = (3x, 3y)$. We have that $H\colon S^1 \times I \longrightarrow \mathbb{R}^2$, given by $H(x, y, t) = (1 - t)f(x, y) + tg(x, y)$ is an isotopy between f and g.
Indeed $H(x, y, 0) = f(x, y)$ and $H(x, y, 1) = g(x, y)$, moreover $H(-, t)$ is an embedding.

Example 1.1.4. We would like to show a homotopy which is not an isotopy.
Let consider $X = [-1, 1]$ and $Y = \mathbb{R}$ and the maps $f(x) = -x$ and $g(x) = x$. If we consider $H(x, t) = 2xt - x$ we have that H is an homotpy between f and g but not an isotopy. Indeed, for $t = \frac{1}{2}$, we have that $H(x, t) = H(x, \frac{1}{2}) = 2x\frac{1}{2} - x = x - x = 0$ which is not an embedding. In particular f and g can't be isotopic since every homotopy needs to exchange endpoints. Moreover f changes the orientation of X while g does not.

In certain case we have that homotopy and isotopy are equivalent. In particular we will see, in the next section, that for simple closed curves on a topological surface the two notions are the same.

1.2 Curves in a surfaces

In this section we will talk about curves in a topological surface S. In particular we will concetrate on the notion of closed curves and some of their properties. The curves will play a fundamental role during our discussion of the Teichmüller space and the Mapping class group. In Chapter 2, the lenght of some set of closed curves on the different points of $Teich(\mathbb{S})$ will allow us to define a set of coordinates on $Teich(\mathbb{S})$ in order to prove that $Teich(\mathbb{S})$ has real dimension $6g - 6$ for $g \geq 2$. In Chapter 3 we will see that some particular element of the mapping class group related to some specific sets of curves on our surface S_g will generate the mapping class group $Mod(S_g)$ of S_g, in particular we will prove that $Mod(S_g)$ is finitely generated as a group.

First of all let us present the definition of closed curve in a surface S.

Definition 1.2.1. i. A *closed curve* in a surface S is a continuous map $\alpha\colon S^1 \longrightarrow S$. We will identify a closed curve with its image $\alpha(S^1) \subset S$.

ii. A closed curve in S is called *simple* if it is embedded in S, i.e. the map $\alpha\colon S^1 \longrightarrow S$ is an embedding.

iii. A closed curve in S is called *essential* if it is not homotopically equivalent to a point, a puncture or a boundary component of S.

Example 1.2.2. The closed curves in Figure 1.1 are example of simple closed curves in a 3-torus.

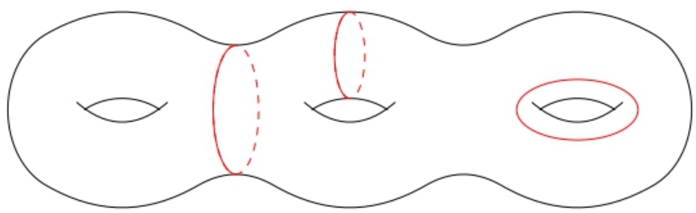


Figure 1.1: Simple closed curves in a 3-torus

Definition 1.2.3. A simple closed curve is called *positively oriented* if when travelling on it one always has the curve interior to the left.
It is called *negatively oriented* if when travelling on it one always has the curve interior to the right.
It is called *oriented* if it is positively oriented or negatively oriented.

As already mentioned before we will work with isotopy classes of curves. In the case of simple closed curves we have that homotopy and isotopy are the same so we can actually think of the isotopy classes of curves as homotopy classes of curves.

Proposition 1.2.4. *Let α and β be two essential simple closed curves in a surface S. Then α is isotopic to β if and only if α is homotopic to β.*

Proof. See Proposition 1.10 of [7]. □

We need to introduce the notion of intersection numbers. We will see two different type of intersection number: one algebraic, that consider the orientation of the curves, and one geometric, that simply counts the number of points in common between the two curves.
The algebraic intersection number will be used to find a particular pair of curves on the torus $\mathbb{T}$ in order to prove that the moduli space of curves of genus 1 can be identified with $\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$ in Chapter 4.
The geometric intersection number will be used in different context. In particular leads to the definition of the notion of minimal position of two curves which will be useful to define a pants decomposition of a topological surface and choose particular representative in isotopy classes of curves. In Chapter 3 we will use the fact that we have pair of curves in minimal position to describe the Alexander method (Proposition

3.3.1), which will give us a way to find a collection of curves $\{\gamma_i\}$ on a surface S such that each mapping class is uniquely determined by its action on this set of curves.
First we will talk about algebraic intersection numbers and then we will see the definition of geometric intersection number and all the notion related.

Definition 1.2.5. If α and β are oriented simple closed curves in S, the *algebraic intersection number* $\hat{i}(\alpha, \beta)$ is the sum of the indices of the intersection point of α and β, where an intersection point is of index $+1$ when the orientation of the intersection agrees with the orientation of S and is -1 otherwise.

Remark 1.2.6. *We have that the algebraic intersection numbers depend only on the homotopy classes of the curves.*

As already mentioned before we can simply count the points in the intersection of two curves without considering their orientation.

Definition 1.2.7. Let a, b be two free homotopy classes of simple closed curves in S, the *geometric intersection number* $i(a, b)$ is the minimal number of intersection points between a representative curve in the class a and a representative curve in the class b :

$$i(a, b) := min\{|\alpha \cap \beta| \mid \alpha \in a,\ \beta \in b\}.$$

Definition 1.2.8. Let a, b be two free homotopy classes of simple closed curves in S, let α and β be two representative of a and b respectively. We say that α and β are in *minimal position* if they realize the minimal intersection in their homotopy classes, i.e. $i(a, b) = |\alpha \cap \beta|$.

In Chapter 4, in order to prove that the action of $Mod(\mathcal{S}_g)$ on $Teich(\mathcal{S}_g)$ is properly discontinuous, we will need a criterion to understand when two curves are in minimal position. In particular we will use the "bigon criterion", that is presented below.

Definition 1.2.9. Two simple closed curves α and β in a surface S form a *bigon* if there is an embedded disk in S whose boundary is the union of an arc of α and an arc of β intersecting in exactly two points. See Figure 1.2.

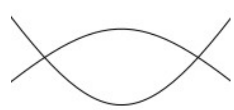


Figure 1.2: A bigon

Proposition 1.2.10. ***The bigon criterion:*** *Two transverse simple closed curves in a surface S are in minimal position if and only if they do not form a bigon.*

Proof. See Proposition 1.7 of [7]. □

The last notion on curves in a surface we will need is the one of filling a surface. In Chapter 3 we will use that a set of curves fills a surface S to describe the Alexander method (Section 3.3). In Chapter 4 we will prove and use the fact that on a surface of genus $g \geq 2$ exists a pair of simple closed curves that fills S_g. Let us give the definition.

Definition 1.2.11. For a surface S with marked points, we say that a collection $\{\gamma_i\}$ of curves and arcs *fills* S if the surface obtained from S by cutting along all γ_i is a disjoint union of disks and once-marked disks.

Remark 1.2.12. *One can prove that a pair of isotopy classes $\{a, b\}$ of simple closed curves in S fills S if for every isotopy class c of essential simple closed curves in S either $i(a, c) > 0$ or $i(b, c) > 0$. We will use this characterization in Chapter 4.*

1.3 Hyperbolic surfaces

As already mentioned in the Introduction the Teichmüller space of a surface $\mathbb{S}$ is the space of hyperbolic structure on $\mathbb{S}$. We then need to recall some fundamental notions of hyperbolic geometry that we will use thoroughly in Chapter 2 in order to define the Teichmüller space of a differentiable surface of genus $g \geq 2$. First of all let us give the definition of hyperbolic metric and hyperbolic surface.

Definition 1.3.1. A surface $\mathbb{S}$ admits a *hyperbolic metric* if there exists a complete, finite-area Riemannian metric on $\mathbb{S}$ of constant curvature -1.
A surface endowed with a hyperbolic metric will be called *hyperbolic surface*.

We denote with $HypMet(S)$ the set of hyperbolic metrics on S.
We will like to give some examples of hyperbolic surfaces.

Example 1.3.2. Let consider the unit disk $\mathbb{B}^2 \subset \mathbb{C}$. And define

$$d(z_1, z_2) = cosh^{-1}\left(1 + \frac{2|z_1 - z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)}\right),$$

for all $z_1, z_2 \in \mathbb{B}^2$.
This metric is a hyperbolic metric and the disk with this metric is called *hyperbolic disk*. In this case the geodesic are the intersections with the disk of euclidean circles and lines meeting the unit disk orthogonally (Figure 1.3).

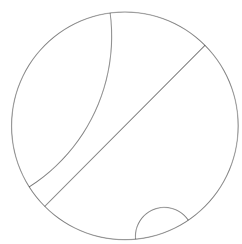


Figure 1.3: Geodesic in the hyperbolic disk

Example 1.3.3. The *hyperbolic plane* is the pair $(\mathbb{H}^2, ds^2)$,
where $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}$ is the Siegel upper-half plane and ds^2 is the metric given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

where $dx^2 + dy^2$ denotes the euclidean metric on $\mathbb{C}$.
In this case the geodesic are semicircles and half-lines perpendicular to the real axis.

In the next chapters we will consider differentiable surfaces of genus $g \geq 2$. The following results will assure us that we can construct a hyperbolic metric on $\mathbb{S}_g$ and so define its Teichmüller space.

Theorem 1.3.4. *Let $\mathbb{S}_g$ be a surface with genus $g \geq 2$. Then $\mathbb{S}_g$ admits a hyperbolic metric.*

Proof. From the Uniformization Theorem (see [1]) we have that $\mathbb{H}^2$ is the universal cover of $\mathbb{S}_g$ (See [14] Chapters 11 and 12 for details). Therefore $\pi_1(\mathbb{S}_g)$ acts on $\mathbb{H}^2$ freely and properly discontinuously, and the quotient is homeomorphic to $\mathbb{S}_g$. Since the action is also isometric, we have that the quotient is endowed with a hyperbolic metric coming from the hyperbolic metric on $\mathbb{H}^2$. □

Remark 1.3.5. *In Theorem 1.3.4 we can construct a metric on* $\mathbb{S}_g$ *from the metric of* $\mathbb{H}^2$ *since when a group acts by isometries on a metric space, the quotient has an induced pseudo-metric. The distance between any two orbits is defined to be the infimum of the distance between any pair of representatives. If the action is also properly discontinuous one has that two orbits have distance zero if and only if they are equal, in other words the induced pseudo-metric is a metric.*

The next result is about the isometry group of a hyperbolic surface. We will use this result in Chapter 4 to show that the stabilizer of a point in $Teich(\mathbb{S}_g)$ under the action of $Mod(\mathbb{S}_g)$ is not trivial but finite. Therefore the action of $Mod(\mathbb{S}_g)$ on $Teich(\mathbb{S}_g)$ is not free.

Proposition 1.3.6. *Let* X *be a hyperbolic surface homeomorphic to* $\mathbb{S}_g$ *with* $g \geq 2$*. Then* $Isom(X)$ *is finite in any hyperbolic metric.*

Proof. First, thanks to an application of the Arzela-Ascoli Theorem (see [2] and [3]), we have that the isometry group of any compact Riemannian manifold is a compact topological group. Therefore it suffices to prove that $Isom(X)$ is discrete or, equivalently, that the connected component in $Isom(X)$ of the identity is trivial. Since the topology in $Isom(X)$ agrees with the subspace topology inherited from $Homeo^+(\mathbb{S}_g)$, it is enough to prove that $Isom(X) \cap Homeo_0(\mathbb{S}_g) = \{1\}$.
Suppose that $\phi \in Isom(X) \cap Homeo_0(\mathbb{S}_g)$. This means that $\phi \in Isom(X)$ is isotopic to the identity. Then ϕ has a lift to $Isom(\mathbb{H}^2)$, since $\mathbb{H}^2$ is the universal cover for X, that is at a bounded distance from the identity map of $\mathbb{H}^2$. By the classification of hyperbolic isometries, any such isometry is equal to the identity. Thus ϕ is the identity. □

1.4 Pants decomposition

In this section we will recall the notion of a pants decomposition of a topological surface of genus $g \geq 2$. We will use its existence to calculate the real dimension of Teichmüller space of $\mathbb{S}_g$, since we are able to describe $Teich(P)$, where P is a pair of pants, up to homeomorphism, as we will see in Section 2.2 of Chapter 2.

First of all we recall what is a pair of pants.

Definition 1.4.1. A *pair of pants* is a compact surface of genus 0 with three boundary components.

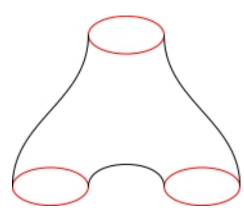


Figure 1.4: A pair of pants in space. Its boundary are in red.

Remark 1.4.2. *Observe that every simple closed curve in a pair of pants is either homotopic to a point or a boundary components.*
A pair of pants P *has Euler characteristic* $\chi(P) = -1$*. Indeed* $\chi(P) = 2 - 2g - b$*, but* $g = 0$ *and* $b = 3$*, since the genus of a pair of pants is* 0 *and it has* 3 *boundary component.*

We can now give the definition of a pants decomposition of a surface S.

Definition 1.4.3. Let S be a surface with genus $g \geq 2$, a *pants decomposition* of S, is a collection of disjoint simple closed curves in S such that when we cut S along these curves, we obtain a disjoint union of pair of pants.

We have the following equivalent definition.

Proposition 1.4.4. *A* pants decomposition *of S is equivalent to a maximal collection of disjoint, essential, simple closed curves in S such that no two of these curves are isotopic.*

Proof. Suppose we have a collection of simple closed curves that cut S into pair of pants. We see that every curve is essential since there are no disk components when we cut S. Further it folllows from Remark 1.4.2 that the given collection is maximal.
Other way round, suppose we have a collection of disjoint, nonisotopic essential closed curves in S. If the surface obtained from S, by cutting along these curves is not a collection of pairs of pants, then it follows from the classification of surfaces and the additivity of Euler characteristic that one component of the cut surface has, either, positive genus or is a sphere with more than three boundary components. On such surface there exists an essential simple closed curve that is not homotopic to a boundary component. Thus the original collection of curves was not maximal. ☐

Remark 1.4.5. *Since a pair of pants has Euler characteristic -1, a pants decomposition of S cuts S in $-\chi(S)$ pairs of pants. Each pair of pants has three boundary curves and, aside from the curves coming from ∂S, these curves match up in pairs to form curves in S. It follows that, for a compact surface S of genus g and with b boundary components, a pants decomposition has*

$$\frac{-3\chi(S) - b}{2} = 3g + b - 3$$

curves. In particular, a pants decomposition of S_g, for $g \geq 2$, has $3g - 3$ curves, cutting S_g in $2g - 2$ pairs of pants.

Remark 1.4.6. *The choice of the family of curves is not unique, an example of two different choices for S_2 is shown in Figure 1.5.*

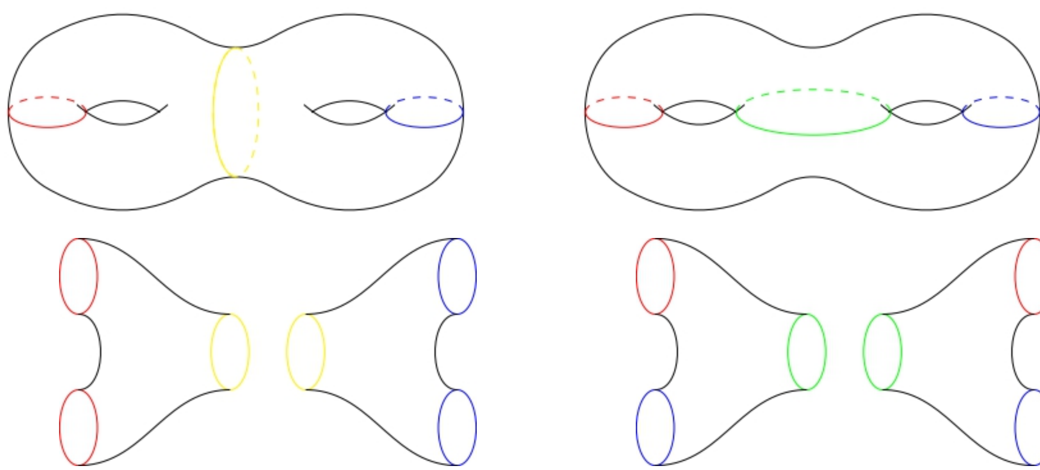


Figure 1.5: Two pants decomposition for S_2

1.5 Complex structure and hyperbolic structure on a surface

As already observed in the Introduction, to describe the moduli space of curves using the Teichmüller approach we will use a one-to-one correspondence between isomorphism classes of Riemann surfaces and isometry classes of hyperbolic surfaces both homeomorphic to a surface S. Let us analyze better this correspondence.

Remark 1.5.1. *Recall that the uniformization theorem gives that any Riemann surface of genus $g \geq 2$ is the quotient of the unit disk Δ by a group Γ of biholomorphic automorphisms acting properly discontinuously and freely on Δ. But any group of biholomorphic automorphisms of Δ preserve the hyperbolic metric on Δ. So Δ/Γ has an induced hyperbolic structure. Conversely any such hyperbolic structure gives a complex structure on X. In other words, for $g \geq 2$ there is a bijective correspondence:*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes} \\ \textit{of Riemann surfaces} \\ \textit{homeomorphic to } S_g \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Isometry classes} \\ \textit{of hyperbolic surfaces} \\ \textit{homeomorphic to } S_g \end{array} \right\}$$

1.6 Quasiconformal and conformal maps

Another notion we need to introduce is the one of quasiconformal map. In Chapter 4 we will use a result on quasiconformal map that will give us a bound on the length of the image of isotopy classes of simple closed curves in hyperbolic surfaces (see Lemma 4.2.7). This Lemma will be one of the technical results which will permit to prove the properly discontinuity of the action of the mapping class group on the Teichmüller space.

First we need to introduce the concept of quasiconformal map on $\mathbb{C}$. Then we will extend the definition in the case of Riemann surfaces.

Let U and V be open subset of $\mathbb{C}$ and let $f\colon U \longrightarrow V$ be a homeomorphism that is smooth oustide of a finite number of points. Using the usual notation for maps and setting $z = x + iy$ we can consider $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as $f(x, y) = (a(x, y), b(x, y))$, where $a, b\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$. Then, where it is defined, the derivate df is described by $df = f_x dx + f_y dy$, where $f_x = (a_x, b_x)$ and $f_y = (a_y, b_y)$.

Switching to the complex notation we have $df = f_z dz + f_{\overline{z}} d_{\overline{z}}$, where $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\overline{z}} = \frac{1}{2}(f_x + if_y)$ and the quantity $\mu_f = f_{\overline{z}}/f_z$ is called the *complex dilatation* of f.

Remark 1.6.1. *We have that f is holomorphic if and only if $\mu_f = 0$.*
Also, since $|f_z|^2 - |f_{\overline{z}}|^2 = a_x b_y - a_y b_x$, we see that f is orientation preserving if and only if $|\mu_f| < 1$.

Definition 1.6.2. Let $f\colon U \longrightarrow V$ be a orientation preserving homeomorphism. Let $p \in U$ at which f is differentiable. The *dilatation* of f at p is defined by

$$K_f(p) = \frac{|f_z(p)| + |f_{\overline{z}}(p)|}{|f_z(p)| - |f_{\overline{z}}(p)|} = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}.$$

Remark 1.6.3. *Observe that the quantity $log(K_f(p))/2$ is precisely the distance between $\mu_f(p)$ and 0 in the Poincaré disk model of $\mathbb{H}^2$, this makes sense since f is orientation-preserving and so $|\mu_f| < 1$.*

There is a geometric interpretation of $K_f(p)$. The map df_p takes the unit circle in $TU_p \simeq \mathbb{C}$ to an ellipse E in $TV_{f(p)}$, and $K_f(p)$ is the ratio of the length of the major axis of E to the length of the minor axis of E. Indeed we parametrize the unit circle in $\mathbb{C}$ as $\theta \longmapsto e^{i\theta}$ for $\theta \in [0, 2\pi]$. The image of this circle under df_p is then the ellipse E and is determined by $E(\theta) = f_z(p)e^{i\theta} + f_{\overline{z}}(p)e^{-i\theta}$ for $\theta \in [0, 2\pi]$. The absolute value of a point $E(\theta)$ is

$$|E(\theta)| = \left|f_z(p)e^{i\theta} + f_{\overline{z}}(p)e^{-i\theta}\right| = |f_z(p)| \left|1 + \mu_f(p)e^{-i2\theta}\right|.$$

Since $1 - |\mu_f(p)| \leq \left|1 + \mu_f(p)e^{-i2\theta}\right| \leq 1 + |\mu_f(p)|$ it follows that the ratio of the maximum absolute value of a point on E to the minimum absolute value of a point on E is precisely $K_f(p)$.

Definition 1.6.4. The *dilatation* of the map f is the number $K_f = sup\ K_f(p)$, where the supremum is taken over over all points p where f is differentiable.

Observe that $1 \leq K_f \leq \infty$.

Definition 1.6.5. If $K_f < \infty$ we say that f is a *quasiconformal*, or K_f*-quasiconformal*, map between the domains U and V of $\mathbb{C}$.

Example 1.6.6. Every biholomorphic map and its inverse are $1-$quasiconformal map.

We can introduce quasiconformal maps between Riemann surfaces. Let $f\colon X \longrightarrow Y$ be a homeomorphism between Riemann surfaces that is smooth outside a finite number of points. Assume also that f respects the orientations induced by the complex structures on X and Y and that f^{-1} is smooth outside a finite number of points. Since the transition map in any atlases for X and Y are biholomorphic and the local expressions for f are orientation preserving, there is a well defined notion of the *dilatation* $K_f(x)$ of f at a point $x \in X$ where f is smooth. We can define, as above, the *dilatation* of f to be $K_f = sup\ K_f(x)$. We will say that f is *quasiconformal*, or K_f*-quasiconformal*, if $K_f < \infty$.
Recall that a map between Riemann surfaces is *holomorphic* if, in any chart, it is given by a holomorphic map from some domain in $\mathbb{C}$ to $\mathbb{C}$.
A bijective holomorphic map between Riemann surfaces is called a *conformal map.* One can prove that conformal maps between Riemman surfaces are biholomorphic.

Lemma 1.6.7. *Let $f\colon X \longrightarrow Y$ be a homeomorphism between Riemann surfaces. Then f is a $1-$quasiconformal homeomorphism if and only if it is a conformal map.*

Proof. $\Longrightarrow$ Suppose that f is $1-$quasiconformal, which is equivalent to $f_{\overline{z}} \equiv 0$ wherever it is defined. Let $A \subset X$ be the set of point where f' is not defined. The restriction of f to $X \setminus A$ is then holomorphic. Since $f_{|X \setminus A}$ is also bijective, it is conformal. Since f is a homeomorphism, its singularities at A must be removable, but f is continuous, so it follows that f is already holomorphic, hence conformal.

$\Longleftarrow$ First of all observe that since f is a homeomorphism, its derivative f' must be non zero at all points where it is defined.

Suppose that f is conformal, in this case f' is defined at every point, and hence f' never vanishes. It follows that f takes circles in the tangent space of X to nondegenerate circles in the tangent space of Y, and so f is $1-$quasiconformal.

□

1.7 Orbifold

In this section we will give an introduction to orbifold, for more details and results the reader can consult [19]. The concept of orbifold is an extension of the concept of manifold, in particular it gives the tools to work with quotient of manifold by properly discontinuous action which are not free. In Chapter 4 we will see that the moduli space of curves $\mathcal{M}_g$ is an example of an orbifold, in particular it is given by the quotient of the Teichmüller space of $\mathbb{S}_g$, which is a topological manifold, by the properly discontinuous action of the mapping class group of $\mathbb{S}_g$.

Definition 1.7.1. Let X be a Hausdorff space covered by a collection of open set $\{U_i\}_{i \in I}$ closed under finite intersection. To each U_i is associated

i. a finite group Γ_i,

ii. an action of Γ_i on an open subset V_i of $\mathbb{R}^n$, for a certain $n \in \mathbb{N}$,

iii. a homeomorphism $\varphi_i\colon V_i/\Gamma_i \longrightarrow U_i$, called an *orbifold chart.*

The collection of orbifold charts is called *orbifold atlas* if the following properties are satisfied.

1. For each inclusion $U_i \subseteq U_j$ there is an injective group homomorphism $f_{ij}\colon \Gamma_i \longrightarrow \Gamma_j$.

2. For each inclusion $U_i \subseteq U_j$ there is a Γ_i−equivariant embedding $\varphi_{ij}\colon V_i \longrightarrow V_j$, called *gluing map.*
3. The gluing map are compatibile with the orbifold charts, i.e. $\varphi_j \circ \varphi_{ij} = \varphi_i$.
4. The gluing map are unique up to composition with elements of Γ_j, i.e. any other possible gluing map from V_i to V_j is of the form $\gamma\varphi_{ij}$ for some $\gamma \in \Gamma_j$.

The orbifold atlas will be said n−dimensional.

Remark 1.7.2. *Note that it is not true generally that $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ when $U_i \subseteq U_j \subseteq U_k$, but there is an element $\gamma \in \Gamma_k$ such that $\gamma\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ and $\gamma \cdot f_{ik}(g) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(g)$ for every $g \in \Gamma_i$.*

Definition 1.7.3. Two orbifold atlases of X are said to be *equivalent* if they can be combined consistently to give a larger orbifold atlas.
An n−dimensional orbifold structure on X is an equivalence class of orbifold atlases of dimension n on X.

Definition 1.7.4. A *n−dimensional orbifold O* consists of a Hausdorff space X_O, called the *underlying space*, equipped with an n−dimensional orbifold structure on it.

We now give some examples of orbifolds.

Example 1.7.5. A closed topological manifold is an orbifold, where each group Γ_i is the trivial group, so that $U_i \simeq V_i$.

In order to construct other examples we need the following result.

Proposition 1.7.6. *Let M be a topological manifold and let Γ be a group acting properly discontinuously on M. Then M/Γ has the structure of an orbifold.*

Proof. For any point $x \in M/\Gamma$, choose $\widetilde{x} \in M$ projecting to x. Let I_x be the isotropy group of $\widetilde{x}$, which is the subgroup of Γ of elements of Γ that leave $\widetilde{x}$ fixed. There is a neighbourhood $\widetilde{U}_x$ of $\widetilde{x}$ invariant by I_x and disjoint from its translates by elements of $\Gamma \setminus I_x$, since the action of Γ on M is properly discontinuous. The projection $U_x \simeq \widetilde{U}_x/I_x$ is a homeomorphism. To obtain a suitable collection of open subset of M/Γ that covers it enlarge the collection composed by the $\{U_x\}$ by adjoining finite intersections, since the collection used to define an orbifold atlas needs to be closed under finite intersections. Whenever $U_{x_1} \cap \cdots \cap U_{x_k} \neq \emptyset$, this means that some set of translates $\gamma_1\widetilde{U}_{x_1} \cap \cdots \cap \gamma_k\widetilde{U}_{x_k}$ has a corresponding non-empty intersection. This intersection may be taken to be $\widetilde{U_{x_1} \cap \cdots \cap U_{x_k}}$, with associated group $\gamma_1 I_{x_1}\gamma_1^{-1} \cap \cdots \cap \gamma_k I_{x_k}\gamma_k^{-1}$ acting on it. In this way we constructed an orbifold atlas on M/Γ. □

Example 1.7.7. A barber shop. Let $M = \mathbb{R}^3$ and consider G the group generated by reflections in the planes $x = 0$ and $x = 1$. Then G is the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, which is the free product of $\mathbb{Z}/2\mathbb{Z}$. The quotient space is the slab $0 \leq x \leq 1$ and it is an orbifold, since the action of G is properly discontinuous.
Physically, this is related to two mirrors on parallel walls, as commonly seen in a barber shop.

Remark 1.7.8. *Observe that each point x in an orbifold O is associated with a group Γ_x, well defined up to isomorphism. Indeed in a local coordinate system $U = V/\Gamma$, Γ_x is the isotropy group of any point in V corresponding to x.*

Definition 1.7.9. The set $\Sigma_O := \{x \ : \ \Gamma_x \neq \{1\}\}$ is the *singular locus* of O.
We say that O is a manifold when $\Sigma_O = \emptyset$.

Remark 1.7.10. *It can happen that the underlying space X_O of an orbifold is a topological manifold, but $\Sigma_O \neq \emptyset$, thus O is not a topological manifold.*

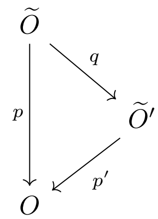
We would like to extend the notion of fundamental group to orbifolds, in this way in Chapter 4 we will be able to study the one of $\mathcal{M}_g$. There are different ways to do so. We will present the one we will use in Chapter 4. In particular we will define the notion of covering orbifold and of universal orbifold covering. We will then use them to define the fundamental group of an orbifold.

Definition 1.7.11. A *covering orbifold* of an orbifold O is an orbifold $\widetilde{O}$, with a projection $p\colon X_{\widetilde{O}} \longrightarrow X_O$ between the underlying spaces, such that each point $x \in X_O$ has a neighbourhood $U \simeq V/\Gamma$, where $V \subseteq \mathbb{R}^n$, for which each component $v_i \in p^{-1}(U)$ is isomorphic to V/Γ_i, where $\Gamma_i \subseteq \Gamma$ is some subgroup. The isomorphism must respect the projection.

Remark 1.7.12. *Note that the underlying space $X_{\widetilde{O}}$ is not generally a covering space of X_O.*

Example 1.7.13. Let Γ be a group acting properly discontinuously on a manifold M, then M is a covering orbifold of M/Γ. In fact, for any subgroup $\Gamma' \subseteq \Gamma$, M/Γ' is a covering orbifold of M/Γ.

Proposition 1.7.14. *An orbifold O has universal cover $\widetilde{O}$. In other words, if $x \in X_O \setminus \Sigma_O$ is a base point for O, $p\colon \widetilde{O} \longrightarrow O$ is a connected covering orbifold with base point $\widetilde{x}$ which projects to x, such that for any other covering orbifold $p'\colon \widetilde{O}' \longrightarrow O$ with base point $\widetilde{x}'$, $p'(\widetilde{x}') = x$, there is a lifting $q\colon \widetilde{O} \longrightarrow \widetilde{O}'$ of p to a covering map of $\widetilde{O}'$.*



Proof. See Proposition 13.2.4 of [19]. □

Example 1.7.15. If $O = M/\Gamma$ then M is the universal orbifold cover and every covering orbifold of O is isomorphic to M/Γ' for some $\Gamma' \subseteq \Gamma$.

Remark 1.7.16. *The universal cover $\widetilde{O}$ of an orbifold O is automatically a regular cover: for any preimage of $\widetilde{y}$ of the base point x there is a deck transformation taking $\widetilde{x}$ to $\widetilde{y}$.*

Definition 1.7.17. The *fundamental group* $\pi_1^{orb}(O)$ of an orbifold O is the group of deck transformations of the unversal cover $\widetilde{O}$.

Example 1.7.18. If $O = M/\Gamma$ then $\pi_1^{orb}(O) \simeq \Gamma$.

Chapter 2

The Teichmüller Space

We now have all the preliminaries notions to introduce the Teichmüller space of a differentiable surface of genus $g \geq 1$. First of all we will analyze the case $g = 1$, we will consider a differentiable torus and we will be able to determine its Teichmüller space up to homeomorphism. This case will be our starting point and it will give us the idea on how to define the Teichmüller space on a differentiable surface of genus $g \geq 2$.

2.1 The case $g = 1$

Let us consider a differentiable surface $\mathbb{S}$ of genus $g = 1$, from the classification of differentiable surfaces we know that $\mathbb{S}$ is diffeomorphic to a torus $\mathbb{T}$. We can then consider $\mathbb{T}$ as our surface. Since $g(\mathbb{T}) = 1$ we have that is not possible to construct a hyperbolic metric on $\mathbb{T}$. But one can prove that a surface $\mathbb{S}$ of genus $g = 1$ always admits flat metrics on it.

Definition 2.1.1. A *flat structure* on $\mathbb{T}$ is a pair (X, ϕ), where X is a differentiable surface endowed with a complete, finite-area flat metric and $\phi\colon \mathbb{T} \longrightarrow X$ is a diffeomorphism.
The diffeomorphism ϕ is called *marking*. The differentiable surface X and the pair (X, ϕ) are called *marked surface*.

Remark 2.1.2. *Since we have that $\phi\colon \mathbb{T} \longrightarrow X$ is a diffeomorphism, in particular injective, it describe a flat metric on $\mathbb{T}$ via pullback. Indeed if we have an injective function $f\colon X \longrightarrow Y$ between differentiable manifolds and Y is endowed with a metric d_Y, we can define a metric d_X on X by putting $d_X(a, b) = d_Y(f(a), f(b))$.*

Observe that different flat structure on $\mathbb{T}$ can give rise to the same flat metric on the torus $\mathbb{T}$. Moreover a flat metric on the torus can have any positive number as its area, but, at the same time, a flat metric on the torus can be multiplied pointwise by a fixed real number so that the area of the resulting metric equals 1. Then we would like to consider only unit-area metric on $\mathbb{T}$ and on the flat structures.
To avoid these problems we define the Teichmüller space of the torus as follows.

Definition 2.1.3. The *Teichmüller space* $Teich(\mathbb{T})$ of the torus is the set of isotopy classes of unit-area flat structures on $\mathbb{T}$, i.e.

$$Teich(\mathbb{T}) = FlMet(\mathbb{T})/Diff_0(\mathbb{T}),$$

where $FlMet(\mathbb{T})$ is the set of unit-area flat metric on $\mathbb{T}$ and the action of $Diff_0(\mathbb{T})$ is by pullback.

We would like to describe $Teich(\mathbb{T})$ with a bijection to some more known topological manifold in order to be able to define also a topology on $Teich(\mathbb{T})$ and study its real dimension. In order to do so we would like to recall that the concept of a torus is tied to the concept of lattice in $\mathbb{C}$. We would like to recall this concept for the reader since we will use it to give a description of the Teichmüller space of a torus.

Definition 2.1.4. A *lattice* in $\mathbb{C}$ is a discrete subgroup Λ of the additive group $\mathbb{C}$ with $\Lambda \simeq \mathbb{Z}^2$.

Remark 2.1.5. *We have that the quotient* $\mathbb{C}/\Lambda$ *is a torus.*

Definition 2.1.6. The *area* of a lattice is the Euclidean area of the torus $\mathbb{C}/\Lambda$.

We have that any lattice in $\mathbb{C}$ is homothetic to a unique unit-area lattice, where a homothety of $\mathbb{C}$ is a map $z \longmapsto \lambda z$ for some $\lambda \in \mathbb{R}_+$. As before we have marked the differentiable surfaces diffeomorphic to $\mathbb{T}$ we would like to mark the lattices in $\mathbb{C}$.

Definition 2.1.7. We will say that a lattice in $\mathbb{C}$ is *marked* if it comes equipped with an ordered set of two generators. Equivalently if it comes equipped with a fixed isomorphism with $\mathbb{Z}^2$.

After recalling the notion of lattices in $\mathbb{C}$, we are ready to prove the following result and describe a bijection between $Teich(\mathbb{T})$ and the hyperbolic plane $\mathbb{H}^2$.

Proposition 2.1.8. *There is a natural bijection between* $Teich(\mathbb{T})$ *and* $\mathbb{H}^2$

Proof.

1. $Teich(\mathbb{T}) \longleftrightarrow \{\text{marked lattices in } \mathbb{C}\}/\sim$, where $\sim$ is the equivalence relation generated by euclidean isometries and homotheties.
   As defined above we have that $Teich(\mathbb{T})$ is the set of isotopy classes of unit-area flat structures on $\mathbb{T}$. We can fix an ordered generating set for $\pi_1(\mathbb{T})$ and an ordered generating set of $\pi_1(\mathbb{C}/\Lambda)$, where Λ is a lattice in $\mathbb{C}$. We can define a diffeomorphism $\phi\colon \mathbb{T} \longrightarrow \mathbb{C}/\Lambda$ that takes the generators of $\pi_1(\mathbb{T})$ in the generators of $\pi_1(\mathbb{C}/\Lambda)$. Then scaling the pullback of the flat metric on $\mathbb{C}/\Lambda$ we obtain a point of $Teich(\mathbb{T})$.
   On the other end let $[(X, \phi)]$ be a point in $Teich(\mathbb{T})$. Then the metric universal cover of X is isometric to $\mathbb{C}$ and the group of deck transformations is a lattice Λ in $\mathbb{C}$ with the image under ϕ of the set of generators for $\pi_1(\mathbb{T})$ as a marking.

2. $\mathbb{H}^2 \longleftrightarrow \{\text{marked lattices in } \mathbb{C}\}/\sim$.
   Let Λ be a marked lattice in $\mathbb{C}$ and $(\nu, \tau) \in \mathbb{C}^2$ the ordered set of generators for Λ. Observe that we can scale and rotate Λ such that $\nu = 1$, i.e. $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ with $\tau \in \mathbb{H}^2$. Then the map $[\Lambda] \mapsto \tau$ is a bijection.

□

Note that we now have a bijection between $Teich(\mathbb{T})$ and $\mathbb{H}^2$, which is a topological manifold of real dimension 2. In this way we can easily endow $Teich(\mathbb{T})$ with a topology. In particular we can define a topology on $Teich(\mathbb{T})$ by imposing that the bijection of Proposition 2.1.8 is a homeomorphism.
This complete the description we need of the Teichmüller space of a differentiable surface of genus $g = 1$.

2.2 Definition of the Teichmüller Space in the case $g \geq 2$

For the rest of the chapter we will consider differentiable surfaces of genus $g \geq 2$.
As proved in Theorem 1.3.4 of Section 1.3 on a differentiable surface $\mathbb{S}$ of genus $g \geq 2$ we can always define a hyperbolic metric. It then comes naturally to use hyperbolic metrics, instead of flat metrics, to describe the Teichmüller space of $\mathbb{S}$. Let $\mathbb{S}$ be a compact surface. In the case of $g = 1$ we have described flat structures on $\mathbb{S}$, analogously we would like to describe a hyperbolic structure on $\mathbb{S}$.

Definition 2.2.1. A *hyperbolic structure* on $\mathbb{S}$ is a pair (X, ϕ), where X is a differentiable surface endowed with a complete, finite-area hyperbolic metric and $\phi\colon \mathbb{S} \longrightarrow X$ is a diffeomorphism.
The diffeomorphism ϕ is called the *marking*. The differentiable surface X and the pair (X, ϕ) are called the *marked hyperbolic surface*.

In the case $g = 1$ we have defined $Teich(\mathbb{T})$ to be the set of isotopy classes of unit-area flat structures on $\mathbb{T}$. We would like to do the same in the case $g \geq 2$, but considering homotopy classes of hyperbolic structures this time. We then need to define when two hyperbolic structures are homotopic.

Definition 2.2.2. Let (X_1, ϕ_1) and (X_2, ϕ_2) be hyperbolic structures on $\mathbb{S}$. We will say that (X_1, ϕ_1) and (X_2, ϕ_2) are *homotopic* if there is an isometry $I\colon X_1 \longrightarrow X_2$ such that the markings $I \circ \phi\colon \mathbb{S} \longrightarrow X_2$ and $\phi_2\colon \mathbb{S} \longrightarrow X_2$ are homotopic.

We can now give the following definition.

Definition 2.2.3. The *Teichmüller space* of $\mathbb{S}$ is the set of homotopy classes of hyperbolic structures on $\mathbb{S}$

$$Teich(\mathbb{S}) = \{(X, \phi)\}/\sim .$$

As we have done in the case of a torus, we can also see $Teich(\mathbb{S})$ as a set of metrics on $\mathbb{S}$. In particular we start by observing that every marking $\phi\colon \mathbb{S} \longrightarrow X$ defines a hyperbolic metrics on $\mathbb{S}$ by the pullback of the hyperbolic metric on X. Thus we can describe the Teichmüller space of $\mathbb{S}$ as the set of isotopy classes of hyperbolic metric on $\mathbb{S}$:

$$Teich(\mathbb{S}) = HypMet(\mathbb{S})/Diff_0(\mathbb{S}),$$

where the action of $Diff_0(\mathbb{S})$ is by pullback.

As defined above the Teichmüller space of $\mathbb{S}$ is the set of homotopy classes of hyperbolic structure. We would like to describe a particular map between two hyperbolic structures on $\mathbb{S}$ that is called *change of marking map.*
Let (X, ϕ), (Y, ψ) be two hyperbolic structure on $\mathbb{S}$. Observe that we have a bijective correspondence between $\mathrm{Homeo}(\mathbb{S})$ and $\mathrm{Homeo}(X, Y)$,

$$\begin{aligned} F_{X,Y}\colon \mathrm{Homeo}(\mathbb{S}) \longrightarrow& \mathrm{Homeo}(X, Y) \\ f \longmapsto& \psi \circ f \circ \phi^{-1} \end{aligned}$$

and the only canonical homeomorphism from $\mathbb{S}$ to $\mathbb{S}$ is the identity.
We can then give the following definition.

Definition 2.2.4. The *change of marking map* of (X, ϕ) and (Y, ψ) is the canonical homeomorphism $\psi \circ \phi^{-1}$, i.e. $F_{X,Y}(id_s)$.

2.3 Topology on the Teichmüller Space

Now that we have defined $Teich(\mathbb{S})$ we would like to endow it with a topology. In the case $g = 1$, to endow $Teich(\mathbb{T})$ with a topology, we have first found a bijection between $Teich(\mathbb{T})$ and a topological space $\mathbb{H}^2$ and then used this bijection to endow $Teich(\mathbb{T})$ with a topology.
We would like to do the same also in this case. In particular we would like to show that there is a bijection between $Teich(\mathbb{S})$ and a topological space, then endow $Teich(\mathbb{S})$ with a topology by imposing that this bijection to be a homeomorphism.
First we need to observe a few things on $\mathrm{PSL}(2, \mathbb{R})$ and recalling a bit of group representation theory.
One can prove that $\mathrm{Isom}^+(\mathbb{H}^2)$ is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Indeed the orientation preserving isometries of $\mathbb{H}^2$ are the Mobiüs transformation that takes $\mathbb{H}^2$ in itself, which are maps of the form $z \longmapsto \frac{az+b}{cz+d}$, where $ad - bc = 1$. The isomorphism between $Isom^+(\mathbb{H}^2)$ and $\mathrm{PSL}(2, \mathbb{R})$ is given by the map
$\left(z \longmapsto \frac{az+b}{cz+d}\right) \longmapsto \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Recall that a representation $\rho\colon G \longrightarrow \mathrm{GL}(V)$ is said to be faithful if it is injective and discrete if $\rho(G)$ is discrete in $\mathrm{GL}(V)$.
Let us consider the set $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2, \mathbb{R}))$ of discrete, faithful representation of $\pi_1(\mathbb{S}_g)$ in $\mathrm{PSL}(2, \mathbb{R})$ which is a connected component of the representation variety $Hom(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2, \mathbb{R}))$, in particular the representations of $\pi_1(\mathbb{S}_g)$ in $\mathrm{PSL}(2, \mathbb{R})$ are given by the holonomy map of hyperbolic structures on $\mathbb{S}_g$. The holonomy map is defined by the marking of the hyperbolic structure (X, ϕ) by

$\phi_*\colon \pi_1(\mathbb{S}_g) \longrightarrow \pi_1(X) \subset Isom^+(\mathbb{H}^2) \simeq \mathrm{PSL}(2,\mathbb{R})$. Note that $\pi_1(X) \subseteq Isom^+(\mathbb{H}^2)$ thanks to the Uniformization theorem.
Let $\mathrm{PGL}(2,\mathbb{R})$ act on $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$ by conjugation: for each $f \in \pi_1(\mathbb{S}_g)$ and for each $h \in \mathrm{PGL}(2,\mathbb{R})$ we define $(h\cdot\rho)(f) = h\rho(f)h^{-1}$.

Proposition 2.3.1. *There is a natural bijective correspondence between $Teich(\mathbb{S}_g)$ and $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$*

Proof.

1. Construction of a map from $Teich(\mathbb{S}_g)$ to $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$.
   Let $[(X,\phi)] \in Teich(\mathbb{S}_g)$. Then, thanks to the properties of universal covers, we have an isometric identification $\eta\colon \widetilde{X} \longrightarrow \mathbb{H}^2$, where $\widetilde{X}$ is the universal cover of X. The marking ϕ identifies $\pi_1(\mathbb{S}_g)$ with $\pi_1(X)$ and $\pi_1(X)$ acts isometrically and properly discontinuously on $\widetilde{X}$.
   This gives rise to a faithful and discrete representation $\rho\colon \pi_1(\mathbb{S}_g) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$, as observed above, ρ is given by the holonomy map.
   We have now to verify that the class of ρ in $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ is well defined. First observe that the choice of η is unique up to postcomposing with $\nu \in \mathrm{Isom}(\mathbb{H}^2)$, so if we choose $\nu\circ\eta$ instead of η we will obtain $\nu\cdot\rho$ that is still in the same class as ρ. Changing (X,ϕ) within its equivalence class is equivalent to changing ϕ within its homotopy class, and this does not affect ρ, since if we lift an isotopy of X to $\widetilde{X} \simeq \mathbb{H}^2$ then points of $\mathbb{H}^2$ move a uniformly bounded distance and so the induced action on $\partial\mathbb{H}^2$ is trivial, but an isometry of $\mathbb{H}^2$ is determined by its action on $\partial\mathbb{H}^2$. At last the isomorphism between $\phi_*(\pi_1(\mathbb{S}_g))$ and $\pi_1(X)$ is well defined up to conjugation classes. So ρ is well defined up to conjugation.

2. Construction of a map from $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ to $Teich(\mathbb{S}_g)$.
   Let $\rho \in DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$. Since ρ is discrete we have that $\rho(\pi_1(\mathbb{S}_g))$ is discrete and its action on $\mathbb{H}^2$ is properly discontinuous. Now if the action of $\rho(\pi_1(\mathbb{S}_g))$ were not free than it would contain a nontrivial elliptic isometry f of $\mathbb{H}^2$, which corresponds to an elliptic point of $\mathrm{PSL}(2,\mathbb{R})$, that is a rotation. But ρ is faithful and $\pi_1(\mathbb{S}_g)$ is torsion free, then f must have infinite order. This is absurd since ρ is discrete.
   Therefore we can construct $X = \mathbb{H}^2/\rho(\pi_1(\mathbb{S}_g))$, which is a surface with fundamental group $\pi_1(\mathbb{S}_g)$, then it is diffeomorphic to $\mathbb{S}_g$. We can recover a homomorphism $\rho_*\colon \pi_1(\mathbb{S}_g) \longrightarrow \pi_1(X)$, and it follows that there is a unique homotopy class that realizes ρ_*. But any homotopy equivalence $\mathbb{S}_g \longrightarrow X$ is homotopic to a diffeomorphism, we set this to be the desired marking.
   This map is well defined since if we replace ρ by one of its $\mathrm{PGL}(2,\mathbb{R})$ conjugates ρ' we will obtain a surface X' isometric to X, so they are the same point in $Teich(\mathbb{S}_g)$.

To conclude just observe that the two maps are one the inverse of the other. □

We use this result to describe a topology on $Teich(\mathbb{S}_g)$. First of all observe that $\pi_1(\mathbb{S}_g)$ is generated by $2g$ element, so every homomorphism $\pi_1(\mathbb{S}_g) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ is determined by the image of the $2g$ generators. so we can see $\mathrm{Hom}(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$ as a subgroup of $\mathrm{PSL}(2,\mathbb{R})^{2g}$. Indeed we know that $\pi_1(\mathbb{S}_g)$ is finitely presented by $2g$ elements with a relation on them, so $\mathrm{Hom}(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$ can be seen as the subgroup of $\mathrm{PSL}(2,\mathbb{R})^{2g}$ defined by the equation corresponding to the relation. On $\mathrm{PSL}(2,\mathbb{R})$ we consider the Lie group topology and the subspace topology on $\mathrm{Hom}(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$. Choosing a different set of generators for $\pi_1(\mathbb{S}_g)$ gives rise to equivalent topology on $\mathrm{Hom}(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$.
Obviously $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$ is a subset of $\mathrm{Hom}(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))$, so it inherits the subspace topology. Finally we endow $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ with the quotient topology. In this way we obtain a topology on $Teich(\mathbb{S}_g)$, which we will call the *algebraic topology*, by imposing that the bijection of Proposition 2.3.1 is a homeomorphism.

The Teichmüller space of a surface $\mathbb{S}$ is a topological space, and can also be endowed with a metric, called the Teichmüller metric. We will not see all the details of the construction of this metric (the reader can consult Chapter 11 of [7] for all the details) but just the definition. In Chapter 4 we will use this metric to prove that the action of the mapping class group over the Teichmüller space is properly discontinuous.

The existence of this metric is not obvious but needs some works on measured foliations and quasiconformal maps. We will just say that there is a preferred class of quasiconformal maps between two points of $Teich(\mathbb{S})$ called Teichmüller mappings in the homotopy class of the change of marking between this two points. The existence of a Teichmüller mapping and good definition of the metric follows from two theorems due to Teichmüller.

Definition 2.3.2. Let $\mathcal{X}$, $\mathcal{Y} \in Teich(\mathbb{S})$. Let $f\colon X \longrightarrow Y$ be the change of marking between two representative X and Y respectively of $\mathcal{X}$ and $\mathcal{Y}$ and let $h\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be the Teichmüller mapping in the homotopy class of f. The *Teichmüller distance* between $\mathcal{X}$ and $\mathcal{Y}$ is given by $d_{Teich}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}log(K_h)$, where K_h is the dilatation of h.

2.4 Dimension count

Now that we have a topology on $Teich(\mathbb{S})$ our goal is to determine the real dimension of $Teich(\mathbb{S})$. To do so we will start by describing the Teichmüller space of a pair of pants and then use the pants decomposition of a surface to calculate the real dimension of $Teich(\mathbb{S})$.
To describe the Teichmüller space of a pair of pants and to find a set of coordinates on $Teich(\mathbb{S})$ we will need a length function on isotopy classes of curves on a hyperbolic surface.
In order to define a length function we need a result on simple closed curves in a hyperbolyc surface.

Proposition 2.4.1. *Let X be a hyperbolic surface and let α be a closed curve in X that is not homotopic to a puncture. Then α is homotopic to a unique geodesic closed curve γ in X.*

Proof. See Proposition 1.3 of [7] □

Let $\mathcal{C}$ be the set of isotopy class of simple closed curves in $\mathbb{S}$ and let $\mathcal{X}$ be a point in $Teich(\mathbb{S})$, and (X, ϕ) one of its representative, we define the length function on $\mathcal{X}$ to be the function

$$\ell_{\mathcal{X}} \colon \mathcal{C} \longrightarrow \mathbb{R}_+,$$

which associate to $c \in \mathcal{C}$ the length of the unique geodesic in X in the isotopy class of $\phi(c)$.
This is well-defined thanks to Proposition 2.4.1. Since we have a topology on $Teich(\mathbb{S}_g)$ we can observe that the length function is continuous. Indeed for $\gamma \in \pi_1(\mathbb{S}_g)$ the function $[\rho] \longmapsto trace(\rho(\gamma))$ is continuous on $DF(\pi_1(\mathbb{S}_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$. Considering $\rho_{\mathcal{X}}$ the representation associated to $\mathcal{X} \in Teich(\mathbb{S}_g)$, then $\ell_{\mathcal{X}}(\gamma) = 2cosh^{-1}(trace(\rho_{\mathcal{X}}(\gamma))/2)$. And so, for c an isotopy class of simple closed curves in $\mathbb{S}$,the function

$$\begin{aligned} \ell\colon Teich(\mathbb{S}) &\longrightarrow \mathbb{R} \\ \mathcal{X} &\longmapsto \ell_{\mathcal{X}}(c) \end{aligned}$$

is continuous. We have all the necessary results on the length function. We will use it to define a set of coordinates on $Teich(\mathbb{S})$ and on the Teichmüller space of a pair of pants.

2.4.2 Teichmüller space of a pair of pants

As already mentioned we need to describe the Teichmüller space of a pair pants to find the real dimension of $Teich(\mathbb{S})$. We will be able to describe $Teich(P)$, where P denotes a pair of pants, up to homeomorphism. First, since P is not a compact differentiable surface without boundary components we need to extends the concept of the Teichmüller space to surfaces with non-empty boundary components.
The idea is to proceed in the same way as the case $g \geq 2$, so first of all we need to say what is a hyperbolic metric on a surface with non-empty boundary.

Definition 2.4.3. Let $\mathbb{S}$ be a surface with boundary $\partial\mathbb{S} \neq \emptyset$. A *hyperbolic metric* on $\mathbb{S}$ is a complete,finite-area Riemannian metric with constant curvature -1 and totally geodesic boundary, which means that the geodesic in $\partial\mathbb{S}$, with respect to the metric induced on $\partial\mathbb{S}$, are geodesics in $\mathbb{S}$.

One can prove that Theorem 1.3.4 can be extended to the case of $\mathbb{S}$ with $\chi(\mathbb{S}) \leq 0$, as is the case of a pair of pants.
Thanks to this results we can define the Teichmüller space of a pair of pants the same way as the one of a differentiable surface of genus $g \geq 2$. So we have the following definition.

Definition 2.4.4. The *Teichmüller space* of a pair of pants P is the set of homotopy classes of hyperbolic structures on P,

$$Teich(P) = \{(X, \phi)\}/\sim .$$

To determine $Teich(P)$ we will use the marked hyperbolic hexagons in $\mathbb{H}^2$. In particular a marked hexagon is a hexagon with one vertex distinguished. Let $\mathcal{H}$ denote the set of marked right-angled geodesic hexagons in $\mathbb{H}^2$, recall that a marked right-angled geodesic hexagon is a hyperbolic hexagon with hyperbolic geodesics as its edges and all is angles are right angles, with the equivalence relation described by the fact that two hexagons are equivalent if there is an orientation preserving isometry of $\mathbb{H}^2$ that takes one hexagon to the other taking the marked point of the first to the marked point of the second.

To describe $Teich(P)$ we first need a preliminary result on $\mathcal{H}$, in particular we want to construct a bijection between $\mathcal{H}$ and $\mathbb{R}^3_+$. After that we will prove that $Teich(P)$ is homeomorphic to $\mathbb{R}^3_+$ showing that there is a bijection between $Teich(P)$ and $\mathcal{H}$.

Proposition 2.4.5. *The map $W\colon \mathcal{H} \longrightarrow \mathbb{R}^3_+$ defined by taking the lengths of every other side of the hexagon, starting at the marked point and traveling counterclockwise, is a bijection.*

Proof. The idea for this proof is to define a two-sided inverse of W, i.e. given an arbitrary triple $(L_\alpha, L_\beta, L_\gamma) \in \mathbb{R}^3_+$ we construct a marked right-angled hexagon H, unique up to marked orientation preserving isometry, that satisfies $W(H) = (L_\alpha, L_\beta, L_\gamma)$.
Recall that given two disjoint geodesic in $\mathbb{H}^2$ with four distinct endpoints at infinity there is a unique geodesic perpendicular to both.
For any $t > 0$ let α_t and β_t be a pair of geodesic in $\mathbb{H}^2$ at distance t apart and let γ'_t be the unique geodesic segment realizing this distance. Let α'_t and β'_t be geodesics on the same side of γ'_t such that α'_t has a perpendicular intersection with β_t at distance L_β from γ'_t and β'_t has a perpendicular intersection with α_t at distance L_α from γ'_t. Lastly we require that if γ'_t is oriented from α_t to β_t then α'_t and β'_t lie to the left of the γ'_t.
Observe that there is $t_0 > 0$ such that α'_{t_0} and β'_{t_0} share an endpoint in $\partial\mathbb{H}^2$. For $t > t_0$ let γ_t the unique geodesic segment perpendicular to α'_t and β'_t. As t varies from t_0 to infinity, the length of γ_t varies continuously from zero to infinity, so there is a t such that the length of γ_t is L_γ. Marking the intersection of α_t and β'_t we obtain the desired point of $\mathcal{H}$.
We just need to verify that H is well defined. The only choice we made is the choice of α_t and β_t, but there is a unique ordered pair of geodesics whose distance is given up to orientation preserving isometry of $\mathbb{H}^2$.
To sum up we constructed a two-sided inverse of W. ☐

We can now determine $Teich(P)$, up to homeomorphism.

Proposition 2.4.6. *Let P be a pair of pants with boundary components $\alpha_1, \alpha_2, \alpha_3$. The map*

$$\begin{aligned} Teich(P) &\longrightarrow \mathbb{R}^3_+ \\ \mathcal{X} &\longmapsto (\ell_{\mathcal{X}}(\alpha_1), \ell_{\mathcal{X}}(\alpha_2), \ell_{\mathcal{X}}(\alpha_3)) \end{aligned}$$

is a homeomorphism.

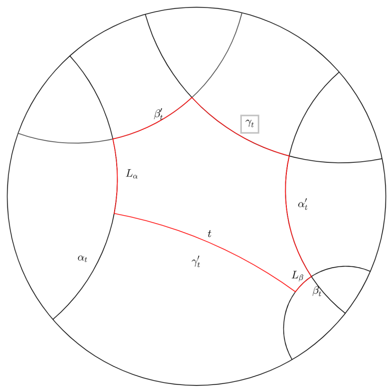


Figure 2.1: Construction of H

Proof. 1. Construction of a bijection between $Teich(P)$ and $\mathcal{H}$.
Let $\mathcal{X} = [(X, \phi)] \in Teich(P)$, so X is an hyperbolic surface with totally geodesic boundary, since X needs to be a hyperbolic surface diffeomorphic to P, and $\phi\colon P \longrightarrow X$ is a diffeomorphism. For each pair of distinct boundary components of X there is a unique isotopy class of arcs connecting them, let $\delta_{ij} = \delta_{ji}$ be the geodesic representative of the arc connecting $\phi(\alpha_i)$ and $\phi(\alpha_j)$. We have that each δ_{ij} is perpendicular to ∂X at both of its endpoints. The closure of the two components of $X \setminus \cup\delta_{ij}$ are hyperbolic right-angled hexagons H_1 and H_2. This two are isometric since the lengths of the δ_{ij} determine the hyperbolic structure on each. Let H be the marked right-angled hexagon in $\mathbb{H}^2$ that is isometric to the image of H_1 where the marked point is $\delta_{13} \cap \phi_{\alpha_1}$ and consider its equivalence class in $\mathcal{H}$.
Given an element of $\mathcal{H}$ and consider one of it representative $H \subseteq \mathbb{H}^2$. Construct a second hexagon H' by reflecting H over the edge lying first in the clockwise direction from the marked point. Label the sides as in Figure 2.2, and then identify the sides labelled δ_{12} and δ_{23} to obtain an hyperbolic pair of pants X. As the marking we take the unique isotopy class of diffeomorphism $P \longrightarrow X$.

2. To conclude we just need to compose the bijection found with the map W from Proposition 2.4.5. In this way we obtain an homeomorphism since if two points of $\mathbb{R}^3_+$ are close then the corresponding hexagons are nearly isometric, so the corresponding representations are close in the algebraic topology on $Teich(P)$.

□

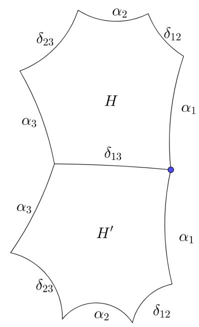


Figure 2.2: Construction of a pair of pants from a marked hexagon

2.4.7 Real dimension of $Teich(\mathbb{S})$

We now want to use the fact that every surface $\mathbb{S}_g$ can be decomposed into pair of pants to calculate the real dimension of $Teich(\mathbb{S}_g)$.
The idea is to use $3g-3$ simple closed curves in $\mathbb{S}_g$ to decompose it into pairs of pants. Then we have $3g-3$ length parameters, that define the hyperbolic structure of the pairs of pants, and $3g-3$ twist parameters, that determine how the pairs of pants are glued together.
To start we need to choose a coordinate system of curves on $\mathbb{S}_g$, which consist of:

- a pants decomposition $\{\gamma_1, \dots, \gamma_{3g-3}\}$ of simple oriented closed curves,
- a set $\{\beta_1, \dots, \beta_{3g-3}\}$ of seams, that is a collection of disjoint simple closed curves in $\mathbb{S}_g$ so that the intersection of the union $\cup\beta_i$ with any pair of pants P determined by the $\{\gamma_j\}$ is a union of three disjoint geodesic arcs connecting the boundary components of P pairwise and perpendicular to them at their endpoints.

See Figure 2.3 for an example of coordinate system.

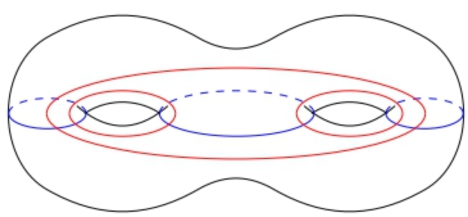


Figure 2.3: An example of a coordinate system for $\mathbb{S}_2$. In blue γ_1, γ_2 and γ_3, in red β_1, β_2 and β_3.

Observe that given a pants decomposition we can construct seams by choosing three disjoint arcs on each pair of pants and matching up the endpoints. Fix a coordinate system of curves on $\mathbb{S}_g$.

Definition 2.4.8. The $3g-3$ length parameters of a point $\mathcal{X} \in Teich(\mathbb{S}_g)$ is the ordered $(3g-3)-$tuple of positive numbers $(\ell_1(\mathcal{X}), \dots, \ell_{3g-3}(\mathcal{X}))$, where $\ell_i(\mathcal{X}) = \ell_{\mathcal{X}}(\gamma_i)$, and $\ell_{\mathcal{X}}$ the length function associated to $\mathcal{X}$.

Thanks to Proposition 2.4.5 the length parameters for a point in $Teich(\mathbb{S}_g)$ determine the isometry types of $2g-2$ pair of pants cut out by the coordinate system of curves for $\mathbb{S}_g$. We need to introduce the twist

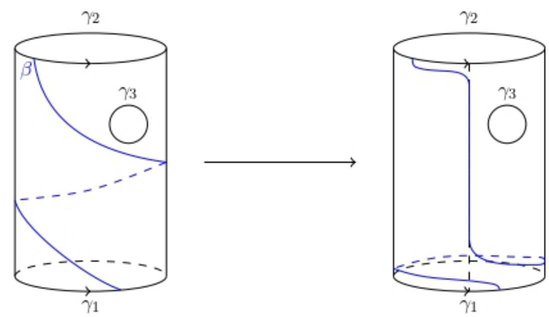


Figure 2.4: Modifying an arc on a pair of pants so that it agrees with a perpendicular arc except near its endpoints

parameters $\theta_i(\mathcal{X})$ to record how these pants are glued together.
Before define the twist parameters we need to define the twisting number. Suppose that β is an arc in a hyperbolic pair of pants P connecting the boundary components γ_1 and γ_2 of P. Let δ be the unique shortest arc connecting γ_1 and γ_2 and N_1, N_2 be regular metric neighbourhoods of γ_1, γ_2, respectively. We can modify β by isotopy so that it agrees with δ outside $N_1 \cup N_2$ (See Figure 2.4).

Definition 2.4.9. The twisting number of β at γ_1 is the signed horizontal displacement of the endpoints $\beta \cap \partial N_1$, which is the shortest length between the point $\beta \cap \partial N_1$ and the point of intersection between ∂N_1 and the curve isotopic to β that agrees with δ outside $N_1 \cup N_2$.
The sign is determined by the orientation of γ_1.
Likewise one can define the twisting number of β at γ_2.

Given $\mathcal{X} = [(X, \phi)] \in Teich(\mathbb{S}_g)$ the $i-$th twist parameter $\theta_i(\mathcal{X})$ is defined as it follows: let β_j be one of the two seams that crosses γ_i. On each side of the $\phi(\gamma_i)$ geodesic there is a pair of pants, and the $\phi(\beta_j)$ geodesic gives an arc in each one of these. Let t and t' be the twisting numbers of each one of these arcs.

Definition 2.4.10. The $i-$th twist parameter of $\mathcal{X}$ is $\theta_i(\mathcal{X}) = 2\pi \dfrac{t - t'}{\ell_{\mathcal{X}}(\gamma_i)}$.

Remark 2.4.11. *We need to check if $\theta_i(\mathcal{X})$ is well-defined, since there were two choices of β_j. As in Proposition 2.4.6 the four geodesic arcs connecting $\phi(\gamma_i)$ to the boundary components of the adjacent pairs of pants are perpendicular to $\phi(\gamma_i)$. Moreover, on each side of $\phi(\gamma_i)$, the two geodesics lie on diametrically opposed points along $\phi(\gamma_i)$. If we modify the seams as in the definition of the twist parameter and then pass to the universal cover of N_i, see Figure 2.5, we obtain that each lift of a seams connects two arcs, and the twist parameter is the signed distance between these arcs. Also the seams do not cross each other. We see that the twist parameter computed by the two seams are the same. See Figure 2.5.*

We can now prove the following theorem by Fricke.

Theorem 2.4.12. *Let $g \geq 2$ and fix any coordinate system of curves on $\mathbb{S}_g$. The map*

$$\begin{aligned} FN \colon Teich(\mathbb{S}_g) &\longrightarrow \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \\ \mathcal{X} &\longmapsto (\ell_1(\mathcal{X}), \ldots, \ell_{3g-3}(\mathcal{X}), \theta_1(\mathcal{X}), \ldots, \theta_{3g-3}(\mathcal{X})) \end{aligned}$$

is a homeomorphism. In particular, $Teich(\mathbb{S}_g) \subset \mathbb{R}^{6g-6}$.

The ordered set of numbers $(\ell_1(\mathcal{X}), \ldots, \ell_{3g-3}(\mathcal{X}), \theta_1(\mathcal{X}), \ldots, \theta_{3g-3}(\mathcal{X}))$ is called the Fenchel-Nielsen coordinates of the point $\mathcal{X} \in Teich(\mathbb{S}_g)$.

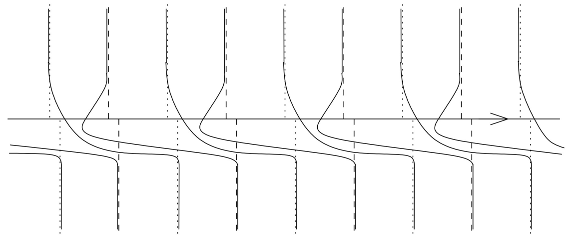


Figure 2.5: Universal cover of N_i. The geodesic arcs are dashed and the modified seams are solid

Proof. To prove the statement we will construct a continuous inverse of the map FN.
Denote the pants decomposition of the fixed coordinate system of curves for $\mathbb{S}_g$ by $\{\gamma_i\}$ and the seams by $\{\beta_j\}$.
Let $(\ell_1, \dots, \ell_{3g-3}, \theta_1, \dots, \theta_{3g-3}) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$. We want to construct $\mathcal{X} \in Teich(\mathbb{S}_g)$ with these Fenchel-Neilsen coordinates with respect to the given coordinate system of curves.
Let $P_{i,j,k}$ be the pair of pants determined by $\gamma_i, \gamma_j, \gamma_k$, which might not be distinct. By Proposition 2.4.6 we can construct a hyperbolic pair of pants $X_{i,j,k}$ whose boundary components have lengths ℓ_i, ℓ_j, ℓ_k, unique up to isometry. Then there is a homeomorphism $P_{i,j,k} \longrightarrow X_{i,j,k}$ taking each γ_i to a boundary curve of length ℓ_i, and the boundary components of $X_{i,j,k}$ inherit orientations from the γ_i.
For each $X_{i,j,k}$ and each pair of its boundary components, we consider the unique geodesic arc that is perpendicular to those boundary components. For each $m \in \{i, j, k\}$, in a small neighbourhood of a boundary component corresponding to the left side of γ_m, we replace each geodesic arc with an arc that travels along that boundary component for an oriented distance of $\dfrac{\theta_m}{2\pi}\ell_m$. The result is unique up to isotopy relative to $\partial X_{i,j,k}$. Given a seam in $P_{i,j,k}$, that is the intersection of some β_l with $P_{i,j,k}$, there is a unique corresponding seam in $X_{i,j,k}$, namely the arc that connects the corresponding boundary components.
Since the boundary curves and seams of $X_{i,j,k}$ are identified with the boundary curves and seams of $P_{i,j,k}$, there is a unique way to construct a quotient $X = \amalg\ X_{i,j,k}/\sim$, where we identify corresponding boundary components of the $X_{i,j,k}$ in such a way that the corresponding seams match up.
Finally we construct a diffeomorphism $\phi\colon \mathbb{S}_g \longrightarrow X$ that respects the identifications of the coordinate system of curves. The marked surface (X, ϕ) is a representative of the desired point in $Teich(\mathbb{S}_g)$.
We have defined a map $FN'\colon \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow Teich(\mathbb{S}_g)$ which is the inverse of the map FN. It follows by the definitions that the two maps are continuous. Thus FN is a homeomorphism. □

We have proved that $Teich(\mathbb{S}_g)$, for $g \geq 2$, has dimension $6g-6$ on $\mathbb{R}$. From this fact it follows that the complex dimension of $Teich(\mathbb{S}_g)$ is $3g-3$.
This conclude our study of the Teichmüller space of a compact differentiable surface.

Chapter 3

The Mapping Class Group

The purpose of this chapter is to introduce the Mapping Class Group of a surface S, of given genus g and with n punctures. We will first see the definition of the mapping class group $Mod(S)$ of S and then compute some simple examples. From this examples we will deduce a method that will work in general called the Alexander method.
Then we will describe a particular class of elements in the mapping class group called Dehn twists. Thanks to this elements and some of their properties we will be able to prove that $Mod(S)$ is finitely generated as a group.

3.1 Definition

Let $Homeo^+(S, \partial S)$ denote the group of orientation-preserving homeomorphism of S that restrict to the identity on ∂S. We endow this group with the compact-open topology. Recall that the compact-open topology is the topology whose subbase is the collection of all the set of the form $V(K, U)$,
where $V(K,U) = \{f \in C(X,Y) \mid f(K) \subseteq U\}$, with $K \subseteq X$ compact and $U \subseteq Y$ open.

Remark 3.1.1. *One can prove that on a compact surface S two homotopic homeomorphisms are isotopic.*

We can now give the definition of mapping class group.

Definition 3.1.2. The mapping class group of S, which is denoted by $Mod(S)$, is the group

$$Mod(S) = \pi_0(Homeo^+(S, \partial S)).$$

The elements of $Mod(S)$ are called mapping classes.
In other words $Mod(S)$ is the group of isotopy classes of elements of $Homeo^+(S, \partial S)$, where isotopies are required to fix the boundary point-wise.

In the case of $\mathbb{S}$ one can prove the following Theorem.

Theorem 3.1.3. *Let $\mathbb{S}$ be a compact surface. Then every homeomorphism of $\mathbb{S}$ is isotopic to a diffeomorphism of $\mathbb{S}$.*

Remark 3.1.4. *Using the previous Theorem we have $Mod(\mathbb{S}) = \pi_0(Diff^+(\mathbb{S}, \partial\mathbb{S}))$, where $Diff^+(\mathbb{S}, \partial\mathbb{S})$ is the group of orientation-preserving diffeomorphism of $\mathbb{S}$ that are the identity on the boundary.*

Let us give some simple examples of mapping classes.

Example 3.1.5. Consider $S = S_g$ and the order g homeomorphism ϕ of S_g, which is given by the rotation of $2\pi/g$, and his class in $Mod(S_g)$ also has order g.

Example 3.1.6. If we represent S_g as a polygon with $4g$ sides with opposite sides identified then we can get mapping classes by rotating the polygon by any number of "clicks". In particular if we rotate by an angle of π we obtain the hyperelliptic involution. For example the rotation by $2\pi/8$ gives an order 8 element of $Mod(S_2)$.

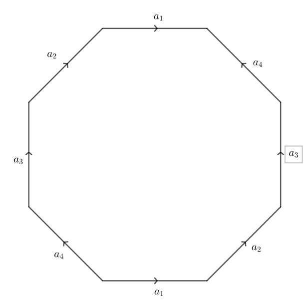


Figure 3.1: The representation of S_2 as an octagon.

Observe that most elements of $Mod(S)$ have infinite order unlike the preceding examples.

3.2 Computation of the mapping class group

In this section we will compute the mapping class group of some simple surfaces, working directly from the definition.

Example 3.2.1. Let consider D^2 the closed disk. We have the following Lemma.

Lemma 3.2.2. ***(Alexander Lemma)*** *The group $Mod(D^2)$ is trivial.*

Proof. First identify D^2 with the closed unit disk in $\mathbb{R}^2$. Let $\phi\colon D^2 \longrightarrow D^2$ be a homeomorphism, with $\phi_{|\partial D^2} = Id_{\partial D^2}$. We need to prove that ϕ is isotopic to the identity. We define

$$F(x,t) = \begin{cases} (1-t)\phi(\dfrac{x}{1-t}) & 0 \leq |x| < 1-t \\ x & 1-t \leq |x| \leq 1 \end{cases}$$

for $t \in [0,1)$, and we define $F(x,1)$ to be the identity map of D^2. Then F is an isotopy from ϕ to the identity. $\square$

The construction of $F(x,t)$ we did in the Alexander Lemma can be thought of as follows: at a time t do the map ϕ on the disk of radius $1-t$ and the identity outside of this disk. This proof is called the "Alexander trick".
One can observe that the Alexander trick works in all dimensions.
We can use the same proof as Alexander Lemma to prove that also the mapping class group of the once-punctured disk is trivial.

Example 3.2.3. Now consider the sphere S^2 and the once-punctured sphere $S_{0,1}$. In the case of $S_{0,1}$ is easy to see that $Mod(S_{0,1})$ is, again, trivial. Indeed we can identify $S_{0,1}$ with $\mathbb{R}^2$ and use the fact that every orientation preserving homeomorphism of $\mathbb{R}^2$ is homotopic to the identity via the straight-line homotopy. For S^2 just observe that every homeomorphism can be modified by isotopy so that it fixes a point, in this way we can apply the case of $S_{0,1}$ to obtain that also $Mod(S^2)$ is trivial.

Example 3.2.4. Consider the three-punctured sphere $S_{0,3}$. First of all observe that we can consider the puncture points as marked points, then the mapping class group is the group of homeomorphisms that leave the set of marked points invariant, modulo isotopies that leave the set of marked points invariant.
In this case we will compute $Mod(S_{0,3})$ by understanding its action on some fixed arcs in $S_{0,3}$.
In order to do this we need some results on simple proper arcs in $S_{0,3}$. Let start by recalling the definition of simple proper arc on a surface.

Definition 3.2.5. A proper arc on S is a continuous map $\alpha\colon [0,1] \longrightarrow S$ such that $\alpha(0)$ and $\alpha(1)$ are either puncture points or in ∂S and $\alpha((0,1))$ is contained in the interior of S. Moreover a proper arc is simple if it is an embedding on $(0,1)$ and essential unless it is homotopic into a puncture.

We now have this result on $S_{0,3}$.

Proposition 3.2.6. *Any two essential simple proper arcs on $S_{0,3}$ with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of $S_{0,3}$ are isotopic.*

Proof. See Proposition 2.2 of [7]. □

We can now compute $Mod(S_{0,3})$.

Proposition 3.2.7. *The natural map $F\colon Mod(S_{0,3}) \longrightarrow \mathfrak{S}_3$ given by the action of $Mod(S_{0,3})$ on the set of marked points of $S_{0,3}$ is an isomorphism.*

Proof. The map F is a surjective homomorphism. So it suffices to show that if a homeomorphism ϕ of $S_{0,3}$ fixes the three marked points, p, q and r, then ϕ is homotopic to the identity.
We choose an arc α in $S_{0,3}$ with distinct endpoints, say p and q. Observe that p and q are endpoints of also $\phi(\alpha)$. Then, by Proposition 3.2.6, we have that α and $\phi(\alpha)$ are isotopic. It follows that ϕ is isotopic to a map that fixes pointwise α, with an abuse of notation we will call ϕ this map.
We can cut $S_{0,3}$ along α and obtain a disk with one marked point, which comes from r. Since ϕ is orientation-preserving, it induces a homemorphism $\overline{\phi}$ of this disk, which is the identity on the boundary (which comes from α). But, we proved that the mapping class group of the once-punctured disk is trivial so there is an homotopy G from $\overline{\phi}$ to the identity. This homotopy G induces a homotopy from ϕ to the identity in $S_{0,3}$. □

Example 3.2.8. Let see an example of an infinite order mapping class group. Let A denote a annulus.

Proposition 3.2.9. *We have $Mod(A) \simeq \mathbb{Z}$.*

Proof. First let construct a map $\rho\colon Mod(A) \longrightarrow \mathbb{Z}$. Let $f \in Mod(A)$, and $\phi\colon A \longrightarrow A$ a homeomorphism representing f. The universal cover of A is the infinite strip $\widetilde{A} \simeq \mathbb{R} \times [0,1]$, and ϕ has a preferred lift $\widetilde{\phi}\colon \widetilde{A} \longrightarrow \widetilde{A}$ fixing the origin. Let δ be an oriented simple proper arc that connects the two boundary components of A. Consider $\widetilde{\delta}$ the unique lift of δ to $\widetilde{A}$ based at the origin. Define $\rho(f)$ to be the endpoint of $\widetilde{\phi}(\widetilde{\delta})$ in $\mathbb{R} \times \{1\} \simeq \mathbb{R}$.
We show that ρ is surjective. The linear transformation of $\mathbb{R}^2$ given by the matrix

$$M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

preserves $\mathbb{R} \times [0,1]$ and is equivariant with respect to the group of deck transformations. The restriction of M to $\mathbb{R} \times [0,1]$ descends to a homeomorphism ϕ of A. It follows from the definition of ρ that $\rho([\phi]) = n$ (See Figure 3.2 for the case $n = -1$).

We show that ρ is injective. Let $f \in Mod(A)$ an element of the kernel of ρ and ϕ a homeomorphism that represents f. Again, let $\widetilde{\phi}$ be the preferred lift of ϕ. Since $\rho(f) = 0$, we have that $\widetilde{\phi}$ acts as the identity on $\partial\widetilde{A}$. To show that ρ is injecive it suffices to show that there is an equivariant homotopy between ϕ and the identity. We claim that the straight line homotopy from $\widetilde{\phi}$ to the identity of $\widetilde{A}$ is equivariant and it fixes the boundary of $\widetilde{A}$, then it descends to an equivariant homotopy from ϕ to the identity which fixes the boundary pointwise.

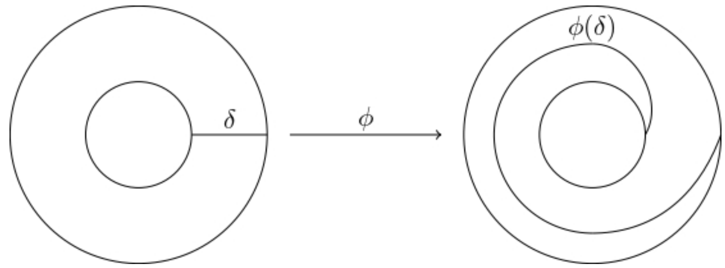


Figure 3.2: Action of ϕ on a proper simple arc δ

To show that the straight line homotopy is equivariant we just need to prove that $\widetilde{\phi}(\tau \cdot x) = \tau \cdot \widetilde{\phi}(x)$ for any deck transformation τ and for any $x \in \widetilde{A}$. From covering space theory we have $\widetilde{\tau \cdot \phi} = \phi_*(\tau) \cdot \widetilde{\phi}(x)$. But ϕ fixes ∂A pointwise, then ϕ_* is the identity automorphism of $\pi_1(A) \simeq \mathbb{Z}$, so $\phi_*(\tau) = \tau$ and the claim is proven.
Thus f is the identity and so ρ is injective. □

Example 3.2.10. Another important example of computation of the mapping class group is the case of a torus $\mathbb{T}$. It will also give us a hint of what to expect in the higher genus case.

Theorem 3.2.11. *The homomorphism*

$$\sigma\colon Mod(\mathbb{T}) \longrightarrow \mathrm{SL}(2, \mathbb{Z})$$

given by the action on $H_1(\mathbb{T}, \mathbb{Z}) \simeq \mathbb{Z}^2$ *is an isomorphism.*

Proof. Any homeomorphism ϕ of $\mathbb{T}$ induces a map $\phi_*\colon H_1(\mathbb{T}, \mathbb{Z}) \longrightarrow H_1(\mathbb{T}, \mathbb{Z})$, but ϕ in invertible, so ϕ_* is an automorphism of $H_1(\mathbb{T}, \mathbb{Z})$. Recall that homotopic map induce the same map on homology, then the map $\phi \longmapsto \phi_*$ induces a map $\sigma\colon Mod(\mathbb{T}) \longrightarrow Aut(\mathbb{Z}^2) \simeq \mathrm{GL}(2, \mathbb{Z})$. The fact that $\sigma(f)$ is an element of $\mathrm{SL}(2, \mathbb{Z})$ can be seen from the fact that the algebraic intersection numbers in $\mathbb{T}$ correspond to determinants, and the fact that orientation preserving homeomorphisms preserve algebraic intersection numbers.
We next prove the surjectivity of σ. Any element $M \in \mathrm{SL}(2, \mathbb{Z})$ induces an orientation preserving linear homeomorphism of $\mathbb{C}$ that is equivariant with respect to the deck transformation group $\mathbb{Z}^2$. Thus, it descends to a homeomorphism ϕ_M of the torus $\mathbb{T} \simeq \mathbb{C}/\mathbb{Z}^2$. Then, thanks to the identification of primitive vectors in $\mathbb{Z}^2$ with homotopy classes of oriented simple closed curves in $\mathbb{T}$, we have $\sigma([\phi_M]) = M$, and so σ is surjective.
At last, we prove that σ is injective. Since $\mathbb{T}$ is a $K(G, 1)$ space, there is a correspondence:

$$\{\text{Homotopy classes of based maps } \ \mathbb{T} \longrightarrow \mathbb{T}\} \longleftrightarrow \{\text{Homomorphism } \ \mathbb{Z} \longrightarrow \mathbb{Z}\}$$

Moreover any element $f \in Mod(\mathbb{T})$ has a representative ϕ that fixes a basepoint for $\mathbb{T}$. Thus, if $f \in ker(\sigma)$, then ϕ is homotopic to the identity, so σ is injective. Likewise the case of the annulus we can contruct the homotopy between ϕ and the identity. □

Remark 3.2.12. *Note that, since* $Mod(\mathbb{T}) \simeq \mathrm{SL}(2, \mathbb{Z})$*, torsion in* $Mod(\mathbb{T})$ *is the same as torsion in* $\mathrm{SL}(2, \mathbb{Z})$*. In particular the group* $\mathrm{SL}(2, \mathbb{Z})$ *has 8 nontrivial conjugacy classes of finite order elements. There are elements of order 2, 3, 4 and 6, given by the matrices*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \textit{and} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

and their inverses. Each of these matrices can be realized as an isometry of the Euclidean Torus.

Example 3.2.13. For the case of the once-punctured torus $S_{1,1}$, we have $H_1(S_{1,1}, \mathbb{Z}) \simeq H_1(\mathbb{T}, \mathbb{Z}) \simeq \mathbb{Z}^2$. Therefore, as in the example of $\mathbb{T}$, there is a homomorphism $\sigma\colon Mod(S_{1,1}) \longrightarrow \mathrm{SL}(2, \mathbb{Z})$.

Indeed σ is surjective since any element of $\mathrm{SL}(2,\mathbb{Z})$ can be realized as a map of $\mathbb{R}^2$ that is equivariant with respect to $\mathbb{Z}^2$ and that fixes the origin. Such a map descends to a homeomorphism of $S_{1,1}$ with the desired action on homology.
To prove that σ is injective, let α and β be simple closed curves in $S_{1,1}$ that intersect in one point. If $f \in ker(\sigma)$ is represented by ϕ, then $\phi(\alpha)$ and $\phi(\beta)$ are isotopic to α and β. We can then modify ϕ by isotopy so that it fixes α and β pointwise. If we cat $S_{1,1}$ along $\alpha \cup \beta$, we obtain a once-punctured disk, and ϕ induces a homeomorphism of this disk fixing the boundary. By the Alexander trick, this homeomorphism of the once-punctured disk is homotopic to the identity by a homotopy that fixes the boundary. It follows that ϕ is homotopic to the identity.

3.3 The Alexander method

The previous examples all follow the same general scheme: find a collection of curves and/or arcs that cut the surface into disks, and apply the Alexander Lemma (Lemma 3.2.2) in order to say that the action of the mapping class group is completely determined by the action on the isotopy classes of these curves and arcs.

This basic setup works for a general surface. In particular the Alexander method (given below) states that, for any S, an element of $Mod(S)$ is often determined by its action on a well-chosen collection of curves and arcs in S. To simplify, we consider only compact surface with finitely many marked points in the interior.

Proposition 3.3.1. ***(Alexander method)*** *Let S be a compact surface, and let $\phi \in Homeo^+(S, \partial S)$. Let $\gamma_1, \dots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S with the following properties:*

i. the γ_i are pairwise in minimal position,

ii. the γ_i are pairwise nonisotopic,

iii. for distinct i, j, k at least one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$ or $\gamma_j \cap \gamma_k$ is empty.

If there is a permutation $\sigma \in \mathfrak{S}_n$ so that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to ∂S.

If we regard $\cup\gamma_i$ as a graph Γ in S, with vertices at the intersection points and at the endpoints of arcs, then the composition of ϕ with this isotopy gives an automorphism ϕ_* of Γ.

Suppose now that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Γ, with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ has a nontrivial power that is isotopic to the identity.

Proof. See Propostion 2.8 of [7]. □

Remark 3.3.2. *A priori the Alexander method only allow us to determine a mapping class up to a finite power. However, on almost every surface, one can choose the collection $\{\gamma_i\}$ so that mapping classes are determined uniquely by their action on the $\{\gamma_i\}$; i.e., one can choose the γ_i so that whenever a homeomorphism ϕ fixes each γ_i up to homotopy, then the induced map ϕ_* of the graph Γ is necessarily the identity.*

3.4 Dehn twists

A particular type of mapping classes are called Dehn twists. These are the simplest infinite order mapping classes, in the sense that they have representatives with the "smallest" possible supports. Thanks to them we can also compute different mapping class group from the one we have already studied in the previous sections, such as the mapping class group of a pair of pants. In the next section we will also be able to use the Dehn twists to prove that $Mod(S)$ is finitely generated as a group.

First of all we give the definition of Dehn twist.

Consider the annulus $A = S^1 \times [0, 1]$. To orient A we embed it in the (θ, r) plane via the map $(\theta, t) \mapsto (\theta, t + 1)$, and take the orientation induced by the standard orientation of the plane.

Let $T\colon A \longrightarrow A$ be the twist map of A given by $T(\theta,t) = (\theta + 2\pi t, t)$. The map T is an orientation preserving homeomorphism that fixes ∂A pointwise. In Figure 3.2 there is an example of the action of the twist map.

Let S be an arbitrary oriented surface and let α be a simple closed curve in S. Let N be a regular neighbourhood of α (see [18]), and choose a orientation preserving homeomorphism $\phi\colon A \longrightarrow N$.

Definition 3.4.1. A *Dehn twist about* α is a homeomorphism $T_\alpha\colon S \longrightarrow S$ described as follows

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in S \setminus N. \end{cases}$$

The Dehn twist T_α depends on the choice of N and ϕ. However, thanks to the uniqueness of regular neighbourhoods, the isotopy class of T_α does not depend on either of these choices. Thus we have the following definition

Definition 3.4.2. Let a denote the isotopy class of α. The *Dehn twist about* a is the mapping class $T_a \in Mod(S)$.

Sometimes, abusing notation, we will write T_α for the mapping class T_a.
Studying the action of Dehn twist on simple closed curves one can prove the following.

Proposition 3.4.3. *Let a be the isotopy class of a simple closed curve α in a surface S. If α in not homotopic to a point or a puncture of S, then the Dehn twist T_a is a nontrivial element of $Mod(S)$.*

Example 3.4.4. Let consider a torus $\mathbb{T}$, as already shown, $Mod(\mathbb{T}) \simeq \mathrm{SL}(2,\mathbb{Z})$. Since $\mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, it follows that $Mod(\mathbb{T})$ is generated by the correspondant elements. This generator are the Dehn twists about the two generators of the fundamental group of the torus, represented in Figure 3.3 below.

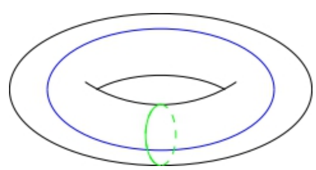


Figure 3.3: Generators of $\pi_1(\mathbb{T})$

Remark 3.4.5. *One can prove that for any $f \in Mod(S)$ and any isotopy class a of a simple closed curve in S we have:*

$$T_{f(a)} = fT_af^{-1}.$$

There is a relation between Dehn twists in $Mod(S)$, called the braid relation.

Lemma 3.4.6. *If a and b are isotopy classes of simple closed curves with $i(a,b) = 1$, then*

$$T_aT_bT_a = T_bT_aT_b$$

Proof. The relation

$$T_aT_bT_a = T_bT_aT_b$$

is equivalent to the relation

$$(T_aT_b)T_a(T_aT_b)^{-1} = T_b.$$

Applying basic property of Dehn twists, one can find that this is equivalent to the relation

$$T_aT_b(a) = b.$$

The computation is shown in Figure 3.4 below. □

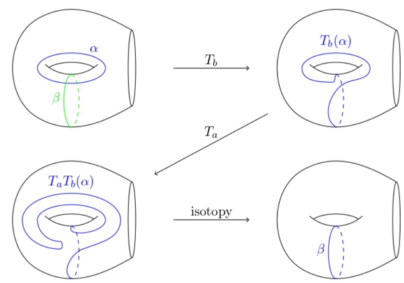


Figure 3.4: The relation $T_aT_b(a) = b$

As mentioned before the Dehn twist are useful to compute mapping class group in particular cases.

Let S be a closed subsurface of a surface S', then there is a natural homomorphism $\eta\colon Mod(S) \longrightarrow Mod(S')$. For $f \in Mod(S)$, we represent it with $\phi \in Homeo^+(S, \partial S)$. Let $\hat{\phi}$ be the element of $Homeo^+(S', \partial S')$ that agrees with ϕ on S and is the identity outside of S, we define $\eta(f) = [\hat{\phi}]$. The map η is obviously well-defined since an homotopy between two elements of f in $Homeo^+(S, \partial S)$ gives a homotopy between the corresponding element in $Homeo^+(S', \partial S')$.

The next theorem describe the kernel of η.

Theorem 3.4.7. *Let S be a closed subsurface of a surface S' and assume S is not homeomorphic to a closed annulus and that no component of $S' \setminus S$ is an open disk. Let $\eta\colon Mod(S) \longrightarrow Mod(S')$ be the induced map. Let $\alpha_1, \ldots, \alpha_m$ denote the boundary components of S that bound once-punctured disks in $S' \setminus S$ and let $\{\beta_1, \gamma_1\}, \ldots, \{\beta_n, \gamma_n\}$ denote the pairs of boundary components of S that bound annuli in $S' \setminus S$.*
Then the kernel of η is the free abelian group

$$ker(\eta) = \langle T_{\alpha_1}, \ldots, T_{\alpha_m}, T_{\beta_1}T_{\gamma_1}^{-1}, \ldots, T_{\beta_n}T_{\gamma_n}^{-1} \rangle.$$

Proof. See Theorem 3.18 of [7]. □

One particular case of Theorem 3.4.7 is the case where $S' \setminus S$ is a once punctured disk. We say that S' is the surface obtained from S by *capping* one boundary component. In this case we have the following proposition.

Proposition 3.4.8. *Let S' be the surface obtained from a surface S by capping the boundary component β with a once-marked disk, and call the marked point in this disk p_0. Denote by $Mod(S, \{p_1, \ldots, p_k\})$ the subgroup of $Mod(S)$ consisting of elements that fix the punctures $p_1, \ldots, p_k$, where $k \geq 0$. Similarly let $Mod(S', \{p_0, \ldots, p_k\})$ denote the subgroup of $Mod(S')$ consisting of elements that fix the marked points $p_0, \ldots, p_k$. Let $\mathcal{C}ap\colon Mod(S, \{p_1, \ldots, p_k\}) \longrightarrow Mod(S', \{p_0, \ldots, p_k\})$ be the induces homomorphism. Then the following sequence is exact:*

$$1 \longrightarrow \langle T_\beta \rangle \longrightarrow Mod(S, \{p_1, \ldots, p_k\}) \xrightarrow{\mathcal{C}ap} Mod(S', \{p_0, \ldots, p_k\}) \longrightarrow 1.$$

Proof. See Proposition 3.19 of [7]. □

Example 3.4.9. Let P denote a pair of pants, recall that $S_{0,3}$ is homemorphic to the interior of P. Denote the subgroup of $Mod(S_{0,3})$ consisting of the elements that fix each of the punctures with $PMod(S_{0,3}) = 1$. From that and applying Proposition 3.4.8 three times we obtain the isomorphism $Mod(P) \simeq \mathbb{Z}^3$.

3.5 Generator for the mapping class group

The aim of this section is to prove that for every $g \geq 0$ the mapping class group $Mod(S_g)$ is generated by finitely many Dehn twists. In particular, in 1964, Lickorish proved that $Mod(S_g)$ is generated by $3g-1$ Dehn twists, later, in 1979, Humphries proved that only $2g+1$ Dehn twists are necessary to generate $Mod(S_g)$.

Remark 3.5.1. *In the case S has punctures there is a problem. Indeed the Dehn twists cannot generate alone the mapping class group since every composition of Dehn twists cannot permute the punctures.*

In the case of a surface with punctures we consider

Definition 3.5.2. The subgroup of $Mod(S_{g,n})$ consisting of elements that fix each puncture individually is called *pure mapping class group* of $S_{g,n}$ and is denoted with $PMod(S_{g,n})$.

Remark 3.5.3. *Observe that, thanks to the action of $Mod(S_{g,n})$, we have the exact short sequence*

$$1 \longrightarrow PMod(S_{g,n}) \longrightarrow Mod(S_{g,n}) \longrightarrow \mathfrak{S}_n \longrightarrow 1$$

Now we want to prove that $PMod(S_{g,n})$ is generated by the set of Dehn twists about nonseparating simple closed curves, and then we will give some example of finite set of generators.

First we need to introduce the complex of curves.

Definition 3.5.4. Let S be a topological surface. The *complex of curves* $\mathcal{C}(S)$ of S is the simplicial complex whose 1-skeleton is given by the following data.

Vertices: there is one vertex of $\mathcal{C}(S)$ for each isotopy class of essential simple closed curves in S.

Edges: there is an edge between any two vertices of $\mathcal{C}(S)$ corresponding to isotopy classes a and b with $i(a,b) = 0$.

More generally, $\mathcal{C}(S)$ has a $k-$simplex for each $(k+1)-$tuple of vertices where each pair of corresponding isotopy classes has geometric intersection number zero.
We have the following result (See Theorem 4.3 of [7]).

Theorem 3.5.5. *If $3g + n \geq 5$, then $\mathcal{C}(S_{g,n})$ is connected.*

We want to consider only nonseparating simple closed curves, so we need to consider the *complex of nonseparating curves* $\mathcal{N}(S)$, which is the subcomplex of $\mathcal{C}(S)$ spanned by vertices corresponding to nonseparating simple closed curves.

Theorem 3.5.6. *If $g \geq 2$, then $\mathcal{N}(S_{g,n})$ is connected.*

Proof. First suppose that $g \geq 2$ and $n \leq 1$. If a and b are arbitrary isotopy classes of nonseparating simple closed curves, then by Theorem 3.5.5 there is a sequence of isotopy classes $a = c_1, \ldots, c_k = b$ with $i(c_i, c_{i+1}) = 0$ for all $i = 1, \ldots, k-1$. We will alter the sequence $\{c_i\}$ so that it consists of isotopy classes of nonseparating simple closed curves. Suppose c_i is separating. Let γ_i be a representative for c_i, and let S' and S'' be the two components of $S_{g,n} - \gamma_i$. By the assumption that $g \geq 2$ and $n \leq 1$, we have that both S' and S'' have positive genus. If c_{i-1} and c_{i+1} have representative that lie in different subsurfaces then $i(c_{i-1}, ci+1) = 0$ so we can remove c_i from the sequence. If c_{i-1} and c_{i+1} have representative that both lies in S', then we replace c_i with the isotopy class of nonseparating simple closed curves in S''. We repeat the above process until each c_i is nonseparating, at which point we have obtained a path between a and b in $\mathcal{N}(S)$.

To prove the thesis in the general case we use induction on n. Assume $n \geq 2$ and proceed as above. The only possible problem is that it might happen that both representative of c_{i-1} and c_{i+1} lies in S' and S'' has genus 0. In this case we have that S' has genus $g \geq 2$, and has fewer punctures then the original surface, so, by induction, we have a path in $\mathcal{N}(S')$ between c_{i-1} and c_{i+1} and we replace c_i by the corresponding sequence of isotopy classes of curves in S. □

We will use a modified complex of nonseparating curves.

Definition 3.5.7. Let $\hat{\mathcal{N}}(S)$ denote the one-dimensional simplicial complex whose vertices are isotopy classes of nonseparating simple closed curves in S, and whose edges correspond to pairs of isotopy classes a, b with $i(a,b) = 1$.

Lemma 3.5.8. *If $g \geq 2$ and $n \geq 0$, then the complex $\hat{\mathcal{N}}(S_{g,n})$ is connected*

Proof. Let a and b be two isotopy classes of simple closed curves in $S_{g,n}$. By Theorem 3.5.6, there is a sequence of isotopy classes $a = c_1, \ldots, b = c_k$ representing vertices of $\hat{\mathcal{N}}(S_{g,n})$ with $i(c_i, c_{i+1}) = 0$. By the change of coordinate principles (see Section 1.3 of [7]), for each c_i, with $i = 1, \ldots, k-1$, one can find an isotopy class d_i of nonseparating simple closed curves with $i(c_i, d_i) = i(d_i, c_{i+1}) = 1$. The sequence $a = c_1, d_1, a_2, \ldots, c_{k-1}d_{k-1}, c_k = b$ represents a path in $\hat{\mathcal{N}}(S_{g,n})$ from a to b. □

We still need another important tool, the *Birman exact sequence*.

Let S be a surface, possibly with puncture but no marked points. Let S^* be the surface obtained from S by marking a point x in the interior of S. There is a natural homomorphism $Forget\colon Mod(S^*) \longrightarrow Mod(S)$ called the *forgetful map*. This map is realized by forgetting that the point x is marked.

Remark 3.5.9. *Note that the forgetful map is surjective. Indeed, any homeomorphism of S can be modified by isotopy so that it fixes x. Than the group $Mod(S, x)$ is isomorphic to the subgroup G of $Mod(S \setminus x)$ preserving the puncture coming from x. The forgetful map can be interpreted as the map $G \longrightarrow Mod(S)$ obtained by filling in the puncture x.*

We would like to study the kernel of $Forget$. Let $f \in Mod(S^*)$ an element of $ker(Forget)$, and let ϕ be a representative of f. We can think of ϕ as a homeomorphism $\overline{\phi}$ of S. Since $Forget(f) = 1$, there is an isotopy from $\overline{\phi}$ to id_S. During this isotopy the image of the point x traces a loop α in S based at x. The idea is that we push x along α dragging the rest of the surface along, as one can see from Figure 3.5.

To make the idea of pushing more precise, let α be a loop in S based at x. We can think of $\alpha\colon [0,1] \longrightarrow S$ as an isotopy of points from x to itself. This isotopy can be extended to an isotopy of all S. Let ϕ_α be the homeomorphism of S obtained at the end of the isotopy. By marking/removing the point x, regarding ϕ_α as a homeomorphism of S^*, and then taking is isotopy class, we obtain a mapping class $Push(\alpha) \in Mod(S^*)$.

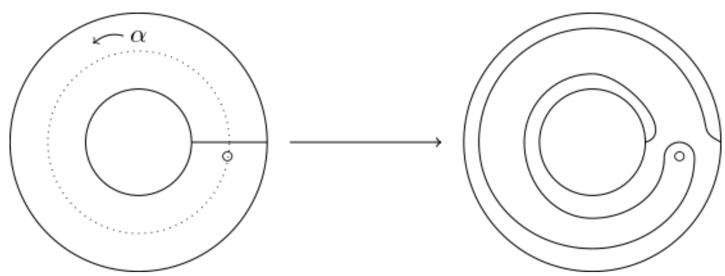


Figure 3.5: Action of the map $Push$

Remark 3.5.10. *Immediately from the definitions we have that for any $h \in PMod(S^*)$ and any $\alpha \in \pi_1(S, x)$, we have*

$$Push(h_*(\alpha)) = hPush(\alpha)h^{-1}.$$

We would like to have that the point-pushing map is well defined $Push\colon \pi_1(S, x) \longrightarrow Mod(S^*)$. This is indeed the case, but is not obvious at all. The Birman exact sequence gives that the point-pushing map is well defined and its image is the kernel of the forgetful map.

Theorem 3.5.11. *Let S be a surface with $\chi(S) < 0$, possibly with punctures and boundary. Let S^* be the surface obtained from S by marking a point x in the interior of S. Then the sequence*

$$1 \longrightarrow \pi_1(S, x) \xrightarrow{Push} Mod(S^*) \xrightarrow{Forget} Mod(S) \longrightarrow 1$$

is exact.

Proof. We will show that there is a fiber bundle

$$\begin{array}{c} Homeo^{+}(S) \\ \big\downarrow \mathcal{E}_x \\ S \end{array}$$

with total space $Homeo^{+}(S)$, base space S and fiber the subgroup of $Homeo^{+}(S)$ consisting of elements that fix x. The map $\mathcal{E}_x$ is the evaluation at the point x. Then the theorem will follow from the long exact sequence of homotopy groups associated to the fiber bundle.

To prove that $\mathcal{E}_x$ is a fiber bundle we need to show that $Homeo^{+}(S)$ is locally homeomorphic to a product of an open set U of S and $Homeo^{+}(S,x)$ so that the restriction of $\mathcal{E}_x$ is the projection to the first factor. Let U be a open neighbourhood of x in S that is homeomorphic to a disk. Given $u \in U$, we can choose $\phi_u \in Homeo^{+}(U)$ so that $\phi_u(x) = u$ and so that ϕ_u varies continuosly as a function of u. We have the following homeomorphism

$$\begin{aligned} U \times Homeo^{+}(S,x) &\longrightarrow \mathcal{E}_x^{-1}(U) \\ (u,\phi) &\longmapsto \phi_u \circ \psi, \end{aligned}$$

and the inverse map is given by $\psi \longmapsto (\psi(x), \phi^{-1}_{\psi(x)} \circ \psi)$. For any other $y \in S$, we can choose a homeomorphism ξ of S taking x to y. Then there is a homeomorphism

$$\begin{aligned} \mathcal{E}_x^{-1}(U) &\longrightarrow \mathcal{E}_x^{-1}(\xi(U)) \\ \psi &\longmapsto \xi \circ \psi, \end{aligned}$$

and so we have verified that $\mathcal{E}_x$ is a fiber bundle.

As observed before the theorem now follows from the long exact sequence of homotopy groups associated to the fiber bundle $\mathcal{E}_x$. In particular the relevant part of the sequence is the following

$$\ldots \longrightarrow \pi_1(Homeo^{+}(S)) \longrightarrow \pi_1(S) \longrightarrow \pi_0(Homeo^{+}(S,x)) \longrightarrow \pi_0(Homeo^{+}(S)) \longrightarrow \pi_0(S) \longrightarrow \ldots$$

One can prove that $\pi_1(Homeo^{+}(S))$ is trivial, and $\pi_0(S)$ is trivial as well. The remaining terms are isomorphic to the terms of the Birman exact sequence and the maps given by the long exact sequence are $Push$ and $Forget$. ☐

Remark 3.5.12. *We can consider the restriction of the Birman exact sequence to any subgroup of $Mod(S,x)$. The most used is $PMod(S,x)$, in this case $Mod(S)$ should be replaced with $PMod(S)$. We can rephrase the Birman exact sequence as follows:*

$$1 \longrightarrow \pi_1(S_{g,n}) \longrightarrow PMod(S_{g,n+1}) \longrightarrow PMod(S_{g,n}) \longrightarrow 1.$$

Now we have all the tools we will need to prove the finite generation of the mapping class group. We will use that $Mod(S)$ acts on $\hat{\mathcal{N}}(S)$, since homeomorphisms take nonseparating simple closed curves to nonseparating simple closed curves and preserve intersection numbers. We need the following results from geometric group theory.

Lemma 3.5.13. *Let G be a group that acts by simplicial automorphisms on a connected, 1-dimensional simplicial complex X. Suppose that G acts transitively on the vertices of X and on pairs of vertices of X that are connected by an edge. Let v, w be two vertices of X that are connected by an edge, and choose $h \in G$ such that $h(w) = v$. Then the group G is generated by the element h together with the stabilizer of v in G.*

Proof. Let $g \in G$. We would like to show that $g \in H$, where H is the subgroup of G generated by the stabilizer of v and h. Since X is connected, there is a sequence of vertices $v = v_0, \ldots, v_k = g(v)$, where adjacent vertices are connected by an edge. Since G acts transitively on the vertices of X, we can choose

elements $g_i \in G$ such that $g_i(v) = v_i$. We take g_0 to be the identity and g_k to be g. We will prove by induction that $g_i \in H$. Obviously $g_0 \in H$. Now assume that $g_i \in H$. We must prove that $g_{i+1} \in H$.
Applying the element g_i^{-1} to the edge between $v_i = g_i(v)$ and $v_{i+1} = g_{i+1}(v)$, we obtain the edge between v and $g_i^{-1}g_{i+1}(v)$. Since G acts transitively on ordered pairs of vertices of X that are connected by an edge, there is an element $r \in G$ that takes the pair $(v, g_i^{-1}g_{i+1}(v))$ to the pair (v, w). In particular r lies in the stabilizer of v and $rg_i^{-1}g_{i+1}(v) = w$. So we have $hrg_i^{-1}g_{i+1}(v) = v$, wich means that $hrg_i^{-1}g_{i+1}$ lies in the stabilizer of v. In particular $hrg_i^{-1}g_{i+1} \in H$. Since h and r lies in H and g_i lies in H by induction, we have that g_{i+1} lies in H. In particular $g_k = g \in H$. We conclude by the arbitrariness of g. □

We can finally prove that $PMod(S_{g,n})$ is finitely generated.

Theorem 3.5.14. *Let $S_{g,n}$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. Then the group $PMod(S_{g,n})$ is finitely generated by Dehn twists about nonseparating simple closed curves in $S_{g,n}$.*

Proof. We will use double induction on genus and the number of punctures of S, with base cases $\mathbb{T} = S_{1,0}$ and $S_{1,1}$.

First we start with the inductive step on the numbers of punctures n. Let $g \geq 1$ and let $n \geq 0$. Asumming that $PMod(S_{g,n})$ is generated by finitely many Dehn twists about nonseparating simple closed curves $\{\alpha_i\}$ in $S_{g,n}$, we will show that $PMod(S_{g,n+1})$ is generated by finitely many Dehn twists about nonseparating simple closed curves in $S_{g,n+1}$. We can assume $(g, n) \neq (1, 0)$ since we know, from Example 3.4.4, that $Mod(\mathbb{T})$ is generated by Dehn twists about nonseparating simple closed curves.
We have the Birman exact sequence

$$1 \longrightarrow \pi_1(S_{g,n}) \longrightarrow PMod(S_{g,n+1}) \longrightarrow PMod(S_{g,n}) \longrightarrow 1.$$

Since $g \geq 1$, we have that $\pi_1(S_{g,n})$ is generated by the classes of finitely many simple nonseparating loops. The image of this loops is a product of two Dehn twists about nonseparating simple closed curves. We begin building a generating set for $PMod(S_{g,n+1})$ by taking each one of this Dehn twists individually. To complete this generating set it remains to choose a lift to $PMod(S_{g,n+1})$ of each Dehn twist generator T_{α_i} of $PMod(S_{g,n})$. But given the nonseparating simple closed curve α_i in $S_{g,n}$ there exists a nonseparating closed curve in $S_{g,n+1}$ that maps to α_i under the forgetful map $S_{g,n+1} \longrightarrow S_{g,n}$. Thus the Dehn twist T_{α_i} in $PMod(S_{g,n})$ has a preimage in $PMod(S_{g,n+1})$ that is a Dehn twist about a nonseparting simple closeed curve in $S_{g,n+1}$. It follows that $PMod(S_{1,n})$ is generated by finitely many Dehn twists about nonseparating simple closed curves for any $n \geq 0$.

We now consider the inductive step on the genus g. Let $g \geq 2$ and suppose that $PMod(S_{g-1,n})$ is generated by finitely many Dehn twists about nonseparating simple closed curves for any $n \geq 0$. Since $\hat{\mathcal{N}}(S_g)$ is connected and $Mod(S_g)$ acts transitevely on ordered pairs of isotopy classes of simple closed curves with geometric intersection number 1, we may apply Lemma 3.5.13 to the case of the action of $Mod(S_g)$ on $\hat{\mathcal{N}}(S_g)$.
Let a be an arbitary isotopy class of nonseparating simple closed curves in S_g, and let b be a isotopy class with $i(a, b) = 1$. Let $Mod(S_g, a)$ denote the stabilizer of a in $Mod(S_g)$. By Lemma 3.4.6, we have that $T_bT_a(b) = a$. Then, by Lemma 3.5.13, $Mod(S_g)$ is generated by $Mod(S_g, a)$ together with T_a and T_b. Thus, it suffices to prove that $Mod(S_g, a)$ is finitely generated by Dehn twists about nonseparating simple closed curves.

Let $Mod(S_g, \vec{a})$ be the subgroup of $Mod(S_g, a)$ consisting of elements that preserve the orientation of a. We have the short exact sequence

$$1 \longrightarrow Mod(S_g, \vec{a}) \longrightarrow Mod(S_g, a) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Since $T_bT_a^2T_b$ switches the orientation of a (it can be proved using the change of coordinates principle), it represents the nontrivial coset of $Mod(S_g, \vec{a})$ in $Mod(S_g, a)$. Thus it remains to show that $Mod(S_g, \vec{a})$ is finitely generated by Dehn twists about nonseparating simple closed curves in S_g.

By Proposition 3.20 of [7], there is a short exact sequence

$$1 \longrightarrow \langle T_a \rangle \longrightarrow Mod(S_g, \vec{a}) \longrightarrow PMod(S_g \setminus \alpha) \longrightarrow 1,$$

where $S_g \setminus \alpha$ is the surface obtained from S_g by deleting a representative α of a. The surface $S_g \setminus \alpha$ is homeomorphic to $S_{g-1,2}$. By our inductive hypothesis, $PMod(S_g \setminus \alpha)$ is finitely generated by Dehn twists about nonseparating simple closed curves. To conclude observe that each such Dehn twist has a preimage in $Mod(S_g, \vec{a})$ that is also a Dehn twist about a nonseparating simple closed curve, therefore it follows that $Mod(S_g, \vec{a})$ is finitely generated by Dehn twists about nonseparating simple closed curves, and we are done. □

We would show some explicit examples of set of generators for $Mod(S_g)$.

Example 3.5.15. The Lickorish generators: The Dehn twists about the $3g-1$ simple closed curves indicated in Figure 3.6 generate $Mod(S_g)$. This generating set was found by Lickorish, so we call these Dehn twists the Lickorish generators.

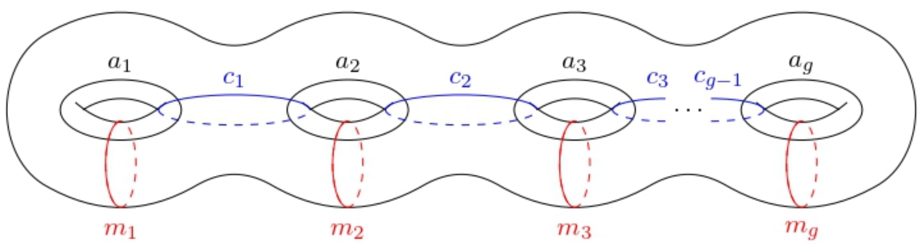


Figure 3.6: The Lickorish generating set

Theorem 3.5.16. *Let S_g be a closed surface if genus $g \geq 1$. Then the Dehn twists about the isotopy classes $a_1, \dots, a_g, m_1, \dots, m_g, c_1, \dots, c_{g-1}$ shown in Figure 3.6 generate $Mod(S_g)$.*

Proof. We will refer to the Dehn twists of the statement as Lickorish twists.

We proceed by induction on g. Since the Lickorish twists in the case of a torus $\mathbb{T}$ are the standard generators for $Mod(\mathbb{T})$, the theorem is true for $g = 1$.
Assume that $g \geq 2$. Analogously as in the proof of Theorem 3.5.14, we can apply Lemma 3.5.13 to the action of $Mod(S_g)$ on $\hat{\mathcal{N}}(S_g)$, and, by Lemma 3.4.6, we have $T_{a_1} T_{m_1} T_{a_1}(m_1) = a_1$., thus it suffices to show that $Mod(S_g, m_1)$, the stabilizer of m_1, lies in the groupgenerated by the Lickorish twists.

Again, let $Mod(S_g, \vec{m_1})$ denote the subgroup of $Mod(S_g, m_1)$ of the elements that preserve the orientation of m_1, then we have the short exact sequence

$$1 \longrightarrow Mod(S_g, \vec{m_1}) \longrightarrow Mod(S_g, m_1) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

Since the product $T_{a_1} T_{m_1}^2 T_{a_1}$ reverses the orientation of m_1, it suffices to show that $Mod(S_g, \vec{m_1})$ lies in the group generated by Lickorish twists. One more time, we have the exact sequence

$$1 \longrightarrow \langle T_{m_1} \rangle \longrightarrow Mod(S_g, \vec{m_1}) \longrightarrow PMod(S_{m_1}) \longrightarrow 1$$

where $S_{m_1} \simeq S_{g-1,2}$ is the surface obtained by deleting a representative of m_1 from S_g. Since T_{m_1} is a Lickorish twist, it suffices to show that $PMod(S_{m_1})$ is generated by the images of the Lickorish twists.

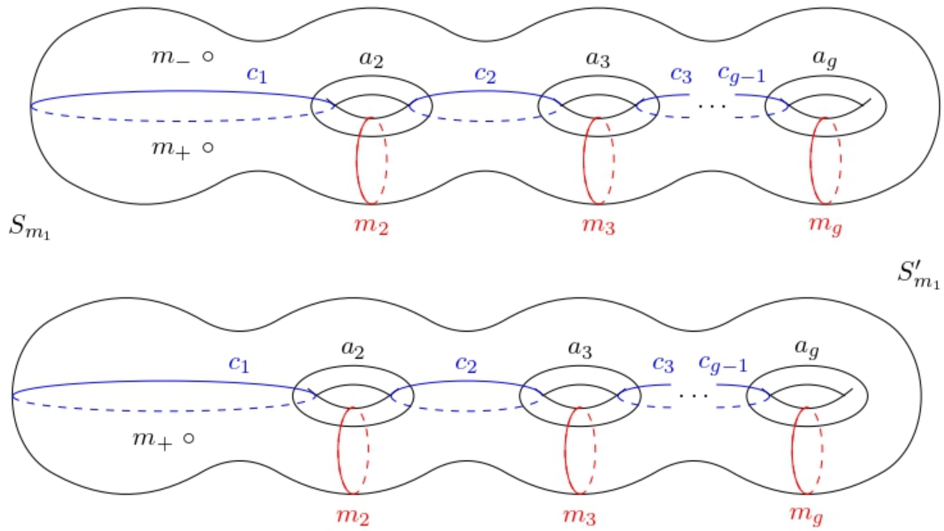


Figure 3.7: The images of the curves from Figure 3.6 in S_{m_1} and S'_{m_1}

We apply the Birman exaxt sequence twice. Let S'_{m_1} be the surface obtained by S_{m_1} by forgetting the firts puncture m_-, and let $S"_{m_1}$ be the surface obtained from S'_{m_1} by forgetting the second puncture m_+. We then have the following maps of exact sequences, where each square commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S'_{m_1}, m_-) & \xrightarrow{Push} & PMod(S_{m_1}) & \longrightarrow & Mod(S'_{m_1}) & \longrightarrow & 1 \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ 1 & \longrightarrow & \pi_1(S_{g-1,1}) & \longrightarrow & PMod(S_{g-1,2}) & \longrightarrow & Mod(S_{g-1,1}) & \longrightarrow & 1, \end{array} \tag{3.1}$$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S"_{m_1}, m_+) & \xrightarrow{Push'} & Mod(S'_{m_1}) & \longrightarrow & Mod(S"_{m_1}) & \longrightarrow & 1 \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ 1 & \longrightarrow & \pi_1(S_{g-1}) & \longrightarrow & Mod(S_{g-1,1}) & \longrightarrow & Mod(S_{g-1}) & \longrightarrow & 1 \end{array} \tag{3.2}$$

We start studying sequence 3.2. The goal is to show that $Mod(S'_{m_1})$ is generated by the images of the Lickorish twists in $Mod(S'_{m_1})$; i.e. we want to show that $Mod(S'_{m_1})$ is generated by the Dehn twists about the simple closed curves shown on the bottom of Figure 3.7. By induction, $Mod(S"_{m_1}) \simeq Mod(S_{g-1})$ is generated by the Dehn twist about the images of these curves in $S"_{m_1}$. So by the exact sequence 3.2, it suffices to show that that each element of $Push'(\pi_1(S"_{m_1})$ is a product of the Dehn twists given in the bottom of Figure 3.7.

Standard generators for $\pi_1(S"_{m_1})$ are shown in Figure 3.8 below. The mapping class $Push'(\alpha_1)$ is equal to the product $T_{c_1}T_{m_2}^{-1}$, so this element is a product of Lickorish twists. Using Lemma 3.4.6 we see that $T_{m_2}T_{a_2}(\alpha_1) = \beta_1$. Thus, by Remark 3.5.10, $Push'(\beta_1)$ is conjugate to $Push'(\alpha_1)$ by a product of Lickorish twists, hence itself is a product of Lickorish twists.

Repeating the conjugation trick, we see that every generator for $\pi_1(S"_{m_1})$ under $Push'$ is a product of the images of Lickorish twists in $Mod(S'_{m_1})$. The required formulas are:

$$(T_{c_i}^{-1}T_{a_{i+1}}^{-1})(T_{a_i}^{-1}T^{-1}c_i)(\beta_{i-1}) = \beta_i$$
$$T_{a_{i+1}}^{-1}T^{-1}m_{i+1}(\beta_i) = \alpha_i.$$

Turning to sequence 3.1, it remains to prove that $Push(\pi_1(S'_{m_1}, m_-))$ lies in the group generated by the Dehn twists about the simple closed curves shown on top of Figure 3.7. The proof is essentialy the same

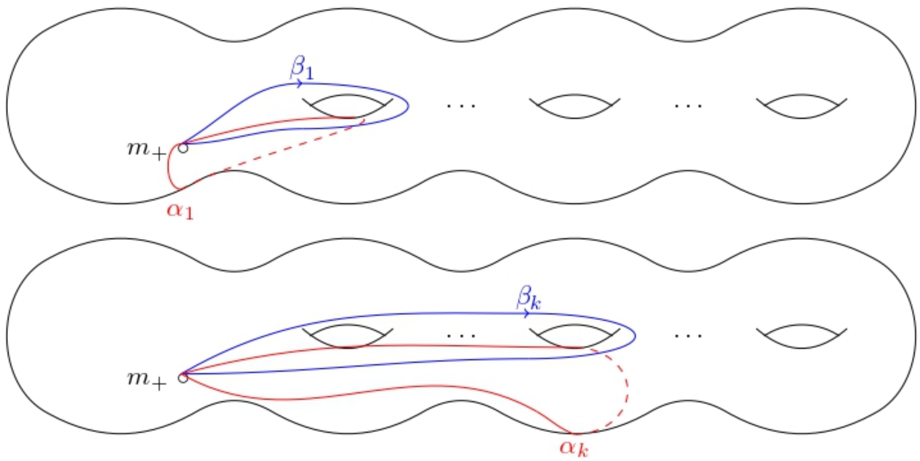


Figure 3.8: Standard generators for $\pi_1(S"_{m_1}, m_+)$

as the previous argument. To facilitate th argument, notice that each $T_{m'_i}$ is a product of Lickorish twists, where the $m'_2, \ldots, m'_{g-1}$ are the isotopy classes shown in Figure 3.9 This follows from the chain relation

$$(T_{m_g} T_{a_g} T_{c_{g-1}} T_{a_{g-1}} T_{c_{g-2}} \ldots T_{a_{k+1}} T_{c_k})^{2(g-k+1)} = T_{m_k} T_{m'_k}.$$

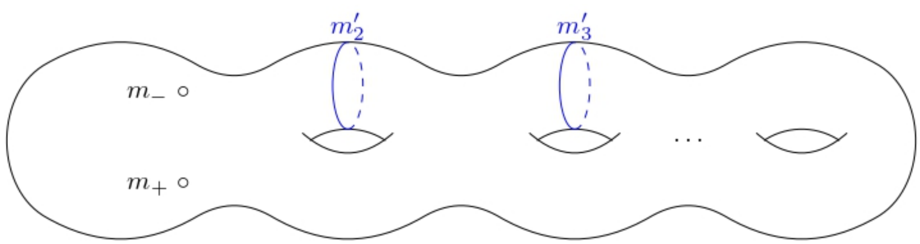


Figure 3.9: Isotopy classes of simple closed cuvers in S_{m_1}

This complete the proof. □

Example 3.5.17. The Humphries generators Another set of generators for $Mod(S_g)$, in the case $g \geq 2$ is the set of Humphries generator.

Theorem 3.5.18. *Let $g \geq 2$. Then the group $Mod(S_g)$ is generated by the Dehn twists about the $2g+1$ isotopy classes of non separating simple closed curves $a_1, \ldots, a_g, c_1, \ldots, c_{g-1}, m_1, m_2$ shown in Figure 3.6.*

Proof. By Theorem 3.5.16 it suffices to show that the Lickorish twists $T_{m_3}, \ldots, T_{m_g}$ can be written in terms of the other Lickorish twists.

For any $1 \leq i \leq g-2$ we will find a product h of Dehn twists about a_i, c_i, and m_{i+1} that takes m_i to m_{i+2}.

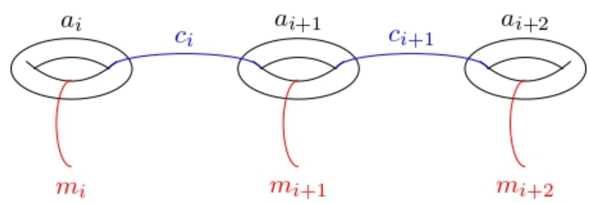


Figure 3.10: Simple closed curves used in the proof

It will then follow that $T_{m_{i+2}} = h_i T_{m_i} h_i^{-1}$ (Remark 3.4.5), and the thesis will be proved.

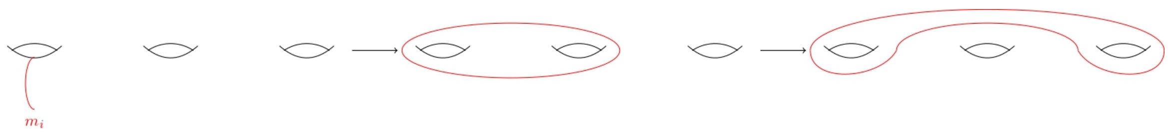


Figure 3.11: Taking m_i to m_{i+2}

The second portion of Figure 3.11 shows $T_{m_{i+1}} T_{a_{i+1}} T_{c_i} T_{a_i}(m_i)$, the last one shows

$$T_{c_{i+1}} T_{a_{i+1}} T_{a_{i+2}} T_{c_{i+1}} T_{m_{i+1}} T_{a_{i+1}} T_{c_i} T_{a_i}(m_i) = d.$$

Note that the last curve is symmetric with respect to the ith and $(i+2)$nd holes. It follows that we can do a similar product of Dehn twists h' in order to take d to m_{i+2}.
Since $h = T_{c_{i+1}} T_{a_{i+1}} T_{a_{i+2}} T_{c_{i+1}} T_{m_{i+1}} T_{a_{i+1}} T_{c_i} T_{a_i}$ used m_{i+1} and no other m_j, it follows that h' will use m_{i+1} and no other m_j. This complete the proof $\square$

Remark 3.5.19. *One can prove that any set of Dehn twist generators for $Mod(S_g)$ must have at least $2g+1$ elements(See section 6.3 of [7]).*
In this sense the set of the Humphries generators is minimal.

Chapter 4

The Moduli Space of Curves

In this chapter we will introduce the moduli space of curves of genus $g \geq 1$. In particular we will define it as the quotient of the Teichmüller space $Teich(\mathbb{S})$ and the mapping class group $Mod(\mathbb{S})$. We will then prove some of its topological properties. To conclude we will see that the moduli space of curves is not compact but we have a description of what it means to "go to infinity", thanks to Mumford's compactness criterion. As we have done in the case of the Teichmüller space of a differentiable surface we will first analyze the case $g = 1$. In this case we will be able to describe the moduli space of a torus providing a description of a fundamental domain for it. Later on we will extend our idea to the case $g \geq 2$. The principal result of this chapter is the fact that the mapping class group acts properly discontinuously on the Teichmüller space, as a corollary, we have that the moduli space of curves is an orbifold for $g \geq 1$ and we will be able to deduce some topological properties from this fact.

4.1 The case $g = 1$

As already observed we have that for the case of genus $g = 1$ we will consider a torus and we will give a description of its moduli space of curves. The moduli space $\mathcal{M}(\mathbb{T})$ of flat, unit area metrics on the torus $\mathbb{T}$ is known as the *modular surface*. It is an important object in mathematics, one reason being that is the moduli space of elliptic curves.
The study of the moduli space $\mathcal{M}(\mathbb{T})$ is a good example of a computable moduli space of curves, since we know how to describe the Teichmüller space of a torus and it6s mapping class group and the action of the latest is well-known.
Recall, from Chapter 2, Section 2.1, that $Teich(\mathbb{T})$ can be identified with the hyperbolic plane $\mathbb{H}^2$. The action of $Mod(\mathbb{T}) \simeq \mathrm{SL}(2,\mathbb{Z})$ on $Teich(\mathbb{T}) \simeq \mathbb{H}^2$ is the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$ by Möbius transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto f(z) = \frac{az-b}{-cz+d}.$$

Proposition 4.1.1. *Let $\sigma\colon Mod(\mathbb{T}) \longrightarrow \mathrm{SL}(2,\mathbb{Z})$ be the isomorphism of Theorem 3.2.11, and let $\eta\colon Teich(\mathbb{T}) \longrightarrow \mathbb{H}^2$ be the identification from Proposition 2.1.8.*
For any $\mathcal{X} \in Teich(\mathbb{T})$ and any $f \in Mod(\mathbb{T})$, we have

$$\eta(f \circ \mathcal{X}) = \sigma(f) \circ \eta(X).$$

Remark 4.1.2. *In other words, Proposition 4.1.1 states that η semiconjugates the action of $f \in Mod(\mathbb{T})$ on $Teich(\mathbb{T})$ to the action of $\sigma(f) \in \mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$.*

Proof. It is enough to check the statement on a set of generators of $Mod(\mathbb{T})$, say

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let α and β be based loops in $\mathbb{T}$ representing generators for $\pi_1(\mathbb{T})$ with $\hat{i}(\alpha,\beta) = 1$, this makes sense if we identify α and β with their images in $H_1(\mathbb{T},\mathbb{Z})$. The isomorphism σ idenifies M with the mapping class T_α^{-1}, thinking of α as an unoriented simple closed curve; it also identifies N with the order 4 mapping class $(T_\alpha T_\beta T_\alpha)^{-1}$, which can be described by cutting $\mathbb{T}$ along α and β, rotating the square by $\frac{\pi}{2}$, and regluing.

Given a point $[(X,\phi)] \in Teich(\mathbb{T})$, we can represent it by a unique marked lattice in $\mathbb{C}$, as in the proof of Proposition 2.1.8, with basis vector 1 corresponding to the oriented curve α and basis vector $\tau \in \mathbb{C}$ in the upper half-plane corresponding to β. We know that

$$T_\alpha^{-1} \circ [(X,\phi)] = [(X, \phi \circ T_\alpha)],$$

where we appropriately regard T_α as either a mapping class or a homeomorphism.
The formula $T_{\phi(\alpha)} = \phi \circ T_\alpha \circ \phi^{-1}$ gives that

$$\begin{aligned}(\phi \circ T_\alpha)(\beta) &\sim T_{\phi(\alpha)}(\phi(\beta))\\ (\phi \circ T_\alpha)(\alpha) &\sim \phi(\alpha),\end{aligned}$$

where $\sim$ denote the isotopy relation. In other words, the effect of T_α^{-1} on the marked lattice is to keep 1 fixed and send τ to $\tau - 1$. But this means that T_α^{-1} acts on $\mathbb{H}^2$ by the Möbius transformation $z \longmapsto z-1$, as we wanted to show.

By similar reasoning, the mapping class associated to N acts on the marked lattice $(1,\tau)$ by sending it to the marked lattice $(-\tau, 1)$. To get the induced action on $\mathbb{H}^2$ we need to put the latter into "standard form" (rotate/flip so the first complex number is 1). If we write $\tau = re^{i\theta}$, then the resulting lattice corresponds to

$$\frac{1}{r}e^{i(\pi-\theta)} = -\frac{1}{r}e^{-i\theta} = -\frac{1}{\tau},$$

which is what we wanted to show. $\square$

Definition 4.1.3. The moduli space of curves of a torus is the space

$$\mathcal{M}(\mathbb{T}) = Teich(\mathbb{T})/Mod(\mathbb{T}) \simeq \mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z}),$$

where the action is given by Proposition 4.1.1.

The kernel of the action on $\mathbb{H}^2$ is $\{\pm I\} = Z(\mathrm{SL}(2,\mathbb{Z}))$, and so $\mathcal{M}(\mathbb{T})$ can also be written as $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$. Note that the action of $Mod(\mathbb{T})$ on $Teich(\mathbb{T})$ is properly discontinuous so $\mathcal{M}(\mathbb{T})$ is an orbifold.

We have given the description of the moduli space of a torus, but we are also able to describe one of its fundamental domains. In particular $\mathcal{M}(\mathbb{T})$ admits $\mathcal{D} = \{\tau \in \mathbb{H}^2 : |Re(\tau)| \leq \frac{1}{2}, |\tau| \geq 1\}$ as a fundamental domain.

Lemma 4.1.4. *The map $\pi\colon \mathcal{D} \longrightarrow \mathcal{M}(\mathbb{T})$ surjects, where π is the natural projection $\pi(\tau) = \mathrm{SL}(2,\mathbb{Z})\tau$.*

Proof. Given $\tau \in \mathbb{H}^2$ it suffices to show that τ is $\mathrm{SL}(2,\mathbb{Z})$-equivalent to some point in $\mathcal{D}$. Repeatedly apply one of $\begin{pmatrix}1 & \pm1\\ 0 & 1\end{pmatrix} : \tau \longmapsto \tau \pm 1$ to translate τ into the vertical strip $\{|Re(\tau)| \leq \frac{1}{2}\}$, and replace τ by this transform. Now if $\tau \notin \mathcal{D}$ then $|\tau| < 1$ and so $Im\left(-\frac{1}{\tau}\right) = Im\left(-\frac{\overline{\tau}}{|\tau|^2}\right) = Im\left(\frac{\tau}{|\tau|^2}\right) > Im(\tau)$; so it suffices to replace τ by $N \circ \tau = -\frac{1}{\tau}$ and repeat the process. Since there are only finitely many integer pairs (c,d) such that $|c\tau + d| < 1$ (because there are only finitely many lattice points inside a disk), the formula

$$Im(\gamma \circ \tau) = \frac{Im(\tau)}{|c\tau + d|^2}, \text{ for } \gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

shows that only finitely many transforms of τ have larger imaginary part. Therefore the algorithm terminates with some $\tau \in \mathcal{D}$. $\square$

One can prove that there are no other identification of interior points, therefore $\mathcal{D}$ is a fundamental domain for $\mathcal{M}(\mathbb{T})$. In particular

Lemma 4.1.5. *Suppose that z_1, z_2 are distinct point in $\mathcal{D}$, and that $z_2 = \gamma z_1$ for some $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Then either*

i. $Re(z_1) = \pm\frac{1}{2}$ *and* $z_2 = z_1 \mp 1$ *or*

ii. $|z_1| = 1$ *and* $z_2 = -\frac{1}{z_1}$.

Proof. We will just give an idea of the proof. We can assume $Im(z_2) \geq Im(z_1)$ by simmetry. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $Im(z_2) \geq Im(z_1)$, we have $|cz_1 + d|^2 \leq 1$, moreover $Im(z_1) \geq \frac{\sqrt{3}}{2}$, since $z_1 \in \mathcal{D}$. Then we have

$$|c|\,\frac{\sqrt{3}}{2} \leq |c|\,Im(z_1) = |Im(cz_1 + d)| \leq |cz_1 + d| \leq 1,$$

since $c \in \mathbb{Z}$, this show $|c| \in \{0,1\}$. If $c = 0$ we have $\gamma = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and $Re(z_2) = Re(z_1) + b$, forcing $|b| = 1$ and $i.$ holds.

If $|c| = 1$ then we obtain the condition $|z_1 + d|^2 \leq 1$, or $(Re(z_1) \pm d)^2 + Im(z_1)^2 \leq 1$, which implies $(Re(z_1) \pm d)^2 \leq \frac{1}{4}$, so $|Re(z_1) \pm d| \leq 1/2$, forcing $|d| \leq 1$.

If also $|d| = 1$ then all inequalities above must be equalities. It follows that $Im(z_1) = \frac{\sqrt{3}}{2}$ and $|Re(z_1) \pm 1| = \frac{1}{2}$, so $Re(z_1) = \pm\frac{1}{2}$ and both $i.$ and $ii.$ hold.

If $d = 0$ then we have $|z_1| \leq 1$, so $|z_1| = 1$, since $z_1 \in \mathcal{D}$, and $Im(z_1) = Im(z_2)$. Therefore also $|z_2| = 1$ by simmetry since z_1 and z_2 have the same conditions on their imaginary parts and on $c-$entries of the matrices transforming each one to the other. Thus z_1 and z_2 have the same absolute value and the same imaginary part but are distinct, forcing their real parts to be opposite and $ii.$ holds. ☐

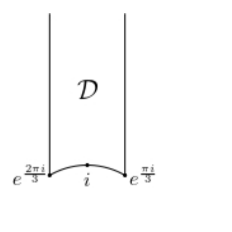


Figure 4.1: The fundamental domain $\mathcal{D}$ for $\mathcal{M}(\mathbb{T})$.

The stabilizer in $Mod(\mathbb{T})$ of a point $\mathcal{X} = [(X,\phi)] \in Teich(\mathbb{T})$ corresponds precisely to the isotopy classes of isometries of X. This can be identified with a finite subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Recall from Example 3.2.10 that, up to powers, there are only two conjugacy classes of finite order elements of $\mathrm{SL}(2,\mathbb{Z})$. The first is that of the matrix N, which fixes the point i and rotates by an angle of π, thus identifying the two halves of the circular boundary of $\mathcal{D}$. This fixed point corresponds to the isometry of the square torus obtained by rotating the square by an angle of $\frac{\pi}{2}$. The second conjugacy class is that of the matrix

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

whose class in $\mathrm{PSL}(2,\mathbb{Z})$ has order 3 and whose unique fixed point in $\mathbb{H}^2$ is the point $e^{\frac{\pi i}{3}}$. This fixed point correspond to the order 3 symmetry of the hexagonal torus.

Since $\mathrm{SL}(2,\mathbb{Z})$ identifies the sides of its fundamental domain, topologically, $\mathcal{M}(\mathbb{T})$ is a punctured sphere. Taking into accounts the fixed points, we see that $\mathcal{M}(\mathbb{T})$ has the structure of an orbifold with signature

$(0; 2, 3, \infty)$, where ∞ signifies the puncture. That is, we can see $\mathcal{M}(\mathbb{T})$ as a punctured sphere with cone points of order 2 and 3.

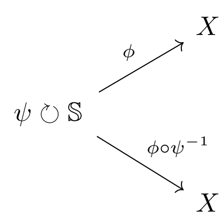
Now that we have given an explicit description of $\mathcal{M}(\mathbb{T})$, we can see that $\mathcal{M}(\mathbb{T})$ is not compact. Indeed the ray $ti \in \mathbb{H}^2 \simeq Teich(\mathbb{T})$, $t \geq 1$, projects to a ray X_t in $\mathcal{M}(\mathbb{T})$ that leaves every compact set. Even more, the distance between X_0 and X_t tends to infinity as t tends to infinity, and so $\mathcal{M}(\mathbb{T})$ has infinite diameter. In particular we can think of X_t as the set of flat tori obtained from the square torus by pinching one of the simple closed curves to ever-smaller lengths.

4.2 Definition of the Moduli Space of Curves

Now that we have studied the case $g = 1$ we would like to extend the theory to the case $g \geq 2$. Again in this section we will suppose that $\mathbb{S}$ is a surface of genus $g \geq 2$.
First of all we need to describe the action of $Mod(\mathbb{S})$ on $Teich(\mathbb{S})$. Let $\mathcal{X}$ be a point in $Teich(\mathbb{S})$. Recall that $\mathcal{X} = [(X, \phi)]$, where X is a hyperbolic surface and $\phi\colon \mathbb{S} \longrightarrow X$ a diffeomorphism. An element $f \in Mod(\mathbb{S})$ acts on $Teich(\mathbb{S})$ as follows. Let $\psi \in Diff^+(\mathbb{S})$ be a representative of f and set

$$f \cdot \mathcal{X} = [(X, \phi \circ \psi^{-1})].$$

The following diagram encode the formula given.



Remark 4.2.1. *The element $[(X, \phi \circ \psi^{-1})]$ is well defined since homotopic markings determine equivalent points of $Teich(\mathbb{S})$.*
Moreover we use ψ^{-1} in order to have a well defined group action.

Note that the action of $Mod(\mathbb{S})$ is by diffeomorphisms.

The orbit of a point $\mathcal{X} = [(X, \phi)] \in Teich(\mathbb{S})$ is the set of points $[(X, \psi)]$, where the marking ψ ranges over all homotopy classes of diffeomorphisms $\mathbb{S} \longrightarrow X$.

Remark 4.2.2. *Note that thinking about $Teich(\mathbb{S})$ as the space of marked hyperbolic surfaces homeomorphic to $\mathbb{S}$, the group $Mod(\mathbb{S})$ acts on $Teich(\mathbb{S})$ simply by changing the markings.*

In order to define the moduli space of curve of genus g as the quotient of the Teichmüller space of $\mathbb{S}$ by the action of the mapping class group of $\mathbb{S}$, we need to prove that the action of $Mod(\mathbb{S})$ is properly discontinuous. In particular the central result of this section will be the following theorem due to Fricke.

Theorem 4.2.3. ***(Fricke)*** *Let $g \geq 1$. The action of $Mod(\mathbb{S}_g)$ on $Teich(\mathbb{S}_g)$ is properly discontinuous.*

We will then use this result to define $\mathcal{M}(\mathbb{S})$ as an orbifold and deduce some of its properties.
Before giving the proof of Theorem 4.2.3 we need some technical results regarding the lengths of curves in a hyperbolic surface.

Definition 4.2.4. Let X be a hyperbolic surface. The *raw length spectrum* of X is the set of positive real numbers

$$rls(X) = \{\ell_X(c) \ : \ c \text{ an isotopy class of simple closed curves in } X\}.$$

Remark 4.2.5. *Note that $rls(X)$ is the set of lengths of simple closed geodesics in X, since we have observed (Proposition 2.4.1) that in every isotopy class there is a geodesic representative.*

The following result will be used to prove Theorem 4.2.3.

Lemma 4.2.6. ***Discreteness of the length spectrum:*** *Let X be a closed hyperbolic surface. Then the set $rls(X)$ is a closed , discrete subset of $\mathbb{R}$. Further, for each $L \in \mathbb{R}$ the set*

$$\{c \ : \ c \text{ an isotopy class of simple closed curves in } X \text{ with } \ell_X(c) \le L\}$$

is finite.

Proof. The hyperbolic surface X is the quotient of $\mathbb{H}^2$ by a free, properly discontinuous isometric action of $\pi_1(X)$. Let $K \subset \mathbb{H}^2$ be a fundamental domain for this action. Since X is closed K is compact. We have that every closed geodesic γ in X has a lift $\widetilde{\gamma}$ that intersect K, since K is a fundamental domain. Then there is a unique, up to sign, $\gamma_0 \in \pi_1(X)$ that acts on $\widetilde{\gamma}$ with translation length $\ell_X(\gamma)$. As a closed loop, γ_0 is frelly homotopic to γ.

Let $R > 0$ be given. Let γ in X be any closed geodesic of length at most R. As above, choose a lift $\widetilde{\gamma}$ that intersects K and let $\langle\gamma_0\rangle$ be the corresponding cyclic subgroup of $\pi_1(X)$, Any point $p \in \widetilde{\gamma} \cap K$ is moved by the hyperbolic translation γ_0 a distance $\ell_X(\gamma)$ in $\mathbb{H}^2$. Let K_R be the closed $R-$neighbourhood of the compact set K. Then K_R is a compact subset of $\mathbb{H}^2$ with the property that $\gamma_0 \cdot K_R \cap K_R \neq \emptyset$. But the action of $\pi_1(X)$ on $\mathbb{H}^2$ is properly discontinuous, then there are only finitely many such γ_0, hence only finitey many such γ. This proves the second statement. The first statement follows directly. ☐

The next lemma we will need is about $K-$quasiconformal maps and their action on the hyperbolic length of closed curves on X.

Lemma 4.2.7. *Let $\phi \colon X_1 \longrightarrow X_2$ be a $K-$quasiconformal homeomorphism between two hyperbolic surfaces X_1, X_2. For any isotopy class c of simple closed curves in X_1, the following inequalities hold:*

$$\frac{\ell_{X_1}(c)}{K} \le \ell_{X_2}(\phi(c)) \le K\ell_{X_1}(c).$$

Proof. Let γ_1 and γ_2 in $Isom^+(\mathbb{H}^2)$ be isometries of $\widetilde{X_1} \simeq \widetilde{X_2} \simeq \mathbb{H}^2$ corresponding to c and $\phi(c)$, respectively. Consider the annuli A_1 and A_2 obtained by taking the quotient of $\mathbb{H}^2$ by $\langle\gamma_1\rangle \simeq \mathbb{Z}$ and $\langle\gamma_2\rangle \simeq \mathbb{Z}$, respectively. Since the map $\pi_1(X_i) \longrightarrow Isom^+(\mathbb{H}^2)$ is well defined up to conjugacy in $\mathrm{PGL}(2,\mathbb{R})$, we can take γ_1 to be the map $z \longmapsto e^{\ell_{X_1}(c)}z$ and γ_2 to be $z \longmapsto e^{\ell_{X_2}(\phi(c))}z$.

We can put the annuli A_1 and A_2 in a standard form, indeed, for each i, we can find the unique open Euclidean annulus A_{m_i} of circumference 1 and height m_i, so that A_i is conformally equivalent to A_{m_i}. We call m_i the modulus of A_i. To find the standard form of A_1 we can choose a branch of the natural logarithm that takes $\mathbb{H}^2$ to the infinite strip of points in $\mathbb{C}$ with imaginary part in $(0, \pi)$. Under this identification the group $\langle\gamma_1\rangle$ corresponds to the infinite cyclic group of translation generated by $z \longmapsto z + \ell_{X_1}(c)$. Since the natural logarithm is a conformal map, A_1 is conformally equivalent to the annulus obtained by identifying vertical sides of a rectangle with width $\ell_{X_1}(c)$ and height π. Thus the modulus $m_1 = \frac{\pi}{\ell_{X_1}(c)}$. Likewise $m_2 = \frac{\pi}{\ell_{X_2}(\phi(c))}$.

The map ϕ lifts to a $K-$quasiconformal map $\widetilde{\phi} \colon A_1 \longrightarrow A_2$. Since $\langle\gamma_i\rangle \subset \pi_1(X_i)$, this is weaker than saying that ϕ lifts to a $K-$quasiconformal map from X_1 to X_2. One can prove that $\widetilde{\phi}$ changes the modulus by at most a multiplicative factor of K. We obtain $\frac{1}{K}m_2 \le m_1 \le Km_2$, and the lemma follows. ☐

The last result we will need is about simple closed curves in S_g.

Proposition 4.2.8. *Let $g \ge 1$. There exists a pair of simple closed curves in S_g that fills S_g*

Proof. Let $\{\alpha_1, \dots, \alpha_k\}$ a maximal collection of pairwise disjoint, non-homotopic, essential simple closed curves in S_g. When we cut S_g along the α_i we obtain a pants decomposition of S_g. We want to construct a simple closed curve β in S_g so that $i(\beta, \alpha_i) > 0$ for each $i = 1, \dots, k$. First we cut S_g along the α_i. On each component of the cut surface we connect by an arc each pair of distinct boundary components coming from the α_i. We can take this arc to be disjoint. In S_g these arcs can be pasted together in order to obtain a collection $\beta_1, \dots, \beta_k$ of pairwise disjoint simple closed curves in S_g. By the bigon criterion each β_j is in minimal position with respect to each α_i and each α_i intersect either one or two of the β_j. Suppose that β_j and $\beta_{j'}$ intersect α_i and are distinct. We can perform a twist about α_i such that β_j and $\beta_{j'}$ become a single curve. Since this process does not creat any bigons, the resulting collection $\{\beta_j\}$ is still in minimal position with each α_i. Continuing in this way we obtain a simple closed curve β that intersects each α_i and is in minimal position with respect to each α_i.

Claim: Let $M = T_{\alpha_1} \dots T_{\alpha_k}$. Then β and $M(\beta)$ fill S_g.

Indeed, let γ be an arbitrary isotopy class of simple closed curves in S_g. We would like to show that either $i(\beta, \gamma) > 0$ or $i(M(\beta), \gamma) > 0$. We have the following inequality (See Proposition 3.4 of [7])

$$\left| i(M(\beta), \gamma) - \sum_{i=1}^{k} i(\alpha_i, \beta) i(\alpha_i, \gamma) \right| \leq i(\beta, \gamma).$$

Suppose that both $i(\beta, \gamma)$ and $i(M(\beta), \gamma)$ are equal to zero. From the inequality above we have $i(\alpha_i, \gamma) = 0$ for each $i = 1, \dots, k$. This means that γ is isotopic to some α_i. But then $i(\beta, \gamma) > 0$ by construction of β, and so we have a contradiction. □

We can now prove Theorem 4.2.3: the proper discontinuity of the action of $Mod(\mathbb{S}_g)$ on $Teich(\mathbb{S}_g)$.

Theorem 4.2.9. *Let $g \geq 1$. The action of $Mod(\mathbb{S}_g)$ on $Teich(\mathbb{S}_g)$ is properly discontinuous.*

Proof. Let B be a compact subset of $Teich(\mathbb{S}_g)$. We need to prove that the set of $f \in Mod(\mathbb{S}_g)$ such that $f \cdot B \cap B \neq \emptyset$ is finite. Let $\mathcal{X} \in B$ be an arbitrary point and let D be the diameter of B.

Let c_1 and c_2 be isotopy classes of essential simple closed curves in $\mathbb{S}_g$ that fill $\mathbb{S}_g$. By the Alexander method (Section 3.3) the set of elements of $Mod(\mathbb{S}_g)$ that fixes $\{c_1, c_2\}$ is finite. Let $L = \max\{\ell_{\mathcal{X}}(c_1), \ell_{\mathcal{X}}(c_2)\}$. Let $f \in Mod(\mathbb{S}_g)$ such that $f \cdot B \cap B \neq \emptyset$. It follows that $d_{Teich}(\mathcal{X}, f \cdot \mathcal{X}) \leq 2D$. By Lemma 4.2.7, we have $\ell_{f \cdot \mathcal{X}}(c_i) \leq KL$ for $i = 1, 2$, where $K = e^{2D}$. But since $\ell_{f \cdot \mathcal{X}}(c_i) = \ell_{\mathcal{X}}(f^{-1}(c_i))$, it follows that $\ell_{\mathcal{X}}(f^{-1}(c_i)) \leq KL$. By Lemma 4.2.6 there are a finite number of isotopy classes of simple closed curves b in $\mathbb{S}_g$ such that $\ell_{\mathcal{X}}(b) \leq KL$. Therefore there are only finitely many possibilities for $f^{-1}(c_1)$ and $f^{-1}(c_2)$. But by our choice of the c_i, there are many finitely choices for f^{-1} once the isotopy classes $f^{-1}(c_i)$ are determined. Then there are finitely many possibilities for f such that $f \cdot B \cap B \neq \emptyset$. □

Remark 4.2.10. *Recall that when a group acts properly discontinuously by homeomorphisms on a manifold, the quotient is a orbifold. Moreover if the original manifold is aspherical, i.e. has contractible universal cover, then also the orbifold is aspherical.*

We can now give the definition of the moduli space of curves of genus $g \geq 2$.

Definition 4.2.11. The *moduli space of curves* $\mathcal{M}(\mathbb{S}_g)$ of given genus g is the quotient of the Teichmüller space of $\mathbb{S}_g$ by the action of the mapping class group of $\mathbb{S}_g$

$$\mathcal{M}(\mathbb{S}_g) = Teich(\mathbb{S}_g)/Mod(\mathbb{S}_g).$$

It follows from the definition and Theorem 4.2.3

Corollary 4.2.12. *For $g \geq 1$ the space $\mathcal{M}(\mathbb{S}_g)$ is an aspherical orbifold.*

Now we would like to study the stabilizer of a point of $Teich(\mathbb{S}_g)$ in $Mod(\mathbb{S}_g)$ in order to prove that the action of $Mod(\mathbb{S}_g)$ is not free, thus $\mathcal{M}(\mathbb{S}_g)$ is not a manifold.
Let $\mathcal{X} \in Teich(\mathbb{S}_g)$ and it is represented by (X, ϕ). Let us determine the stabilizer of $\mathcal{X}$ in $Mod(\mathbb{S}_g)$. Let $h \in Mod(\mathbb{S}_g)$ and say that it is represented by a diffeomorphism ψ. We have $h \cdot \mathcal{X} = \mathcal{X}$ if and only if the marked surfaces (X, ϕ) and $(X, \phi \circ \psi^{-1})$ are equivalent, which is the case if and only if $\phi \circ \psi \circ \phi^{-1} \colon X \longrightarrow X$ is isotopic to an isometry τ_h of X. Note that τ_h is well defined since no two isometries of a hyperbolic surface are isotopic. Also, τ_h is orientation-preserving since ψ is. The correspondence $h \longleftrightarrow \tau_h$ is an isomorphism between the stabilizer of a point $\mathcal{X}$ in $Mod(\mathbb{S}_g)$ and $Isom^+(X)$. In particular by Proposition 1.3.6 the stabilizer of $\mathcal{X}$ in $Mod(\mathbb{S}_g)$ is finite but, in general, not trivial.

The last thing we want to observe is that, since the action of $Mod(\mathbb{S})$ is isometric and properly discontinuous, $\mathcal{M}(\mathbb{S})$ inherits a metric from the Teichmüller metric on $Teich(\mathbb{S})$.

4.3 Compactness criterion for $\mathcal{M}(\mathbb{S}_g)$

In this section we will see that $\mathcal{M}(\mathbb{S}_g)$ is not compact for every $g \geq 2$, similarly to what we have done in the case of $g = 1$. Later on we will prove the Mumford's compactness criterion that affirms that the $\varepsilon-$thick part of $\mathcal{M}(\mathbb{S}_g)$ is compact for every $\varepsilon > 0$. This result will be true for $g \geq 1$, and it will prove that $\mathcal{M}(\mathbb{S}_g)$ is covered by a collection of compact subset.

We would like to use a similar idea as we have done in Section 4.1 to demonstrate that $\mathcal{M}(\mathbb{S}_g)$ is not compact. First we need to introduce a function on $\mathcal{M}(\mathbb{S}_g)$.

Definition 4.3.1. Let $X \in \mathcal{M}(\mathbb{S}_g)$. The *injectivity radius* of X at a point x is the largest r for which the $r-$ball in X centered at x is isometrically embedded.
The *injectivity radius* of X is the infimum of these injectitivity radii over all points of X.

Remark 4.3.2. *A related function is $\ell(X)$, the length of the shortest essential closed geodesic in X. One can see that $\ell(X)$ is twice the injectivity radius of X, and that any geodesic realizing $\ell(X)$ is simple. It follows from Lemma 4.2.6 that $\ell(X)$ is positive.*

We want to show that the diameter of $\mathcal{M}(\mathbb{S}_g)$ with respect to the Teichmüller metric is infinite. To do so we want to construct a sequence of points in $\mathcal{M}(\mathbb{S}_g)$ leaving every compact set.

Let $X \in \mathcal{M}(\mathbb{S}_g)$ and let $\mathcal{X} \in Teich(\mathbb{S}_g)$ be some lift. Let γ be a simple geodesic in $\mathbb{S}_g$ such that $\ell_{\mathcal{X}}(\gamma) = \ell(X)$. We can use γ as part of a coordinate system of curves for Fenchel-Nielsen coordinates on $Teich(\mathbb{S}_g)$. Then for $t \geq 1$ construct $\mathcal{X}_t \in Teich(\mathbb{S}_g)$ with the property that $\ell_{\mathcal{X}_t}(\gamma) = \frac{\ell(X)}{t}$. Let X_t denote the image of $\mathcal{X}_t$ in $\mathcal{M}(\mathbb{S}_g)$. We have $\ell(X_t) \leq \frac{\ell(X)}{t}$. It follows from Lemma 4.2.7 and the definition of the Teichmüller metric that the distance between X and X_t in $\mathcal{M}(\mathbb{S}_g)$ tends to infinity as t tends to infinity.

Theorem 4.3.3. *The diameter of $\mathcal{M}(\mathbb{S}_g)$ with respect to the Teichmüller metric is infinite.*

Proof. It is enough to consider X a point in $\mathcal{M}(\mathbb{S}_g)$ and X_t as above. □

It follows directly

Corollary 4.3.4. *The space $\mathcal{M}(\mathbb{S}_g)$ is not compact.*

Now that we have proved that $\mathcal{M}(\mathbb{S}_g)$ is not compact for every $g \geq 1$ we want to prove the compactness criterion of Mumford. To do so we need to define the $\varepsilon-$thick part of $\mathcal{M}(\mathbb{S}_g)$.

Definition 4.3.5. The $\varepsilon-$*thick part* of $\mathcal{M}(\mathbb{S}_g)$ is the set

$$\mathcal{M}_\varepsilon(\mathbb{S}_g) = \{X \in \mathcal{M}(\mathbb{S}_g) \ : \ \ell(X) \geq \varepsilon\}.$$

Remark 4.3.6. *Thanks to Lemma 4.2.6 we have that the length spectrum of each closed hyperbolic surface is discrete, thus* $\{\mathcal{M}_\varepsilon(\mathbb{S}_g) \ : \ \varepsilon > 0\}$ *is an exhaustion of* $\mathcal{M}(\mathbb{S}_g)$*, i.e.*

$$\mathcal{M}(\mathbb{S}_g) = \bigcup_\varepsilon \mathcal{M}_\varepsilon(\mathbb{S}_g).$$

Before state Mumford's compactness criterion we need two results that we will use in the proof. The first one is due to Mahler and will give us the proof of Mumford's compactness criterion in the case $g = 1$. The second one is due to Bern and gives a superior bound on the length of the curves of a pants decomposition of a surface S.

The Mahler's compactness criterion is a result on lattices in $\mathbb{R}^n$. We will only prove this result in the case $n = 2$, since it corresponds to the case $g = 1$.
First recall what is a lattice in $\mathbb{R}^n$. A *lattice* in $\mathbb{R}^n$ is the $\mathbb{Z}-$span of a basis for $\mathbb{R}^n$. The lattice is *marked* if it comes equipped with a basis as a $\mathbb{Z}-$module.

Definition 4.3.7. The *injectivity radius* of a lattice $\Lambda \subset \mathbb{R}^n$ is the length of the shortest nonzero vector in Λ.
The *volume* of Λ is the Riemannian volume of $\mathbb{R}^n/\Lambda$.

The group $\mathrm{SL}(n, \mathbb{R})$ acts transitively on the space of marked unit volume lattices in $\mathbb{R}^n$.

Definition 4.3.8. The *moduli space of unit volume lattices in* $\mathbb{R}^n$ is the quotient $\mathcal{L}_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ endowed with the qoutient topology from the Lie group $SL(n, \mathbb{R})$.

We can define $\mathcal{L}_n(\varepsilon)$ to be the subspace of $\mathcal{L}_n$ consisting of lattices with injectivity radius bounded below by 2ε.

Theorem 4.3.9. ***Mahler's compactness criterion:*** *Let* $n \geq 1$. *For any* $\varepsilon > 0$ *the space* $\mathcal{L}_n(\varepsilon)$ *is compact.*

Proof. We will prove only the case $n = 2$ since it corresponds to $\mathcal{M}(\mathbb{S}_g)$ in the case $g = 1$.

Let $\Lambda \simeq \mathbb{Z}^2$ be a lattice in $\mathbb{R}^2$ with injectivity radius bounded below by ε. Let v be the shortest nonzero vector in Λ, and let w be the vector with shortest nonzero distance to the real subspace spanned by v. Observe that there are no points of Λ in the interior of the parallelogram spanned by v, w, and so v, w generate Λ.

To ensure that any infinite sequence of lattices has a convergent subsequence we will show that the norms of v and w are bounded above by a function of ε, indipendent of Λ. Then we will show that w", the projection of w to $v^\perp$, is bounded below by a function of ε, this will ensure that the limiting lattice is nondegenerate.

Let w' be the projection of w to the subspace spanned by v. We have $|v| \leq |w| \leq |w'| + |w"| \leq \frac{1}{2}|v| + |w"|$ and so $|w"| \geq \frac{|v|}{2} \geq \varepsilon$. Since $|v|\,|w"| = 1$, we have $|v| = \frac{1}{|w"|} \leq \frac{1}{\varepsilon}$ and $|w"| = \frac{1}{|v|} \leq \frac{1}{2\varepsilon}$. Without loss of generality, we can choose w to be the shortest among $\{w + kv \ : \ k \in \mathbb{Z}\}$, and so we may assume $|w'| \leq \frac{|v|}{2} \leq \frac{1}{2\varepsilon}$. And the thesis follows. □

The last result we will need to prove Mumford's compactness criterion is the following theorem.

Theorem 4.3.10. *Let* S *be a compact surface with* $\chi(S) < 0$. *There is a constant* $L = L(S)$ *such that for any hyperbolic surface* X *homeomorphic to* S*, there is a pants decomposition* $\{\gamma_i\}$ *of* X *with* $\ell_X(\gamma_i) \leq L$ *for each* i.

Definition 4.3.11. The smallest L that satisfies the thesis of the Theorem 4.3.10 is called *Bers' constant.*

Proof. See Theorem 12.8 of [7]. □

We can finally state and prove Mumford's compactness criterion.

Theorem 4.3.12. ***Mumford's compactness criterion:*** *Let $g \geq 1$ For each $\varepsilon > 0$ the space $\mathcal{M}_\varepsilon(\mathbb{S}_g)$ is compact.*

Proof. As already observed the case $g = 1$ is the case $n = 2$ of Mahler's compactness criterion. So we can assume $g \geq 2$.

Since $\mathcal{M}(\mathbb{S}_g)$ inherits the Teichmüller metric from $Teich(\mathbb{S}_g)$, it suffices to show that $\mathcal{M}_\varepsilon(\mathbb{S}_g)$ is sequentially compact for $\varepsilon > 0$. Let $\{X_i\}_{i\in I}$ be a sequence in $\mathcal{M}_\varepsilon(\mathbb{S}_g)$ and $\mathcal{X}_i \in Teich(\mathbb{S}_g)$ a lift of X_i for each $i \in I$. We want to show that there is a subsequence of $\{X_i\}_{i\in I}$ converges in $\mathcal{M}_\varepsilon(\mathbb{S}_g)$. To do so we will show that for a fixed choice of Fenchel-Nielsen coordinates, the $\mathcal{X}_i$ can be chosen to lie in a compact rectangular region of the Euclidea space $(\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3}$.
By Theorem 4.3.10, for each $\mathcal{X}_i$ there is a pants decomposition $\mathcal{P}_i$ of $\mathbb{S}_g$ with $\ell_{\mathcal{X}_i}(\gamma) \in [\varepsilon, L]$ for each $\gamma \in \mathcal{P}_i$, where L is Bers' constant. Since there are only finitely many topological types of pants decomposition of $\mathbb{S}_g$, we can choose a subsequence, also denoted $\{\mathcal{X}_i\}$, and a sequence $f_i \in Mod(\mathbb{S}_g)$ so that $f_i(\mathcal{P}_i) = \mathcal{P}_1$.
In the Fenchel-Nielsen coordinates adapted to $\mathcal{P}_1$ the $\mathcal{Y}_i = f_i \cdot \mathcal{X}_i$ have length parameters in $[\varepsilon, L]$.
Since Dehn twists about the curves of $\mathcal{P}_1$ change the twist parameters by 2π, there is a product h_i of Dehn twists about the curves of $\mathcal{P}_1$ so that the twist parameters of $h_i \cdot \mathcal{Y}_i$ lie in the interval $[0, 2\pi]$ for each i. This concludes the proof. □

4.4 The topology at infinity of $\mathcal{M}(\mathbb{S})$

As suggested by the title in this section we will study some properties of $\mathcal{M}(\mathbb{S})$ due to its noncompactness. In particular we will compute the group π_0 and π_1 "at infinity". We will also study some of its topological properties that follows from its definition as the quotient of $Teich(\mathbb{S})$ by the action of $Mod(\mathbb{S})$.

The first result we will see will give us as corollaries various connectedness properties for $\mathcal{M}(S)$.

Proposition 4.4.1. *Let $g \geq 2$. Let $\mathcal{X}$, $\mathcal{Y} \in Teich(\mathbb{S}_g)$, and suppose that their images X, $Y \in \mathcal{M}(\mathbb{S}_g)$ lie in $\mathcal{M}(\mathbb{S}_g) \setminus \mathcal{M}_\varepsilon(\mathbb{S}_g)$. Then there is a path from $\mathcal{X}$ to $\mathcal{Y}$ in $Teich(\mathbb{S}_g)$ whose projection to $\mathcal{M}(\mathbb{S}_g)$ lies in $\mathcal{M}(\mathbb{S}_g) \setminus \mathcal{M}_\varepsilon(\mathbb{S}_g)$.*

Proof. By the assumption on $\mathcal{X}$ and $\mathcal{Y}$, there are nontrivial simple closed curves α and β in $\mathbb{S}_g$ such that $\ell_{\mathcal{X}}(\alpha) < \varepsilon$ and $\ell_{\mathcal{Y}}(\beta) < \varepsilon$.
By Theorem 3.5.5, there is a sequence of essential simple closed curves $\alpha = \gamma_1, \ldots, \gamma_n = \beta$ such that $i(\gamma_i, \gamma_{i+1}) = 0$. for all i.

Take γ_1 and γ_2 to be part of the Fenchel-Nielsen coordinate system of curves. Decreasing only the length parameter of γ_2 and keeping the other parameter fixed in this coordinate system, we obtain a connected path in $Teich(\mathbb{S}_g)$, starting at $\mathcal{X}$ and ending at some point $\mathcal{X}_2$ with the property that $\ell_{\mathcal{X}_2}(\gamma_2) < \varepsilon$ and $\ell_{\mathcal{Z}}(\gamma_1) < \varepsilon$ for all points $\mathcal{Z}$ on the path.
Repeating this procedure from γ_2 to γ_3, etc., we obtain a path in $Teich(\mathbb{S}_g)$ from $\mathcal{X}$ to some $\mathcal{Y}'$, where each point on the path projects to $\mathcal{M}(\mathbb{S}_g) \setminus \mathcal{M}_\varepsilon(\mathbb{S}_g)$, and in particular where the length of $\gamma_n = \beta$ in $\mathcal{Y}'$ is less than ε. We can then vary the last set of Fenchel-Nielsen coordinates to obtain a path from $\mathcal{Y}'$ to $\mathcal{Y}$ where the length of β remains less than ε. The concatenation of these paths satisfies the thesis of the proposition. □

Remark 4.4.2. *Note that Proposition 4.4.1 tells us that, given any two points of $Teich(\mathbb{S}_g)$, each of which has some short essential closed curve, these points are connected by a path in $Teich(\mathbb{S}_g)$ every point of which has some short essential closed curve.*

We now want to talk about ends of $\mathcal{M}(S)$. In particular the theory of ends of spaces is a way to count the "noncompact directions" of a space. First we will see what means to have one end.

Definition 4.4.3. A conncected, locally compact topological space X has *one end* if for every compact set $B \subset X$ the space $X \setminus B$ has only one component whose closure is noncompact.

Remark 4.4.4. *Let X be a connected, locally compact metric space, and let $\{X_i\}$ be a exhaustion of X by compact sets with $X \setminus X_i$ path connected. Then X has one end.*

Example 4.4.5. Let $X = \mathbb{R}^d$ with $d \geq 2$ and X_i the ball of radius i about any fixed point.

Example 4.4.6. Compact spaces do not have one end.

The space $X = \mathbb{R}$ does not have one end. Indeed the complement of a closed interval has two unbounded components.

Thanks to Proposition 4.4.1 we have the following

Corollary 4.4.7. *Let $g \geq 1$. The moduli space $\mathcal{M}(\mathbb{S}_g)$ has one end.*

Proof. In the case $g = 1$, the result follows from the explicit description of $\mathcal{M}(\mathbb{T})$ given in Section 4.1.

Let $g \geq 2$. From Proposition 4.4.1 we have that $\mathcal{M}(\mathbb{S}_g) \setminus \mathcal{M}_\varepsilon(\mathbb{S}_g)$ is path connected for any $\varepsilon > 0$. Since $\{\mathcal{M}_\varepsilon(\mathbb{S}_g) \ : \ \varepsilon > 0\}$ forms an exhaustion of $\mathcal{M}(\mathbb{S}_g)$ by compact set, we conclude that $\mathcal{M}(\mathbb{S}_g)$ has one end. □

We already observed that $\mathcal{M}(\mathbb{S}_g)$ is path connected, we will show that the topological space underlying $\mathcal{M}(\mathbb{S}_g)$ is simply connected for all $g \geq 1$.

Proposition 4.4.8. *The topological space underlying $\mathcal{M}(\mathbb{S}_g)$ is simply connected for all $g \geq 1$.*

Proof. If $g = 1$ than this follows from the fact that $\mathcal{M}(\mathbb{T})$ is the $(0; 2, 3, \infty)$ hyperbolic orbifold, so that the underlying topological space is a once-punctured sphere, i.e. is homeomorphic to $\mathbb{R}^2$. For $g \geq 2$ the thesis follows from the following three facts:

i. $Mod(\mathbb{S}_g)$ is generated by finite order elements (See Theorem 7.16 of [7]).

ii. The action of each finite order element on $Teich(\mathbb{S}_g)$ has a fixed point.

iii. The cover $Teich(\mathbb{S}_g) \longrightarrow \mathcal{M}(\mathbb{S}_g)$ satisfies the path-lifting property, since $\mathcal{M}(\mathbb{S}_g)$ is the quotient of $Teich(\mathbb{S}_g)$, which is simply connected, by a properly discontinuous action.

To get the simply connectivity we take any loop in $\mathcal{M}(\mathbb{S}_g)$ based at the image in $\mathcal{M}(\mathbb{S}_g)$ of a fixed point of one of the generators of $Mod(\mathbb{S}_g)$. The lift of this loop is a closed loop in $Teich(\mathbb{S}_g)$, and any null-homotopy in $Teich(\mathbb{S}_g)$ descends to a null-homotopy in $\mathcal{M}(\mathbb{S}_g)$. □

The next topological invariant we will study is the orbifold foundamental group.
Recalling that $Teich(\mathbb{S})$ is simply connected and that $Mod(\mathbb{S})$ acts properly discontinuously on $Teich(\mathbb{S})$, we have

$$\pi_1^{orb}(\mathcal{M}(\mathbb{S})) \simeq Mod(\mathbb{S}).$$

From that we can also obtain information about homotopy classes of loops in $\mathcal{M}(\mathbb{S})$.

Example 4.4.9. The orbifold $\mathcal{M}(\mathbb{T})$ has a unique homotopy class that can be freely homotoped outside every compact subset of $\mathcal{M}(\mathbb{T})$; namely the free homotopy class represented by the conjugacy class of the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

The case $g \geq 2$ is different. In particular we have

Corollary 4.4.10. *Let $g \geq 2$. Any loop in $\mathcal{M}(\mathbb{S}_g)$ can be freely homotoped outside every compact set in $\mathcal{M}(\mathbb{S}_g)$.*

Proof. It suffices to consider loops that are in essential and compact sets of the form $\mathcal{M}_\varepsilon(\mathbb{S}_g)$. Let $\varepsilon > 0$ be given. Let α be a essential loop in $\mathcal{M}(\mathbb{S}_g)$, and consider $X \in \mathcal{M}(\mathbb{S}_g) \setminus \mathcal{M}_\varepsilon(\mathbb{S}_g)$. Since $\mathcal{M}(\mathbb{S}_g)$ is path connected, α can be freely homotoped to a loop β based at X. As above β can be lifted to a path $\widetilde{\beta}$ in $Teich(\mathbb{S}_g)$. Thanks to Proposition 4.4.1, there is a path γ between the endpoints of $\widetilde{\beta}$ with projection $\overline{\gamma}$ in $\mathcal{M}(\mathbb{S}_g) \setminus \mathcal{M}_\varepsilon(\mathbb{S}_g)$. Any homotopy from $\widetilde{\beta}$ to γ descends to a homotopy from α to $\overline{\gamma}$. □

Remark 4.4.11. *Another way to state Corollary 4.4.10 is that, for $g \geq 2$ the fundamental group of $\mathcal{M}(\mathbb{S}_g)$ "relative to infinity" is trivial, more formally the inclusion map $\mathcal{M}(\mathbb{S}_g) \setminus \mathcal{M}_\varepsilon(\mathbb{S}_g) \hookrightarrow \mathcal{M}(\mathbb{S}_g)$ induces an isomorphism of orbifold fundamental groups.*

4.5 Cohomology of $\mathcal{M}(\mathbb{S})$

Studying the cohomology of the moduli space of curves directly is not easy. Instead we will use the theory of classifying space to obtain an isomorphism, in the case $g \geq 2$, between the rational cohomology of $\mathcal{M}(\mathbb{S}_g)$ and the cohomology of the classifying space of $Mod(\mathbb{S}_g)$.

Recall that the action of $Mod(\mathbb{S}_g)$ on $Teich(\mathbb{S}_g)$ is not free, thus $\mathcal{M}(\mathbb{S}_g)$ is not a classifying space for $Mod(\mathbb{S}_g)$. But we can consider the diagonal action of $Mod(\mathbb{S}_g)$ on the product $EMod(\mathbb{S}_g) \times Teich(\mathbb{S}_g)$, where $EMod(\mathbb{S}_g)$ is the universal cover of the classifying space of $Mod(\mathbb{S}_g)$. Note that this action is free and properly discontinuous. Denote the quotient, which is a $K(Mod(\mathbb{S}_g), 1)$ space, by $BMod(\mathbb{S}_g)$.

The projection map $EMod(\mathbb{S}_g) \times Teich(\mathbb{S}_g) \longrightarrow Teich(\mathbb{S}_g)$ is $Mod(\mathbb{S}_g)$-equivariant, so it induces a map $h\colon BMod(\mathbb{S}_g) \longrightarrow \mathcal{M}(\mathbb{S}_g)$. In particular if $\mathcal{X} \in Teich(\mathbb{S}_g)$ maps to $X \in \mathcal{M}(\mathbb{S}_g)$, then $h^{-1}(X)$ is a classifying space for the stabilizer of $\mathcal{X}$ in $Mod(\mathbb{S}_g)$.
Analogously, using Γ, a finite-index, torsion-free normal subgroup of $Mod(\mathbb{S}_g)$ which acts freely and properly discontinuously on $Teich(\mathbb{S}_g)$, we obtain a continuous map $\widetilde{h}\colon B\Gamma \longrightarrow Teich(\mathbb{S}_g)/\Gamma$, where $B\Gamma$ denotes a $K(\Gamma, 1)$ space. By Whitehead's theorem, we have that $\widetilde{h}$ is a homotopy equivalence, since $B\Gamma$ and $Teich(\mathbb{S}_g)/\Gamma$ are classifying spaces and $\widetilde{h}_\star\colon \pi_1(B\Gamma) \longrightarrow Teich(\mathbb{S}_g)/\Gamma$ is an isomorphism.
We can give the following result on the rational cohomology of $\mathcal{M}(\mathbb{S}_g)$.

Theorem 4.5.1. *Let $g \geq 2$, and let $h\colon BMod(\mathbb{S}_g) \longrightarrow \mathcal{M}(\mathbb{S}_g)$ be the map constructed above. Then the induced homomorphism*

$$h^\bullet\colon H^\bullet(\mathcal{M}(\mathbb{S}_g), \mathbb{Q}) \longrightarrow H^\bullet(BMod(\mathbb{S}_g), \mathbb{Q})$$

is an isomorphism.

Proof. Let $G = Mod(\mathbb{S}_g)/\Gamma$. The finite group G acts by covering space automorphisms on $B\Gamma$ and $Teich(\mathbb{S}_g)/\Gamma$. By construction the map $\widetilde{h}$ is $G-$equivariant. We thus have the following commutative diagram:

$$\begin{array}{ccc} B\Gamma & \xrightarrow{\widetilde{h}} & Teich(\mathbb{S}_g)/\Gamma \\ \downarrow & & \downarrow \\ BMod(\mathbb{S}_g) & \xrightarrow{h} & \mathcal{M}(\mathbb{S}_g) \end{array}$$

Since $\widetilde{h}$ is a $G-$equivariant homotopy equivalence, it induces a $G-$equivariant isomorphism

$$\widetilde{h}^\bullet\colon H^\bullet(Teich(\mathbb{S}_g)/\Gamma, \mathbb{Q}) \longrightarrow H^\bullet(B\Gamma, \mathbb{Q}).$$

Since $\widetilde{h}^\bullet$ is $G-$equivariant, it restricts to an ismorphism of the corresponding invariants.
Moreover, the covering map $B\Gamma \longrightarrow BMod(\mathbb{S}_g)$ induces an isomorphism

$$H^\bullet(BMod(\mathbb{S}_g), \mathbb{Q}) \longrightarrow H^\bullet(B\Gamma, \mathbb{Q})^G$$

and the covering map $Teich(\mathbb{S}_g)/\Gamma \longrightarrow \mathcal{M}(\mathbb{S}_g)$ induces an isomorphism

$$H^\bullet(\mathcal{M}(\mathbb{S}_g), \mathbb{Q}) \longrightarrow H^\bullet(Teich(\mathbb{S}_g)/\Gamma, \mathbb{Q})^G.$$

These two isomorphisms come from the transfer argument in cohomology (See Proposition 3G.1 of [10]). The thesis follows from the isomorphism $\widetilde{h}^\bullet$. □

This concludes our study of the moduli space of curves and its topological properties. The theory of moduli space is still expanding these days and a lot more can be said on $\mathcal{M}(\mathbb{S})$.

Appendix A

Other constructions for $\mathcal{M}(\mathcal{S})$

From now on C will be a smooth, complete, connected curve of genus g over $\mathbb{C}$.
In this work we constructed $\mathcal{M}(\mathcal{S})$ as the quotient of the Teichmüller space of a surface and its mapping class group. This technique is known as the Teichmüller approach. But this is not the only way to describe $\mathcal{M}(S)$. In this appendix we will just give the idea of two others possible approaches: the Hodge theory approach and the geometric invariant theory approach.
The general idea behind these three different approaches is to consider complex curves with some additional structure, so that a parameter space can be described, and then taking the quotient of this space by the relation that identifies these additional structures.

Each approach has different advantages and gives different information on $\mathcal{M}(\mathcal{S})$. The Teichmüller approach gives us some important topological information on $\mathcal{M}(\mathcal{S})$ as we have observed in Chapter 4.

A.1 The Hodge Theory approach

In this approach the idea is to associate to C the data of its polarized Jacobian, which is equivalent to giving a complex vector space V of dimension g with lattice $\Lambda \simeq \mathbb{Z}^{2g}$ and skew-symmetric form Q. Respectively, these ingredients are obtained from C as: the dual of $H^0(C, K_C)$, the first homology group $H_1(C, \mathbb{Z})$ and the intersection pairing. Choose a a symplectic basis $\beta = \{a_1, \dots, a_g, b_1, \dots, b_g\}$ for $H_1(C, \mathbb{Z})$ and a complex basis $\omega_1, \dots, \omega_g$ of $H^0(C, K_C)$ whose period matrix with respect to the $a-$cycles is Id_g, then we can associate the period matrix $P \in \mathrm{M}_g(\mathbb{C})$, given by integrating the ω's around the $b-$cycles. The Riemann bilinear relations say that P is symmetric with positive definite imaginary part. Thanks to these two conditions we obtain, respectevely, a subspace and a open subset of the space $\mathrm{M}_g(\mathbb{C})$ whose intersection is called the *Siegel upper halfspace* of dimension g and is denoted $\mathfrak{h}_g$.
We have that the group $\mathrm{Sp}(2g, \mathbb{Z})$ of symplectic changes of base acts on $\mathfrak{h}_g$ and this action corresponds to the choice of symplectic basis made above.
One can prove that period matrices of curves form a locally closed subset $\mathfrak{c}_g$ of $\mathfrak{h}_g$. If we consider all $\mathfrak{h}_g$ and we quotient it by $\mathrm{Sp}(2g, \mathbb{Z})$ we obtain $\mathcal{A}_g$, which is the moduli space of abelian varieties of dimension g. While if we consider the restriction of this action to the locus $\mathfrak{c}_g$, we obtain $\mathcal{M}(\mathcal{S}_g)$.

This approach again describe $\mathcal{M}(\mathcal{S}_g)$ only as an analytic space, as the Teichmüller approach, but the group $\mathrm{Sp}(2g, \mathbb{Z})$ is more approachable then $Mod(S_g)$. At the same time we can say much less about the space $\mathfrak{c}_\mathfrak{g}$. In particular describing the locus $\mathfrak{c}_g$ in $\mathfrak{h}_g$, or $\mathcal{M}(\mathcal{S}_g)$ in $\mathcal{A}_g$ is known as the Schottky problem.

Another advantage of this method is that we see $\mathcal{M}(\mathcal{S}_g)$ contained in $\mathcal{A}_g$, which has a Baily-Borel compactification $\widetilde{\mathcal{A}}_g$, called Satake compactification. Taking the closure of $\mathcal{M}(\mathcal{S}_g)$ in $\widetilde{\mathcal{A}}_g$ yields a compactification of $\mathcal{M}(\mathcal{S}_g)$, (in this case when we say that $\overline{\mathcal{M}}$ is a compactification of $\mathcal{M}$ we mean that $\overline{\mathcal{M}}$ is a compact

analytic variety that contains $\mathcal{M}$ as an analytic open subset), which we will denote by $\widetilde{\mathcal{M}}(\mathcal{S}_g)$ and also called Satake compactification. The only problem with this compactification of $\mathcal{M}(\mathcal{S}_g)$ is that it is not modular, which means that $\widetilde{\mathcal{M}}(\mathcal{S}_g)$ is not a moduli space for any moduli functor of curves that contains the moduli functor of smooth curves as an open subfunctor. Indeed one can prove that the points in $\widetilde{\mathcal{M}}(\mathcal{S}_g) \setminus \mathcal{M}(\mathcal{S}_g)$ correspond to isomorphism classes of smooth curves of lower genus, and these do not naturally fit into families of curves of genus g.
However the Satake compactification is still important and gives an information about $\mathcal{M}(S_g)$. As a matter of fact $\widetilde{\mathcal{M}}(\mathcal{S}_g)$ has two important properties. First $\widetilde{\mathcal{M}}(\mathcal{S}_g)$ is projective, and second the codimension of the complement $\widetilde{\mathcal{M}}(\mathcal{S}_g) \setminus \mathcal{M}(\mathcal{S}_g)$ in $\widetilde{\mathcal{M}}(\mathcal{S}_g)$ is equal to 2 for $g \geq 3$. Considering the intersection of $\mathcal{M}(\mathcal{S}_g)$ with generic divisors in some large multiple $\mathcal{O}(n)$ of a very ample sheaf on $\widetilde{\mathcal{M}}(\mathcal{S}_g)$ through any point, we see that through any point of $\mathcal{M}(\mathcal{S}_g)$ passes a complete curve lying entirely in $\mathcal{M}(\mathcal{S}_g)$. Actually, there is a complete curve through any finite collection of points of $\mathcal{M}(\mathcal{S}_g)$. Using a curve through two points, on which any holomorphic function must be constant, we see that there are no nonconstant functions on $\mathcal{M}(\mathcal{S}_g)$. These facts show that $\mathcal{M}(\mathcal{S}_g)$ is neither projective nor affine.

A.2 The Geometric Invariant Theory (G.I.T.) approach

This approach is slightly different from the previous ones. Indeed in the two approach presented above the extra information attached to a curve was analytic. Correspondingly, the parameter space of curves with this extra data was not algebraic but a complex analytic variety, and the group acting on this space was not an algebraic group. In the G.I.T. approach, however, everything is algebraic.

We will now see the idea behind. Recall that for any integer $n \geq 3$, any curve C can be embedded as a curve of degree $2(g-1)n$ in the projective space $\mathbb{P}^N = \mathbb{P}^{(2n-1)(g-1)-1}$ by the complete linear series $|nK_C|$. We attach to a curve C the data of such an embedding, i.e. we consider pairs consisting of a curve C and an $n-$canonical embedding $\varphi\colon C \longrightarrow \mathbb{P}^N$. One can prove that the family of all such pairs corresponds to a locally closed subset $\mathcal{K}$ of the Hilbert scheme $\mathcal{H} = \mathcal{H}_{2(g-1)n,g,(2n-1)(g-1)-1}$ of smooth curves of degree $2(g-1)n$ and genus g in $\mathbb{P}^N$. Observe that the ambiguity in choosing the map φ is simply a matter of choosing a basis for the space $H^0(C, K_C^{\otimes n})$ of $n-$canonical differentials on C, in other words, the group $\mathrm{PGL}(N+1,\mathbb{C})$ acts on $\mathcal{K}$, and the quotient, if one exists, should be $\mathcal{M}(\mathcal{S}_g)$.

One problem with this approach is that, since the group $\mathrm{PGL}(N+1,\mathbb{C})$ is continuous, the existence of a "nice" quotient is by no means assured. This can be proved using the techniques of the geometric invariant theory. We will not enter the details here but the reader can see [9] and [15] if interested in more details. We will simply assume that we have constructed this quotient.
This approach has two significant advantages over the previous ones.
First it exhibits $\mathcal{M}(\mathcal{S}_g)$ as an algebraic variety, second it leads to an explicit modular projective compactification of $\mathcal{M}(\mathcal{S}_g)$.

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