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Holographic map for the D1-D5 system

Mappa olografica
per il sistema D1-D5

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Sunto

L'equivalenza olografica tra teorie di gravità in spazi anti-de Sitter e teorie di campo conformi permette il calcolo di funzioni di correlazione della teoria di campo nel regime di accoppiamento forte in termini della teoria classica di (super)gravità. Un passaggio preliminare necessario per questi calcoli è la costruzione della mappa olografica che lega gli operatori conformi con i campi duali di supergravità. Questa tesi si concentra sulla teoria conforme D1-D5, emergente dal limite a basse energie di uno stato legato di D1- e D5-brane, e si propone di stabilire la mappa olografica per operatori conformi che appartengono a diversi multipletti tensoriali.

Abstract

The holographic duality between gravity theories in anti-de Sitter spaces and conformal field theories (CFT) allows the computation of CFT correlators in the strong-coupling regime in terms of the classical (super)gravity theory. A necessary preliminary step for this computation is the construction of the holographic map linking the CFT operators with the dual supergravity fields. This thesis focuses on the D1-D5 CFT emerging from the low-energy limit of a bound state of D1- and D5-branes and aims at establishing the holographic map for conformal operators belonging in different tensor multiplets.

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1 Introduction

Holography is one of the most important discoveries of high-energy theoretical physics in recent years. The term “holography” comes from holograms, which are two-dimensional objects that store information about all three dimensions of the object they represent. The holographic principle, proposed by Gerard ’t Hooft and Leonard Susskind, states that the physical phenomena on a volume of space (the bulk) are somehow encoded in a theory on the boundary of that space. The first hint for holography comes from studying the thermodynamics of black holes, which predicts that the black hole entropy is proportional to the surface area of the object, meaning that a quantity of a three-dimensional object (the black hole) depends on its two-dimensional surface area, rather than on its bulk volume.

The most important and useful application of the holographic principle is the anti-de Sitter/conformal field theory correspondence, AdS/CFT for short, introduced by Juan Martín Maldacena in 1997. AdS/CFT is a relationship between two seemingly unrelated physical theories that live in spaces with different dimensions. One is a quantum gravity theory defined on anti-de Sitter space (AdS) which is the maximally symmetric space with negative curvature. The other is a conformal field theory (CFT) that is a quantum field theory that enjoys conformal symmetry, which is composed of the transformations that preserve angles. The quantum gravity theories that are most used in this context are those formulated in terms of string theory. String theory was instrumental in the inspiration and formulation of the correspondence for two reasons. One is that string theory is as of now the best candidate for the theory of quantum gravity, because one of the vibrational modes of the strings it describes can be interpreted as the graviton, the carrier for the gravitational interaction. The other reason is that string theory, besides closed and open strings contains other extended physical objects, which are called D-branes: they are the hypersurfaces on which open strings end. The crucial point that hinted at the presence of some kind of correspondence between field theories and gravity theories comes right from D-branes. One can describe a system of D-branes in two different ways: in terms of open strings and closed strings. The theory defined on the brane worldvolume arising from open strings is a field theory, while the theory due to closed strings is a gravity theory: this allows a gravitational description of a system of D-branes. In the so-called decoupling limit, interactions between the two theories are turned off, so they decouple and become independent: the field theory becomes conformal and the spacetime becomes anti-de Sitter space. Maldacena realized that the D-brane systems in this decoupling limit can be described in terms of the conformal theory or the gravity theory on AdS, and that the two descriptions have to be equivalent. This is the heart of the AdS/CFT correspondence. Generally, AdS/CFT is difficult to apply, because it is complicated to do quantum-gravity calculations in string theory and, on the other side, field theory at strong coupling is impossible to treat with modern tools. Fortunately, there are regimes in which strongly coupled field theory is dual to effectively classical gravity, namely supergravity. The fact that AdS/CFT relates complicated theories with simpler theories has been applied even in other branches of physics like condensed matter and nuclear physics: examples are respectively high-temperature superconductivity and the quark-gluon plasma, that are both strongly coupled systems intractable using standard techniques and theories.

Operationally, AdS/CFT relates chiral primary operators in the field theory with supergravity fields. The so-called "holographic dictionary" consists in determining all pairs of chiral primary operators and supergravity fields. These fields are small perturbations to the background spacetime. Chiral primary operators are referred to as "light operators". However, it is possible to take particular combinations of chiral primary operators that are called "heavy operators" and are dual to whole nontrivial geometries. An important aspect of AdS/CFT is in correlators with chiral primary operators, which are fundamental quantities in any quantum and conformal field theory. However, when the theory is strongly coupled, correlators becomes involved and difficult to compute. Holographic dualities provide a powerful tool for treating them by studying, according to a known prescription, their supergravity counterpart, namely the equations of motion of fields in a background geometry, where both the background geometry and the fields depend on the operators contained in the correlator.

There exist multiple examples of AdS/CFT dual pairs in various number of dimensions. The most popular and understood correspondence comes from the system of D3-branes: the two dual descriptions of this system are type IIB string theory on the 10-dimensional product space $AdS_5 \times \mathbb{S}^5$—where $\mathbb{S}^5$ is the five-dimensional sphere—and the so-called $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on the four-dimensional boundary. Note that string theory lives on a 10-dimensional space so, in order to properly apply the duality, we reduce the dimensionality of the space via a process called compactification, and effectively consider AdS_5. Other systems of D-branes give rise to other dualities. For instance, this work is focused on the D1-D5 system, which in a certain limit is described by type IIB supergravity on $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$—where $\mathbb{T}^4$ is the four-dimensional torus—and a two-dimensional conformal field theory known as D1-D5 CFT. This system consists in a number of D1-branes and D5-branes that wrap around the compact dimensions of spacetime. The D1-D5 system exhibits a peculiarity which distinguishes it from D3 system; the latter has the maximum number of supersymmetries, so each field is related via supersymmetry to every other, i.e. there is only one field multiplet. Conversely, in the former system the presence of the torus $\mathbb{T}^4$ breaks half the supersymmetries and causes the supergravity fields—and the CFT operators—to organize in five tensor multiplets, also called flavours. There is a SO(5) symmetry that rotates between the multiplets; this obviously applies also to correlators: one can show that a three-point correlator with two different-flavour states is zero and non-zero otherwise. Fields or operators that have the same dimensions but different flavours have "similar" properties. Among the five operators with different flavours but equal dimensions, three transform nontrivially under the torus and the other two are invariant.

In this work, we will consider the two torus-invariant operators and in particular their descendants, which are obtained by acting with two supercharges on the original operator, that is sometimes called "ancestor". In the context of correlators, descendants are useful because they make correlators easier to compute. That said, the goal of this thesis is to determine the holographic map between these two descendants and their corresponding supergravity dual fields. Generally, descendants are dual to six-dimensional scalar fields, but the fact that the two descendants are somewhat similar—they have same dimensions and are both torus invariant—makes it difficult to pin down the precise holographic relation. In the literature, some indirect arguments have been provided to establish the precise map. Our approach involves doing a direct holographic calculation, exploiting the aforementioned SO(5) flavour symmetry.

Let us briefly explain the procedure. For concreteness we call the two fields F and G and the two descendants A and B. We consider the two three-point correlators that contain an unflavoured operator, the ancestor of A, and A or B. On the supergravity side, this corresponds to two calculations, one with F and one with G. As we said before, for symmetry reasons the correlator with A is going to be zero, implying that the corresponding supergravity calculation has to be zero as well. This means that the field that makes the supergravity calculation vanish is the field that is dual to A, leading obviously to identify the other one as the dual to B.

We now give the outline of the thesis. Chapter 2 provides the basic principles of any conformal field theory. Chapter 3 introduces string theory: the relativistic string, its quantization and the string spectrum, as well as superstring theory; even if we do not really use string theory in the following, it is useful to explain the basis for supergravity, which is the subject of Chapter 4. Subsequently, in Chapter 5, we describe AdS/CFT correspondence, giving some motivations and the general statement, as well as the prescription to compute correlators in the gravity side. We then describe the D1-D5 system in Chapter 6, starting from the conformal side and deriving the geometries from their dual CFT states. Finally, Chapter 7 contains the original contribution of the thesis, namely the holographic map we discussed above, which is commented upon in the final Chapter 8.

2 | Conformal field theory

Conformal field theories (CFT) are invariant under conformal transformations. Conformal transformations are transformations that preserve angles, not lengths. This implies that the theory looks the same at all length scales. Since the end of the twentieth century, conformal field theories have become increasingly important because they play a role in the AdS/CFT correspondence.

In this chapter we provide an introduction to conformal field theory. For more details, we refer to standard references such as [1–3].

2.1 Conformal invariance

A *conformal transformation* is a transformation of the coordinates that leaves the metric invariant up to a scale:

$$g'_{\mu\nu}(\boldsymbol{x}) = \Lambda^2(\boldsymbol{x})\, g_{\mu\nu}(\boldsymbol{x}). \tag{2.1}$$

For the special case where $\Lambda^2(\boldsymbol{x}) = 1$, the transformation reduces to Poincaré transformation, so we deduce that the Poincaré group is a subgroup of the conformal one. Conformal transformations are also sometimes called angle-preserving, because they preserve angles between curves, even after a local dilation.

Under a generic infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu(\boldsymbol{x})$, the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{2.2}$$

In order for (2.2) to be conformal, the definition (2.1) requires that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(\boldsymbol{x})\, g_{\mu\nu} = \frac{2}{d}\, \partial_\rho \epsilon^\rho\, g_{\mu\nu} \tag{2.3}$$

where the factor $f(\boldsymbol{x})$ is determined by taking the trace on both sides.

It can be shown that, for dimensions $d > 2$, ϵ_μ is at most quadratic in the coordinates x^μ and can be written in the generic form

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{2.4}$$

where $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Depending on the coefficients, we can identify four kinds of conformal transformations: translation, rotation, dilation and special conformal transformation.

In classical field theory, a spinless field $\phi(x)$ will transform under a conformal transformation $x \to x'$ as

$$\phi(x) \to \phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \phi(x) \tag{2.5}$$

where Δ is the conformal dimension of ϕ and $\left|\partial x'/\partial x\right|$ is the Jacobian of the conformal transformation of the coordinates. A field transforming like (2.5) is "quasi-primary".

An important quantity in any field theory is the *energy-momentum tensor*. Conformal invariance implies it is traceless. Under a generic change of coordinates $x^\mu \to x^\mu + \epsilon^\mu$, the action changes as follows:

$$\delta S = \int d^d x\, T^{\mu\nu} \partial_\mu \epsilon_\nu = \frac{1}{2} \int d^d x\, T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{2.6}$$

The conformal transformations imply

$$\delta S = \frac{1}{d} \int d^d x\, T^\mu_\mu \partial_\rho \epsilon^\rho$$

so the action is invariant under conformal transformation if $T^{\mu\nu}$ is traceless.

Let us now see how conformal invariance is implemented in quantum field theory, especially in correlation functions. We will mostly list the results without proofs, which can be found in [1]. Consider a two-point function $\langle \phi_1(x_1)\phi_2(x_2) \rangle$. For spinless fields, conformal invariance yields

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \left| \frac{\partial x'}{\partial x} \right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x_1)\phi_2(x_2) \rangle \tag{2.7}$$

If we specialize to a scale transformation $x \to \lambda x$, we obtain

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\phi_2(\lambda x_2) \rangle \tag{2.8}$$

Furthermore, rotation and translation invariance imply $\langle \phi_1(x_1)\phi_2(x_2) \rangle = f(|x_1 - x_2|)$ or, in other words,

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.9}$$

Finally, invariance under special conformal transformation implies

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{2.10}$$

For three-point functions, we have

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_1+\Delta_3-\Delta_2}} \tag{2.11}$$

where for brevity $x_{ij} = \left|x_i - x_j\right|$. However, these totally constrained forms end with 4-point functions, for which

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{23}x_{14}} \right) \prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j} \tag{2.12}$$

where $\Delta = \sum_i^4 \Delta_i$. The function f depends on the so-called *anharmonic ratios*, which are invariant under any conformal transformation, but it is not determined by conformal invariance.

2.2 Two-dimensional conformal field theory

Conformal field theory is particular in two dimensions, especially if we consider local transformations, which are transformations that are not defined everywhere or necessarily invertible. There are an infinity of two-dimensional coordinate transformations that are locally conformal: they are the *holomorphic mappings*.

In two dimensions, we consider the plane of coordinates (z^0, z^1). A conformal transformation is any change of coordinates $z^\mu \to w^\mu(x)$ where $w^\mu(x)$ is a holomorphic function, i.e. it satisfies the Cauchy-Riemann equations:

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \qquad \text{and} \qquad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{2.13}$$

Given $z = z^0 + iz^1$ and $w = w^0 + iw^1$, the holomorphic Cauchy-Riemann equations become simply

$$\partial_{\bar{z}} w(z, \bar{z}) = 0 \tag{2.14}$$

whose solution is any holomorphic mapping $z \to w(z)$.

2.2.1 Conformal group

All that we have inferred above is *local*, that is, we have not imposed the condition that conformal transformations be defined everywhere and be invertible. Strictly speaking, in order to form a group, the mappings must be invertible, and must map the whole plane (Riemann sphere) into itself. We must therefore distinguish *global conformal transformations*, which satisfy these requirements, from local ones, which do not.

The set of global transformation, called the *special conformal group*, is the set of mappings

$$f(z) = \frac{az + b}{cz + d} \tag{2.15}$$

with $ad - bc = 1$. We can associate to each f a matrix with determinant 1, that is a matrix in $\mathrm{SL}(2, \mathbb{C})$. So the global conformal group in two dimensions is isomorphic to $\mathrm{SL}(2, \mathbb{C})$ which in turn is isomorphic to the Lorentz group $\mathrm{SO}(3, 1)$. Then we found the global conformal group has 6 real parameters in two dimensions.

2.2.2 Conformal generators

Local properties are more useful than the global ones, so the local group, made up of all (not necessarily invertible) holomorphic mappings is of great importance. We now find the *algebra of its generators*.

Any holomorphic infinitesimal transformation may be expressed as

$$z' = z + \epsilon(z) \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1} \tag{2.16}$$

The effect of such mapping on a spinless and dimensionles field $\phi(z, \bar{z})$ is

$$\begin{aligned} \phi'(z', \bar{z}') &= \phi(z, \bar{z}) \\ &= \phi(z', \bar{z}') - \epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z', \bar{z}') \end{aligned} \tag{2.17}$$

The variation of the field under this transformation is

$$\begin{aligned}\delta\phi &= -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi \\ &= \sum_n \left[c_n \ell_n\, \phi(z,\bar{z}) + \bar{c}_n \bar{\ell}_n\, \phi(z,\bar{z})\right]\end{aligned} \tag{2.18}$$

in which we Laurent-expanded $\epsilon(z)$ and defined

$$\ell_n = -z^{n+1}\partial_z \qquad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{2.19}$$

These are the generators, and they obey the following relations which characterize the *conformal algebra*:

$$\begin{aligned}[][\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\ [\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m} \\ [\ell_n, \bar{\ell}_m] &= 0\end{aligned} \tag{2.20}$$

Thus the conformal algebra is the direct sum of two independent isomorphic algebras, called *Witt algebras*.

The infinite algebra made of ℓ_n contains a finite subalgebra generated by ℓ_{-1}, ℓ_0 and ℓ_1 which is associated with the *global conformal group*. Indeed, it is manifest from the definition (2.19) that $\ell_{-1} = -\partial_z$ generates translations, $\ell_0 = -z\partial_z$ generates rotations and scale transformations, while $\ell_1 = -z^2\partial_z$ generates special conformal transformations.

2.2.3 Correlation functions

Before writing correlators in two dimensions, we define what we mean by "primary field". A field ϕ with scaling dimension Δ and planar spin s, possesses the so-called *conformal dimensions*, defined as

$$h = \frac{1}{2}(\Delta + s) \qquad \bar{h} = \frac{1}{2}(\Delta - s) \tag{2.21}$$

Under the transformations $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$, ϕ is a *primary field* if it transforms as

$$\phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \tag{2.22}$$

which is the generalization of (2.5) for $s \neq 0$. If the mapping is infinitesimal, that is $w = z + \epsilon(z)$ and $\bar{w} = \bar{z} + \bar{\epsilon}(\bar{z})$, the variation of primary fields is

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}} &\equiv \phi'(z,\bar{z}) - \phi(z,\bar{z}) \\ &= -(h\phi\partial_z\epsilon + \epsilon\partial_z\phi) - (\bar{h}\phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\phi)\end{aligned} \tag{2.23}$$

The correlation function of n primary fields ϕ_i with conformal dimensions $(h_i, \bar{h}_i)$ reads

$$\big\langle \phi_1(w_1,\bar{w}_1)\cdots\phi_n(w_n,\bar{w}_n)\big\rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i} \big\langle \phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\big\rangle \tag{2.24}$$

which fixes the form of two- and three-point functions. The novelty is spin, given by $h_i - \bar{h}_i$. Note that equations (2.10) and (2.11) are still valid in two dimensions. In complex coordinates we have

$$\big\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\big\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \qquad \text{if} \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h}\end{cases} \tag{2.25}$$

Note that it vanishes if the conformal dimensions are different; this comes from rotation invariance with spins: the sum of spins should be zero.

Equation (2.11) becomes

$$\begin{aligned}\left\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\right\rangle = \\ C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3}\,z_{23}^{h_2+h_3-h_2}\,z_{13}^{h_3+h_1-h_2}}\frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\,\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_2}\,\bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}\end{aligned} \tag{2.26}$$

Four-points functions are not fixed neither in two dimensions, but the anharmonic ratios live on the same plane, so we have relations between them:

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \qquad 1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}} \qquad \frac{\eta}{1-\eta} = \frac{z_{12}z_{34}}{z_{14}z_{23}} \tag{2.27}$$

so that the general (2.12) translates into

$$\left\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\right\rangle = f(\eta,\bar{\eta})\prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j}\,\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \tag{2.28}$$

2.2.4 Ward identities

Ward identities are the quantum counterpart of Noether's theorem and as such, they arise from symmetries. Let us see which identities come from conformal invariance. We report here a set of Ward identities associated with translation, rotation and scale invariance [1]. They are

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\left\langle T^\mu{}_\nu(x)X\right\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)\,\frac{\partial}{\partial x_i^\nu}\left\langle X\right\rangle \\ \varepsilon_{\mu\nu}\left\langle T^{\mu\nu}(x)X\right\rangle &= -i\sum_{i=1}^{n}s_i\delta(x-x_i)\left\langle X\right\rangle \\ \left\langle T^\mu_\mu X\right\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\left\langle X\right\rangle\end{aligned} \tag{2.29}$$

where X stands for a string of n primary fields defined at points x_i. Using the relation,

$$\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} \tag{2.30}$$

we can rewrite the identities (2.29) in complex coordinates, which are

$$\begin{aligned}2\pi\partial_z\left\langle T_{\bar{z}z}X\right\rangle + 2\pi\partial_{\bar{z}}\left\langle T_{zz}X\right\rangle &= -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\left\langle X\right\rangle \\ 2\pi\partial_z\left\langle T_{\bar{z}\bar{z}}X\right\rangle + 2\pi\partial_{\bar{z}}\left\langle T_{z\bar{z}}X\right\rangle &= -\sum_{i=1}^{n}\partial_z\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\left\langle X\right\rangle \\ 2\left\langle T_{z\bar{z}}X\right\rangle + 2\left\langle T_{\bar{z}z}X\right\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\left\langle X\right\rangle \\ -2\left\langle T_{z\bar{z}}X\right\rangle + 2\left\langle T_{\bar{z}z}X\right\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)s_i\left\langle X\right\rangle\end{aligned} \tag{2.31}$$

If we add and subtract the last two equations of (2.31), we find

$$\begin{aligned} 2\pi \langle T_{\bar{z}z} X \rangle &= -\sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{z - w_i} h_i \langle X \rangle \\ 2\pi \langle T_{z\bar{z}} X \rangle &= -\sum_{i=1}^{n} \partial_{z} \frac{1}{\bar{z} - \bar{w}_i} \bar{h}_i \langle X \rangle \end{aligned} \tag{2.32}$$

Inserting these two inside the first two equations of (2.31), we find holomorphic and antiholomorphic derivatives set to zero. Using $T = -2\pi T_{zz}$, the holomorphic one is:

$$\partial_{\bar{z}} \left\{ \big\langle T(z,\bar{z}) X \big\rangle - \sum_{i=1}^{n} \left[\frac{1}{z - w_i} \partial_{w_i} \langle X \rangle + \frac{h_i}{(z - w_i)^2} \langle X \rangle \right] \right\} = 0 \tag{2.33}$$

and this allows us to write

$$\big\langle T(z) X \big\rangle = \sum_{i=1}^{n} \left\{ \frac{1}{z - w_i} \partial_{w_i} \langle X \rangle + \frac{h_i}{(z - w_i)^2} \langle X \rangle \right\} + \text{reg.} \tag{2.34}$$

Similar expressions hold for the antiholomorphic counterpart, where $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$. The Ward identities (2.29) can be grouped in a single relation. We will simply write the final result, while the procedure is explained in [1].

$$\boxed{\delta_{\epsilon,\bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \big\langle T(z) X \big\rangle + \frac{1}{2\pi i} \oint_C d\bar{z} \bar{\epsilon}(\bar{z}) \big\langle T(\bar{z}) X \big\rangle} \tag{2.35}$$

where $\delta_{\epsilon,\bar{\epsilon}} \langle X \rangle$ is the variation of X under a local transformation of parameters ϵ and $\bar{\epsilon}$.

For the Ward identity to work, we must require that the energy-momentum tensor be finite for every z. Since it is symmetric and traceless, it has spin $s = 2$; furthermore, since it is an energy density, it has scaling dimension $\Delta = 2$, leading to conformal dimensions $h = 2$ and $\bar{h} = 0$. Under a global transformation $w = 1/z$ we have

$$T'(w) = \left(\frac{dw}{dz} \right)^{-2} T(z) = z^4 T(z) \tag{2.36}$$

and since $T'(w)$ is as regular as $T(z)$, the latter must decay like z^{-4} as $z \to \infty$.

2.2.5 Operator product expansion

Correlation functions typically are singular in points where fields coincide, and this arises from the fluctuating nature of quantum fields.

The *operator product expansion*, or OPE, is the representation of a product of fields (at positions z and w) as a sum of terms made up of a regular operator times a diverging function for $z \to w$.

The OPE of T with a primary field ϕ is simply the correlator (2.34) without the regular terms:

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \phi(w,\bar{w}) \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} \phi(w,\bar{w}) \end{aligned} \tag{2.37}$$

In general, the OPE of fields $A(z)$ and $B(w)$ is

$$A(z)B(w) \sim \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{2.38}$$

where $\{AB\}_n(w)$ is some expression non-singular at $w = z$. In (2.37), for example, $\{T\phi\}_1 = \partial_w \phi(w)$.

Two famous examples of fields are the free boson and the free fermion. Denoted respectively by ∂X and ψ, they are primary fields of dimension $h = 1$ and $h = 1/2$. Their OPEs are

$$\partial X(z)\partial X(w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \qquad \psi(z)\psi(w) \sim \frac{1}{2\pi g}\frac{1}{z-w} \tag{2.39}$$

Note that that the structure of the OPEs reflects the nature of their corresponding field: the bosonic one is invariant under switch of the two fields while the fermionic one has a sign change. Moreover, the OPEs with the two fields with their corresponding energy-momentum tensor are

$$T(z)\partial X(w) \sim \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} \qquad T(z)\psi(w) \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w} \tag{2.40}$$

Finally we compute the OPEs of the two T's with themselves:

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{2.41}$$

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{2.42}$$

where g denotes an unspecified coupling constant of the boson and fermion fields. By comparing (2.41) and (2.42) with (2.37), we see that these energy-momentum tensors are *not* primary fields, because of the term proportional to $1/(z-w)^4$ which does not appear in the OPE of T with primary fields (2.37). We see shortly that the numerator of the additional term is proportional to the *central charge* c.

2.2.6 Central charge

The OPEs of T for the theory of the free boson and the free fermion (2.41, 2.42) can be generalized to

$$\boxed{T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}} \tag{2.43}$$

where c is a model-dependent constant called *central charge*. The central-charge term is what makes T *non-primary*: without that term, we would say that it is primary with $h = 2$.

Transformation of energy-momentum tensor

So T is not primary unless $c = 0$, and all interesting theories have $c > 0$. Let's see what this implies for the transformation of the energy-momentum tensor. Given the Ward identity (2.35), T transforms as

$$\delta T(w) = -\operatorname{Res}[\epsilon(z)\, T(z)\, T(w)] \tag{2.44}$$

$$= -\operatorname{Res}\left[\epsilon(z)\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T(w)}{z-w} + \cdots\right)\right] \tag{2.45}$$

If $\epsilon(z)$ contains no singular terms, we can expand

$$\epsilon(z) = \epsilon(w) + \epsilon'(w)(z-w) + \frac{1}{2}\epsilon''(z-w)^2 + \frac{1}{6}\epsilon'''(w)(z-w)^3 + \cdots \tag{2.46}$$

from which we find

$$\delta T(w) = -\epsilon(w)\,\partial T(w) - 2\epsilon'(w)\,T(w) - \frac{c}{12}\epsilon'''(w) \tag{2.47}$$

This is the infinitesimal version. Under a finite conformal transformation $z \to \tilde{z}(z)$, T is transformed as

$$\tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \left[T(z) - \frac{c}{12}\,S(\tilde{z}, z)\right] \tag{2.48}$$

where $S(\tilde{z}, z)$ is known as the *Schwarzian* and is defined by

$$S(\tilde{z}, z) = \left(\frac{d^3\tilde{z}}{dz^3}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{d^2\tilde{z}}{dz^2}\right)^2\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \tag{2.49}$$

It is simple to check that the Schwarzian has the right infinitesimal form to give (2.47). Its key property is that it preserves the group structure of successive conformal transformations.

Note further that the extra term in the transformation (2.48) is constant for all states: it can be considered as the Casimir energy of the system.

Physical meaning

The central charge can be viewed as a quantum anomaly of the theory, causing a symmetry breaking due to the introduction of a macroscopic length scale. An example is the boundary condition when the CFT is mapped from the plane to a cylinder of length L. The central charge induces a change on the vacuum energy [1]:

$$\langle T_{\text{cyl}} \rangle = -\frac{c\pi^2}{6L^2} \tag{2.50}$$

while $\langle T_{\text{plane}} \rangle = 0$. This is, as we said above, proportional to the Casimir energy.

There are other interpretations of c [1, 3], but we report here the basic idea behind the *c-theorem* stated by Zamolodchikov. For starters, note that we can perturb a CFT by adding an extra term on the action:

$$S = S + \alpha \int \phi(\sigma)\, d^2\sigma \tag{2.51}$$

where ϕ is some operator and α is simply a constant. The effect of the new term depends on the dimension Δ of ϕ.

- If $\Delta < 2$, the parameter α has positive dimension and the perturbation is called *relevant*. Renormalization group (RG) flow causes to move away from the original CFT;
- If $\Delta = 2$, the perturbation is called *marginal* and generates a new CFT;
- If $\Delta > 2$, the ultraviolet region is altered, while the infrared region is described by the original CFT. This type of perturbation is called *irrelevant*.

Naturally, information is lost when we flow from UV region to IR region. The c-theorem states that there exists a function c on the space of all theories which monotonically decreases along RG flows. At the fixed points, theories are conformal, and the function c coincides with the central charge of the conformal theory.

2.3 Quantum conformal field theory

Our discussion has been about fields. We now talk about quantum states. We start by quantizing the theory.

2.3.1 Radial quantization

In quantizing a two-dimensional field theory, it is natural to parameterize the plane by two Cartesian coordinates that we call "time" and "space". However, it is common in CFT to define the theory on a infinite cylinder, where the time t goes along the axis of the cylinder and space around it ($x \in [0, L]$). The map from it to the plane is given by $z = \mathrm{e}^{2\pi\xi/L}$ where $\xi = t + ix$ is the complex coordinate on the cylinder.

2.3.2 Asymptotic states

In order to construct a Hilbert space of states, we have to define a vacuum state $|0\rangle$. We define asymptotic fields

$$\phi_{\text{in}} \propto \lim_{t \to -\infty} \phi(x, t)|0\rangle \iff \phi_{\text{in}} = \lim_{z,\bar{z} \to 0} \phi(z, \bar{z})|0\rangle \tag{2.52}$$

and we define the out state as the Hermitian conjugate of the in state. The Hermitian conjugate in radial quantization is defined as

$$\phi^\dagger(z, \bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \phi(1/\bar{z}, 1/z) \tag{2.53}$$

The out state $\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger$ has a well-defined inner product with $|\phi_{\text{in}}\rangle$. It is

$$\begin{aligned} \langle\phi_{\text{out}} \mid \phi_{\text{in}}\rangle &= \lim_{z,\bar{z},w,\bar{w} \to 0} \langle 0|\phi(z, \bar{z})^\dagger \phi(w, \bar{w})|0\rangle \\ &= \lim_{z,\bar{z},w,\bar{w} \to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\phi(1/\bar{z}, 1/z)\phi(w, \bar{w})|0\rangle \\ &= \lim_{\xi,\bar{\xi} \to \infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0|\phi(\bar{\xi}, \xi)\phi(0, 0)|0\rangle \end{aligned} \tag{2.54}$$

where we defined $\xi = 1/z$. Note that the prefactors in (2.53) render the inner product well defined for $z \to \infty$, according to the form of two-point functions as shown in (2.25). We can interpret the last vacuum expectation value as a correlator since the operators are already time-ordered.

A conformal field ϕ of dimensions $(h, \bar{h})$ can be mode expanded as follows:

$$\phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n} \tag{2.55}$$

where

$$\phi_{m,n} = \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z}\, \bar{z}^{n+\bar{h}-1} \phi(z, \bar{z}) \tag{2.56}$$

On the real surface ($\bar{z} = z^*$) the Hermitian conjugation is straightforward:

$$\phi(z, \bar{z})^\dagger = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi^\dagger_{m,n} \tag{2.57}$$

which is compatible with the conjugation according to the definition (2.53) provided $\phi^\dagger_{m,n} = \phi_{-m,-n}$.

In the following we shall only consider the holomorphic component, thanks to the decoupling of the two characteristic of conformal theories.

$$\phi(z) = \sum_{m\in\mathbb{Z}} z^{-m-h}\phi_m \qquad \phi_m = \frac{1}{2\pi i}\oint dz z^{m+h-1}\phi(z) \tag{2.58}$$

2.3.3 Radial ordering

Within radial quantization, the time ordering is *radial ordering*, explicitly defined by

$$\mathcal{R}\,\Phi_1(z)\Phi_2(w) = \begin{cases} \Phi_1(z)\Phi_2(w) & |z| > |w| \\ \Phi_2(w)\Phi_1(z) & |z| < |w| \end{cases} \tag{2.59}$$

In order for OPEs to have an operatorial meaning the fields must be radially ordered.

We now relate OPEs to commutation relations. Let $a(z)$ and $b(w)$ be two holomorphic fields and consider

$$\oint_w dz a(z)b(w) \tag{2.60}$$

where z goes around w. We further define the contour integrals of $a(z)$ and $b(z)$ as

$$A = \oint a(z)\,dz \qquad B = \oint b(z)\,dz \tag{2.61}$$

We split (2.60) in the following terms:

$$\oint_w dz a(z)b(w) = \oint_{C_1} dz a(z)b(w) + \oint_{C_2} dz b(w)a(z) = [A, b(w)] \tag{2.62}$$

where C_1 and C_2 are circles centered in the origin of radii $|w| + \epsilon$ and $|w| - \epsilon$, respectively. We can add an arbitrary number of fields in the integral, provided $b(w)$ is the only one to have an OPE with $a(z)$ singular at $z \to w$. In practice, integral (2.62) is evaluated by substituting the OPE $a(z)b(w)$, of which only the term in $1/(z-w)$ contributes, by the theorem of residues. We can also write

$$[A, B] = \oint_0 dw \oint dz a(z)b(w) \tag{2.63}$$

where A and B are integrals of a and b around some contour. These formulas relate OPE with commutation relations.

2.3.4 Virasoro algebra

We apply (2.62) and (2.63) to the conformal Ward identity (2.35). We consider an infinitesimal transformation with holomorphic coordinate $\epsilon(z)$ and define

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz \epsilon(z)T(z) \tag{2.64}$$

With (2.62), the Ward identity translates into

$$\delta_\epsilon \Phi(w) = -[Q_\epsilon, \Phi(w)] \tag{2.65}$$

which means that the Q_ϵ is the generator of conformal transformations, the *conformal charge*; cf. Di Francesco, Eq. (2.167).

We expand T according to (2.55).

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z) \tag{2.66}$$

and the same for $\bar{T}(\bar{z})$ and $\bar{L}_n$. We also expand $\epsilon(z)$ as

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\,\epsilon_n \tag{2.67}$$

so putting this into (2.64),

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n\, L_n \tag{2.68}$$

from which we conclude that L_n are generators of local conformal transformations in the Hilbert space, exactly like l_n generate conformal mappings in the function space. Likewise, L_{-1}, L_0 and L_1 generate SL$(2,\mathbb{C})$ in the Hilbert space. Furthermore, $L_0 + \bar{L}_0$ generates dilations $(z,\bar{z}) \to \lambda(z,\bar{z})$, which are equivalent to *time translations* within radial quantization: $L_0 + \bar{L}_0$ is proportional to the Hamiltonian of the system.

While the classical generators of local conformal transformations obey the Witt algebra (2.20), the quantum generators obey the *Virasoro algebra*, which is

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
[L_n, \bar{L}_m] &= 0 \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}
\end{aligned} \tag{2.69}$$

where c is the central charge of the theory. The commutators derive from the T expansion (2.66), the TT OPE (2.43) and the integral (2.63). The vanishing commutator comes from $T(z)\bar{T}(w) \sim 0$.

The vacuum state must be invariant under global conformal transformations: this means that it must be annihilated by L_{-1}, L_0 and L_1. This comes as a particular of the condition that $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ are well-defined as $z, \bar{z} \to 0$, which implies

$$L_n|0\rangle = \bar{L}_n|0\rangle = 0 \qquad n \geq -1 \tag{2.70}$$

and also implies the vanishing of the VEV of the energy-momentum tensor:

$$\langle 0|T(z)|0\rangle = 0 \tag{2.71}$$

Primary fields on the vacuum state produce eigenstates of the Hamiltonian, i.e. $L_0 + \bar{L}_0$. To see this, we write, from the OPE of $T(z)$ with a primary field (2.37), the following:

$$\begin{aligned}
[L_n, \phi(w,\bar{w})] &= \frac{1}{2\pi i}\oint_w dz z^{n+1} T(z)\phi(w,\bar{w}) \\
&= \frac{1}{2\pi i}\oint_w dz z^{n+1}\left[\frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{z-w} + \text{reg.}\right] \\
&= h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \qquad n \geq -1
\end{aligned} \tag{2.72}$$

and similarly for the antiholomorphic counterpart. After applying this to the state $|h,\bar{h}\rangle = \phi(0,0)|0\rangle$ we conclude that

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle \tag{2.73}$$

which indeed means that it is an eigenstate of the Hamiltonian. Likewise, $L_n|h,\bar{h}\rangle = \bar{L}_n|h,\bar{h}\rangle = 0$ for $n > 0$.

Expanding the field in modes according to (2.55) we easily find

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{n+m} \tag{2.74}$$

of which a special case is

$$[L_0, \phi_m] = -m\phi_m \tag{2.75}$$

This means that operators ϕ_m act as ladder operators for the eigenstates of L_0. Applying ϕ_{-m} for $m > 0$ increases the conformal dimension of the state by m. The same holds for L_{-m}, in fact

$$[L_0, L_{-m}] = mL_{-m} \tag{2.76}$$

This implies that excited states are obtained by applying L_k multiple times. The resulting state

$$L_{-k_1} \cdots L_{-k_n}|h\rangle \qquad 1 \le k_1 \le \cdots \le k_n \tag{2.77}$$

will be an eigenstate of L_0 with $h' = h + k_1 + \cdots + k_n = h + N$, where N is called the level. These states are called *descendant* of the state $|h\rangle$.

The subspace of Hilbert space generated by the state $|h\rangle$ and its descendants is called a *Verma module.*

2.3.5 Normal ordering

For free fields, like the free boson or the free fermion, the OPE contains only one singular term, and the regularization can be done simply by subtracting the corresponding VEV (which in terms of modes this is equivalent to the ordinary normal ordering). For other fields, e.g. energy-momentum tensor, subtracting the VEV leaves some singular terms. We need a new definition of normal ordering. Given the general OPE

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{2.78}$$

then the new normal ordering is

$$(AB)(w) = \{AB\}_0(w) \tag{2.79}$$

which is the non-singular term in the OPE. We accordingly define the contraction to include all singular terms

$$\overbrace{A(z)B}(w) \equiv \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{2.80}$$

so that we rewrite the normal ordering as

$$(AB)(w) = \lim_{z \to w} \left[A(z)B(w) - \overbrace{A(z)B}(w) \right] \tag{2.81}$$

and the OPE as

$$A(z)B(w) = \overbrace{A(z)B}(w) + (AB)(w) \tag{2.82}$$

Using contour integrals the normal ordering can be expressed as

$$(AB)(w) = \frac{1}{2\pi i} \oint_w dz \frac{1}{z-w} A(z)B(w) \tag{2.83}$$

As an aside, note that it is possible to mode-expand fields around an arbitrary point w rather than 0, to obtain

$$\phi(z) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-h}\phi_n(w) \tag{2.84}$$

and, in particular,

$$T(z) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-2}L_n(w) \tag{2.85}$$

in which

$$L_n(w) = \frac{1}{2\pi i}\oint_w dz(z-w)^{n+1}T(z) \tag{2.86}$$

This re-writing allows us to have a different expression for the OPE

$$T(z)A(w) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-2}(L_nA)(w) \tag{2.87}$$

The mode version of (2.83) is

$$(AB)_m = \sum_{n\leq -h_A} A_nB_{m-n} + \sum_{n>-h_A} B_{m-n}A_n \tag{2.88}$$

wherein the modes $(AB)_n$ are defined by

$$(AB)(z) = \sum z^{-n-h_A-h_B}(AB)_n \tag{2.89}$$

This new normal ordering requires a new formulation of the Wick theorem.

2.4 Conformal families and operator algebra

2.4.1 Descendant fields

Primary fields are fundamental in CFT. The state $|h\rangle = \phi(0)|0\rangle$ created by a primary field with conformal dimension h gives birth to a tower of states of higher conformal dimensions.

Each descendant state can be viewed as a descendant field acting on the vacuum $|0\rangle$. For example,

$$L_{-n}|h\rangle = L_{-n}\phi(0)|0\rangle = \frac{1}{2\pi i}\oint dz z^{1-n}T(z)\phi(0)|0\rangle \tag{2.90}$$

is equivalent to $(L_n\phi)(0)|0\rangle$ according to (2.87). Descendant states may be obtained by acting on the vacuum with the regular terms in the OPE $T(z)\phi(0)$.

The natural definition of the descendant field $\phi^{(-n)}$ associated with $(L_{-n}\phi)$ is

$$\phi^{(-n)}(w) = (L_{-n}\phi) = \frac{1}{2\pi i}\oint_w dz\frac{1}{(z-w)^{n-1}}T(z)\phi(w) \tag{2.91}$$

The physical properties i.e. the correlation functions of these fields can be derived from the "ancestor" primary field. Indeed, consider the correlator $\langle (L_n\phi)(w)\, X\rangle = \left\langle \phi^{(-n)}(w)\, X\right\rangle$ where X denotes as usual a string of primary fields with conformal dimensions h_i and position w_i. This correlator may be calculated by substituting the definition of the descendant (2.91), in which the contour circles w only, excluding the positions w_i of the other fields. The residue theorem may be applied by reversing the contour and summing the contributions from the poles at w_i, with the help of the OPE (2.37) of T with primary fields:

$$\begin{aligned}\left\langle \phi^{(-n)}(w)X \right\rangle &= \frac{1}{2\pi i}\oint_w dz(z-w)^{1-n}\left\langle T(z)\phi(w)X\right\rangle \\ &= -\frac{1}{2\pi i}\sum_i \oint_{\{w_i\}} dz(z-w)^{1-n}\left\{\frac{1}{z-w_i}\partial_{w_i}\left\langle \phi(w)\,X\right\rangle + \right. \\ &\quad \left. \frac{h_i}{(z-w_i)^2}\left\langle \phi(w)\,X\right\rangle\right\} \\ &\equiv \mathcal{L}_{-n}\left\langle \phi(w)\,X\right\rangle\end{aligned} \tag{2.92}$$

where we have defined $\mathcal{L}$ as the following differential operator:

$$\mathcal{L}_{-n} = \sum_i \left\{\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right\} \tag{2.93}$$

so we conclude that a correlator of descendant fields is simply the correlator of primary fields on which we apply a differential operator.

2.4.2 Conformal families

A *conformal family* is the set comprising a primary field ϕ and its descendants and is denoted $[\phi]$. Since under a conformal transformation fields in $[\phi]$ transform among themselves, we can say that the OPE of $T(z)$ with any member of $[\phi]$ will be composed of other members of $[\phi]$.

For instance, we calculate the OPE of $T(z)$ with $\phi^{(-n)}$. Eq. (2.87) implies

$$T(z)\phi^{(-n)}(w) = \sum_{k\geq 0}(z-w)^{k-2}(L_{-k}\phi^{(-n)})(w) + \sum_{k>0}\frac{1}{(z-w)^{k+2}}(L_k\phi^{(-n)})(w) \tag{2.94}$$

where the first term is the non-singular part and contains the descendants $\phi^{(-k,-n)}$, while the second term contains the singular factors and can be derived by using the ordinary OPE

$$\begin{aligned}T(z)\phi^{(-n)} &= \frac{1}{2\pi i}\oint_w dx\frac{1}{(x-w)^{n-1}}T(z)T(x)\phi(w) \\ &\sim \frac{1}{2\pi i}\oint_w dx\frac{1}{(x-w)^{n-1}}\left\{\frac{c/2}{(z-x)^4} + \frac{2T(x)}{(z-x)^2} + \frac{\partial T(x)}{z-x}\right\}\phi(w) \\ &= \frac{cn(n^2-1)/12}{(z-w)^{n+2}}\phi(w) + \oint_w dx\frac{1}{(x-w)^{n-1}}\sum_{l=0}^{\infty}\phi^{(-l)}(w)\times \\ &\qquad \left\{\frac{2(x-w)^{l-2}}{(z-x)^2} + \frac{(l-2)(x-w)^{l-3}}{z-x}\right\} \\ &= \frac{cn(n^2-1)/12}{(z-w)^{n+2}}\phi(w) + \sum_{l=0}^{n+1}\frac{2n-l}{(z-w)^{n+2-l}}\phi^{(-l)}(w)\end{aligned} \tag{2.95}$$

where we have used the equality

$$\frac{1}{2\pi i}\oint_w dx\frac{1}{(x-w)^n}\frac{F(w)}{(z-x)^m} = \frac{(n+m-2)!}{(n-1)!(m-1)!}\frac{F(w)}{(z-w)^{n+m-1}} \tag{2.96}$$

Finally, we write

$$T(z)\phi^{(-n)}(w) = \frac{cn(n^2-1)/12}{(z-w)^{n+2}}\phi(w) + \sum_{k=1}^{n}\frac{n+k}{(z-w)^{k+2}}\phi^{(k-n)}(w) + \sum_{k\geq 0}(z-w)^{k-2}\phi^{(-k,-n)}(w) \tag{2.97}$$

where we changed the summed index in the second term and used $\phi^{(-k,-n)} = (L_k\phi^{(-n)})$ in the third. This applies, for instance, to $\phi^{(-1)} = \partial\phi$, in the following way

$$T(z)\partial\phi(w) \sim \frac{2h\phi(w)}{(z-w)^3} + \frac{(h+1)\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w} \tag{2.98}$$

Note that descendants of primary fields are *not* primary, and are instead called *secondary fields.* A secondary field $A(z)$ transforms, under $z \to f(z)$, as

$$A(z) \to \left(\frac{df}{dz}\right)^{h'} A(f(z)) + \text{extra terms} \tag{2.99}$$

where $h' = h + N$ if the ancestor has dimension h. The extra terms transforms into poles in the OPE of $A(w)$ with $T(z)$ as is shown above.

2.4.3 Operator algebra

The main object of any field theory are the correlation functions, since they are observable. In conformal field theory, although symmetry provides some constraints, additional dynamical information is needed to determine the correlators. The required information is the *operator algebra*: the OPEs of all primary fields with each other, including regular terms.

Firstly, we have to determine the field normalization, namely the two-point function:

$$\left\langle \phi_\alpha(w,\bar{w})\phi_\beta(z,\bar{z})\right\rangle = \frac{C_{\alpha\beta}}{(w-z)^{2h}(\bar{w}-\bar{z})^{2\bar{h}}} \tag{2.100}$$

Since the coefficients $C_{\alpha\beta}$ are symmetric, we can choose a basis of primary fields such that $C_{\alpha\beta} = \delta_{\alpha\beta}$. Conformal families with different ϕ_α's will be orthogonal in the sense of the two-point functions and this applies also to Verma modules. In fact, we can always, by a global conformal transformation, bring the states to asymptotic positions (e.g. $w = \infty$ and $z = 0$), so that the two-point function becomes a bilinear product; given two fields ϕ and ϕ' we can write

$$\left\langle \phi(w,\bar{w})\phi'(z,\bar{z})\right\rangle \to \lim_{w,\bar{w}\to\infty} w^{2h}\bar{w}^{2\bar{h}}\left\langle \phi(w,\bar{w})\phi'(0,0)\right\rangle = \langle h \mid h'\rangle\langle \bar{h} \mid \bar{h}'\rangle \tag{2.101}$$

thus the orthogonality between the highest state implies the same for all the descendants, and the Verma module as a whole.

Invariance under scaling transformations requires the operator algebra to have the form

$$\phi_1(z,\bar{z})\phi_2(0,0) = \sum_p \sum_{\{k,\bar{k}\}} C_{12}^{p\,\{k,\bar{k}\}}\, z^{h_p-h_1-h_2+K}\, \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2+\bar{K}}\, \phi_p^{\{k,\bar{k}\}}(0,0) \tag{2.102}$$

where K and $\bar{K}$ are the sum of all k_i and $\bar{k}_i$ respectively, and $\{k\}$ means a collection of indices. Basically, $\phi_p^{\{k,\bar{k}\}}$ is a descendant of primary field ϕ_p upon which we acted with lowering operators with all indexes in the set $\{k,\bar{k}\}$.

We take the correlator of this with a third primary field $\phi_r(w,\bar{w})$ of dimensions h_r e $\bar{h}_r$.

$$\begin{aligned}\langle\phi_r|\phi_1(z,\bar{z})|\phi_2\rangle &= \lim_{w,\bar{w}\to\infty} w^{2h_r}\bar{w}^{2\bar{h}_r}\left\langle\phi_r(w,\bar{w})\phi_1(z,\bar{z})\phi_2(0,0)\right\rangle \\ &= \frac{C_{r12}}{z^{h_1+h_2-h_r}\, z^{\bar{h}_1+\bar{h}_2-\bar{h}_r}}\end{aligned} \tag{2.103}$$

where for the last equality we took the limit $w\to\infty$ in the original formula for the three-point function (2.26). On the OPE side, the only contributing term is $p\{k,\bar{k}\}=r\{0,0\}$ (i.e. the primary field ϕ_r) because of the orthogonality of the Verma modules. We conclude that

$$C_{12}^{p\{0,0\}} \equiv C_{12}^{p} = C_{p12} \tag{2.104}$$

Note that the adopted normalization eliminates the distinction between covariant and contravariant indices.

Since the correlators of descendants depends on those of primaries, we have

$$C_{12}^{p\{k,\bar{k}\}} = C_{12}^{p}\,\beta_{12}^{p\{k\}}\beta_{12}^{p\{\bar{k}\}} \tag{2.105}$$

which means that the coefficient of a descendant field is equal to the one for the primary field times functions which depend on the primary field, the other fields in the correlator, and the lowering indices.

Let's do a simple case where $h_1=h_2=h$. When applying (2.102) on the vacuum, we find

$$\phi_1(z,\bar{z})|h,\bar{h}\rangle = \sum_p C_{p12}\, z^{h_p-2h}\,\bar{z}^{\bar{h}_p-2\bar{h}}\varphi(z)\bar{\varphi}(\bar{z})|h_p,\bar{h}_p\rangle \tag{2.106}$$

wherein we introduced

$$\varphi = \sum_{\{k\}} z^K \beta_{12}^{p\{k\}} L_{-k_1}\cdots L_{-k_N} \tag{2.107}$$

and a similar $\bar{\varphi}$, to separate the holomorphic part from the anti-holomorphic part. For the holomorphic part, we define

$$|z,h_p\rangle \equiv \varphi(z)|h_p\rangle = \sum_{N=0}^{\infty} z^N|N,h_p\rangle \tag{2.108}$$

where we wrote it as a series of descendants $|N,h_p\rangle$ at level N belonging to the module $V(h_p)$ with primary field $|h_p\rangle=|0,h_p\rangle$. Thus

$$L_0|N,h_p\rangle = (h_p+N)|N,h_p\rangle \tag{2.109}$$

We now apply L_n on both sides of (2.106) for $n>0$. The l.h.s. yields:

$$\begin{aligned}L_n\phi_1(z,\bar{z})|h,\bar{h}\rangle &= [L_n,\phi_1(z,\bar{z})]|h,\bar{h}\rangle \\ &= (z^{n+1}\partial_z + (n+1)hz^n)\phi_1(z,\bar{z})|h,\bar{h}\rangle\end{aligned} \tag{2.110}$$

where we applied (2.72). The r.h.s. gives

$$\begin{aligned}&\sum_p C_{p12}\, z^{h_p-2h}\,\bar{z}^{\bar{h}_p-2\bar{h}} L_n|z,h_p\rangle|\bar{z},\bar{h}_p\rangle = \\ &\quad\sum_p C_{p12}\, z^{h_p-2h}\,\bar{z}^{\bar{h}_p-2\bar{h}}\left[(h_p+h(n+1))z^n + z^{n+1}\partial_z\right]|z,h_p\rangle|\bar{z},\bar{h}_p\rangle\end{aligned} \tag{2.111}$$

We now substitute the power series (2.108) into the states and obtain

$$L_n|N+n,h_p\rangle = (h_p + (n-1)h + N)|N,h_p\rangle \tag{2.112}$$

This relation, together with the Virasoro algebra, allows the recursive calculation of all the $|N,h_p\rangle$ and hence of all $\beta_{12}^{p\{k\}}$.

Let us calculate explicitly the lowest coefficients. First, we know that, using (2.107) for $z=1$ and $\{k\}=1$,

$$|1,h_p\rangle = \beta_{12}^{p\{1\}} L_{-1}|h_p\rangle \tag{2.113}$$

Applying L_1 on this state yields

$$L_1|1,h_p\rangle = h_p|h_p\rangle = \beta_{12}^{p\{1\}} L_1 L_{-1}|h_p\rangle \tag{2.114}$$

and, since $L_1 L_{-1}|h_p\rangle = [L_1, L_{-1}]|h_p\rangle = 2h_p|h_p\rangle$, we find

$$\beta_{12}^{p\{1\}} = \frac{1}{2} \tag{2.115}$$

Note that we assumed $h_1 = h_2$, otherwise

$$\beta_{12}^{p\{1\}} = \frac{h_p + h_1 - h_2}{2h_p} \tag{2.116}$$

2.4.4 Conformal blocks

Four-point functions can be reduced to three-point functions with the help of the operator algebra (2.102). We will see in detail which part of the correlator is fixed by conformal invariance and which is not. Consider

$$\big\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\big\rangle \tag{2.117}$$

Such a function depends on the ratios

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \qquad \bar{x} = \frac{(\bar{z}_1-\bar{z}_2)(\bar{z}_3-\bar{z}_4)}{(\bar{z}_1-\bar{z}_3)(\bar{z}_2-\bar{z}_4)} \tag{2.118}$$

Since these ratios are invariant under global transformations,we perform one in which $z_4 = 0$, $z_1 = \infty$ and $z_2 = 1$, so that $z_3 = x$ and we can write

$$\lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\big\langle \phi_1(z_1,\bar{z}_1)\phi_2(1,1)\phi_3(x,\bar{x})\phi_4(0,0)\big\rangle = G_{34}^{21}(x,\bar{x}) \tag{2.119}$$

wherein we defined

$$G_{34}^{21}(x,\bar{x}) = \langle h_1,\bar{h}_1|\phi_2(1,1)\phi_3(x,\bar{x})|h_4,\bar{h}_4\rangle \tag{2.120}$$

We have related a four-point function to a matrix element between two asymptotic states. We apply operator algebra (2.102) to compute

$$\phi_3(x,\bar{x})\phi_4(0,0) = \sum_p C_{34}^p x^{h_p-h_3-h_4}\,\bar{x}^{\bar{h}_p-\bar{h}_3-\bar{h}_4}\Psi_p(x,\bar{x}\mid 0,0) \tag{2.121}$$

in which

$$\Psi_p(x,\bar{x}\mid 0,0) = \sum_{\{k,\bar{k}\}} \beta_{34}^{p\{k\}}\bar{\beta}_{34}^{p\{\bar{k}\}} x^K \bar{x}^{\bar{K}} \phi_p^{\{k,\bar{k}\}}(0,0) \tag{2.122}$$

The function $G_{34}^{21}(x,\bar{x})$ now assumes the form

$$G_{34}^{21}(x,\bar{x}) = \sum_p C_{34}^p C_{12}^p A_{34}^{21}(p \mid x,\bar{x}) \tag{2.123}$$

where again we introduced a function:

$$A_{34}^{21}(p \mid x,\bar{x}) = (C_{12}^p)^{-1} x^{h_p-h_3-h_4}\bar{x}^{\bar{h}_p-\bar{h}_3-\bar{h}_4}\langle h_1,\bar{h}_1|\phi_2(1,1)\Psi_p(x,\bar{x}\mid 0,0)|0\rangle \tag{2.124}$$

We have rewritten the four-point function as a sum over intermediate conformal families, labeled by p. This expression has a straightforward diagrammatic interpretation.

This can be factorized into the holomorphic and antiholomorphic part:

$$A_{34}^{21}(p \mid x,\bar{x}) = \mathscr{F}_{34}^{21}(p \mid x)\bar{\mathscr{F}}_{34}^{21}(p \mid \bar{x}) \tag{2.125}$$

where

$$\mathscr{F}_{34}^{21}(p \mid x) = x^{h_p-h_3-h_4}\sum_{\{k\}}\beta_{34}^{p\{k\}} x^K \frac{\langle h_1|\phi_2(1)L_{-k_1}\cdots L_{-k_N}|h_p\rangle}{\langle h_1|\phi_2(1)|h_p\rangle} \tag{2.126}$$

in which the denominator is simply $(C_{21}^p)^{1/2}$, recalling (2.26) for $z=1$. The functions $\mathscr{F}$ are called *conformal blocks*. They can be calculated simply from the knowledge of the conformal dimensions and the central charge, by commuting the Virasoro generators over the field $\phi_2(1)$ one after the other. The field normalizations and coefficients C_{mn}^p drop out of the conformal block at the end of this process.

Going back to the partial wave decomposition (2.123), we see that the conformal blocks represent the element in four-point functions that can be determined from conformal invariance. They depend on the anharmonic ratios through a series expansion. The remaining elements are the three-point function coefficients C_{12}^p and C_{34}^p which are not fixed by conformal invariance. Therefore, the four-point function (2.120) is expressed as

$$\boxed{G_{34}^{21}(x,\bar{x}) = \sum_p C_{34}^p C_{12}^p \mathscr{F}_{34}^{21}(p \mid x)\bar{\mathscr{F}}_{34}^{21}(p \mid \bar{x})} \tag{2.127}$$

An explicit form for the conformal blocks in generally unknown and the application of the formula (2.126) is often tedious.

2.4.5 Crossing symmetry and conformal bootstrap

In defining $G_{34}^{21}(x,\bar{x})$, we have chosen an order for the fields in the correlator. However, except for a sign change in fermions, the order does not matter, and we could have put z_2 to 0 and z_4 to 1. Then $z_3 = 1-x$ and

$$G_{32}^{41}(1-x,1-\bar{x}) = G_{34}^{21}(x,\bar{x}) \tag{2.128}$$

We can analogously interchange ϕ_1 with ϕ_4 to get

$$G_{34}^{21}(x,\bar{x}) = \frac{1}{x^{2h_3}\bar{x}^{2\bar{h}_3}} G_{31}^{24}(1/x,1/\bar{x}) \tag{2.129}$$

All of these are representations of *crossing symmetry*.

We express (2.128) in terms of the conformal blocks

$$\sum_p C_{21}^p C_{34}^p \mathscr{F}_{34}^{21}(p \mid x)\bar{\mathscr{F}}_{34}^{21}(p \mid \bar{x}) = \sum_q C_{41}^q C_{32}^q \mathscr{F}_{32}^{41}(q \mid x)\bar{\mathscr{F}}_{32}^{41}(q \mid \bar{x}) \tag{2.130}$$

Assuming the conformal blocks are known, this equation provides constraints that can determine the coefficients C_{mn}^{p} and the conformal dimension h_p. The procedure of computing correlators based on crossing symmetry is called the *bootstrap approach*. The bootstrap hypothesis is the *sole* "dynamical input" necessary to fully determine the correlators other than the part fixed by conformal invariance. The constraints imposed by crossing symmetry do not exclude interesting theories.

3 | String theory

String theory is a theory that, starting from the principle that the fundamental object are one-dimensional strings, encompasses and unifies all the fundamental interactions, including gravity, providing a description of quantum gravity. In this chapter we introduce some basic, standard concepts of string theory and its quantization. Actually, we will not work with string theory, but it is important to understand where supergravity, its low-energy limit, comes from.

3.1 The relativistic string

In this section we describe a string in special relativity. In string theory, the fundamental objects are the strings, which are one-dimensional objects that can vibrate. Each different mode of vibration gives rise to a different particle.

3.1.1 The relativistic point particle

Let us start from the action of a relativistic point particle, written in a way that makes easy the generalization to a string. We demand that the action be invariant under Lorentz transformations; in all worldlines Γ, the quantity agreed upon by all observers is the *proper time* or, equivalently, the spacetime interval, ds. Let's write it:

$$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 = -\eta_{\mu\nu}\, dx^\mu\, dx^\nu \tag{3.1}$$

To have the correct dimensions, we multiply this by the rest mass times c^2, which is a unit of energy. We now have the action:

$$S = -mc^2 \int_\Gamma d\tau = -mc \int_\Gamma ds \tag{3.2}$$

Other than being Lorentz invariant, S is also *reparameterization invariant*, a feature crucial in string theory. To see this, we express the integrand in terms of the coordinates of the worldline x^μ, which depend on τ. We get

$$S = -mc \int_\Gamma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau \tag{3.3}$$

where we use the convention for which

$$\left(\frac{dx^\mu}{d\tau}\right)^2 = -1 \tag{3.4}$$

We can now check rather easily that S is reparameterization invariant by going from τ to τ': the derivatives change but also the differential transforms accordingly. This action yields the following equations of motion:

$$\frac{dp^\mu}{d\tau} = 0 \tag{3.5}$$

which means that the four-momentum is conserved along the worldline.

3.1.2 The Nambu-Goto action

We proceed to the string by analogy: as a moving particle traces out a line, so a string traces out a surface—the *worldsheet*. As proper time (length) is constant to all observers, the equivalent for a string is the proper area.

We parameterize the worldsheet with coordinates σ and τ: the transverse (space) and longitudinal (time) coordinates, respectively; as such, σ is in $[0, \sigma_1]$ while τ runs along the whole real line. We introduce X^μ, that are the coordinates of the string in spacetime, and their derivatives $\dot{X}^\mu$ and X'^μ with respect to τ and σ, respectively. Finally, the area of the worldsheet is

$$\iint \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}\, d\sigma\, d\tau \tag{3.6}$$

For dimensional consistency, the area must be multiplied by a mass per time or force per velocity to get an action. The natural choice falls upon T/c where T is the string tension; however, it is usually preferred to use $\alpha' = 1/(2\pi\hbar c T)$ which is related to the *string length* ℓ_s by $\ell_\mathrm{s} = \hbar c\sqrt{\alpha'}$. Finally the *Nambu-Goto action* reads

$$S = -\frac{1}{2\pi\alpha'} \iint \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}\, d\sigma\, d\tau \tag{3.7}$$

where $\hbar = c = 1$. Since it is important that this action is reparameterization invariant, it is worth to write it in a manifestly reparameterization-invariant way. To this end, we introduce coordinates $(\xi^1, \xi^2) = (\tau, \sigma)$ and write the induced metric on the worldsheet, which reads

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \tag{3.8}$$

or, in matrix form,

$$\gamma = \begin{pmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{pmatrix} \tag{3.9}$$

It is clear that the radicand in the action is the determinant of γ. Now the desired invariance is manifest:

$$S = -\frac{1}{2\pi\alpha'} \iint \sqrt{-\det\gamma}\, d\xi^1\, d\xi^2 \tag{3.10}$$

which is a form we can generalize easily to describe other objects.

The equations of motion obtained by varying the action w.r.t. X^μ are

$$\frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} = 0 \tag{3.11}$$

where we defined the conjugate momenta:

$$P^\tau_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{3.12}$$

$$P^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{3.13}$$

Actually, the Nambu-Goto action is highly nonlinear in X^μ and thus it is customary to use the *Polyakov action*.

3.1.3 The Polyakov action

The Polyakov action reads

$$S = -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma \sqrt{-g} g^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu\,\eta_{\mu\nu} \tag{3.14}$$

where $g \equiv \det g$ is the determinant of $g_{\alpha\beta}$. This action introduces a new field, $g_{\alpha\beta}$, that can be seen as a dynamical metric on the worldsheet, which will have its own equations of motion. The equation of motion for X^μ is

$$\partial_\alpha(\sqrt{-g} g^{\alpha\beta}\partial_\beta X^\mu) = 0 \tag{3.15}$$

which can be shown to coincide with the equation of motion (3.11) from the Nambu-Goto action, except that now also $g_{\alpha\beta}$ is to be fixed by its equation. Recalling $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\alpha\beta}\delta g^{\alpha\beta} = +\frac{1}{2}\sqrt{-g} g^{\alpha\beta}\delta g_{\alpha\beta}$, we have

$$\delta S = -\frac{T}{2}\int d\tau\, d\sigma \delta g^{\alpha\beta}\left(\sqrt{-g}\,\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta} g^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu\right)\eta_{\mu\nu} = 0 \tag{3.16}$$

which determines that

$$g_{\alpha\beta} = 2f(\tau,\sigma)\,\partial_\alpha X\cdot\partial_\beta X = 2f(\tau,\sigma)\,\gamma_{\alpha\beta} \tag{3.17}$$

Hence the γ and g metrics are related by an arbitrary factor f which drops out of the equation of motion (3.15) for X^μ, concluding that both actions yield the same equation of motion for X^μ. In fact, if we replace the metric in the action (3.14) with its equation (3.17), the f factor drops out again and we recover Nambu-Goto action (3.10).

Note that the Polyakov action possesses, as the Nambu-Goto's, both Poincaré invariance and reparameterization invariance, which is akin to a gauge invariance. Moreover it satisfies the so-called *Weyl invariance*, that is the invariance of the action under the change of the metric by a scale factor:

$$g_{\alpha\beta}(\sigma) \to \Omega^2 g_{\alpha\beta}(\sigma) \tag{3.18}$$

As for f, the Ω^2 factor cancels out in the action between $g^{\alpha\beta}$ and $\sqrt{-g}$.

3.1.4 Solving the equation of motion

The equation of motion (3.15) is pretty complicated for its dependence on g. We can simplify it a bit by fixing a convenient gauge. Thanks to reparameterization invariance, we can set

$$g_{\alpha\beta} = \mathrm{e}^{2\phi}\eta_{\alpha\beta} \tag{3.19}$$

and with a suitable Weyl transformation we obtain $g_{\alpha\beta} = \eta_{\alpha\beta}$, that is, we turned the worldsheet metric into a flat metric. This allows us to greatly simplify the Polyakov action, which reduces to the action of D free scalar fields, that is

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\partial_\alpha X\cdot\partial^\alpha X \tag{3.20}$$

and the equation of motion for X^μ reduces to a wave equation

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{3.21}$$

Note however that we still need to make sure that $g_{\alpha\beta}$ satisfies its equation of motion: we have to compute the variation of the action with respect to the metric (3.16), but this is proportional to the *stress-energy tensor*:

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}} \tag{3.22}$$

so the equation for $g_{\alpha\beta}$ is $T_{\alpha\beta} = 0$, which induces constraints on $\dot{X}$ and X'

$$T_{01} = \dot{X}\cdot X' = 0 \qquad T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \tag{3.23}$$

It is convenient to switch to lightcone coordinates on the worldsheet,[1] so we introduce $\sigma^\pm = \tau \pm \sigma$, and the wave equation becomes

$$\partial_+ \partial_- X^\mu = 0 \tag{3.24}$$

The most general solution of this is

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \tag{3.25}$$

where $X^\mu_L(\sigma^+)$ and $X^\mu_R(\sigma^-)$ are arbitrary functions that describe left-moving and right-moving waves, respectively. This solution obeys both the constraints (3.23) and the periodicity condition

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \tag{3.26}$$

Since it is periodic, it can be expanded in Fourier modes, like

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n\, \mathrm{e}^{-in\sigma^+} \\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n\, \mathrm{e}^{-in\sigma^-} \end{aligned} \tag{3.27}$$

where some normalizations are chosen for later convenience. This mode expansion will come in handy in quantization. Let us make a few remarks.

- X_L and X_R do not individually satisfy the periodicity condition due to the terms linear in $\sigma^\pm$. However, the sum of them is invariant under $\sigma \to \sigma + 2\pi$ as required.
- The variables x^μ and p^μ are the position and momentum of the center of mass of the string. This can be checked, for example, by studying the Noether currents arising from the spacetime translation symmetry $X^\mu \to X^\mu + c^\mu$. One finds that the conserved charge is indeed p^μ.
- Reality of X^μ requires that the coefficients of the Fourier modes, α^μ_n and $\tilde{\alpha}^\mu_n$, obey

$$\alpha^\mu_n = (\alpha^\mu_{-n})^\star \qquad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^\star \tag{3.28}$$

[1] Note that worldsheet lightcone coordinates are different from spacetime lightcone coordinates, that we will define below.

3.1.5 The constraints in lightcone coordinates

The constraints (3.23) in the worldsheet lightcone coordinates become

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \tag{3.29}$$

which in turn give constraints on the momenta p^μ and the Fourier modes. Let's compute the second one using the mode expansion. For starters, we have

$$\partial_- X^\mu = \partial_- X_R^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu \, \mathrm{e}^{-in\sigma^-} \tag{3.30}$$

For consistency we can define the zero mode α_0^μ as $\sqrt{\alpha'/2} p^\mu$, which accounts for the translation of the string. The constraint then reads

$$\begin{aligned} (\partial_- X)^2 &= \frac{\alpha'}{2} \sum_{m,p} \alpha_m \cdot \alpha_p \, \mathrm{e}^{-i(m+p)\sigma^-} \\ &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} \, \mathrm{e}^{-in\sigma^-} \\ &\equiv \alpha' \sum_n L_n \, \mathrm{e}^{-in\sigma^-} = 0 \end{aligned} \tag{3.31}$$

where we have introduced the Virasoro modes L_n, defined as

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m \tag{3.32}$$

The same applies for left-moving modes, where we define analogous $\tilde{L}_n$, and the left-moving zero mode $\tilde{\alpha}_0^\mu$ to be equal to $\alpha_0^\mu = \sqrt{\alpha'/2}\, p^\mu$. The apparently natural fact that the zero modes are equal is going to be crucial in quantization.

The mode-expanded solutions (3.27) must obey the constraints

$$L_n = \tilde{L}_n = 0 \tag{3.33}$$

for all $n \in \mathbb{Z}$. Note that these Virasoro modes are closely related to the Virasoro algebra of conformal field theory.

The Virasoro modes are related to the string effective mass: this is because they contain the spacetime momentum p^μ and $p_\mu p^\mu = -M^2$. The mass is then equal to

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-m} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-m} \tag{3.34}$$

Note that we have two expressions for the mass, one with left-moving modes and one with right-moving modes. They have to be equivalent, which is a condition known as *level matching.*

3.2 Quantum closed strings

There are several ways to quantize the closed string. Two of those are

Covariant quantization we quantize the system and then impose the constraints. This is akin to Gupta-Bleuler quantization of QED

Lightcone quantization we impose the constraints classically and then we quantize only the physical degrees of freedom. In QED, the equivalent way is using the Coulomb gauge.

Of course the two methods should yield the same results. We are going to discuss briefly the first method and switch to the second one when we encounter difficulties.

3.2.1 Covariant quantization

As we said before, covariant quantization means we quantize the theory then impose the constraints

$$\dot{X} \cdot X' = \dot{X}^2 + X'^2 = 0 \tag{3.35}$$

We quantize the quantities X^μ and P_μ by promoting them to operators and determining the usual (equal-time) commutation relations:

$$\begin{aligned} \big[X^\mu(\sigma,\tau), P_\nu(\sigma',\tau)\big] &= i\delta(\sigma-\sigma')\delta^\mu_\nu \\ \big[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\big] &= \big[P_\mu(\sigma,\tau), P_\nu(\sigma',\tau)\big] = 0 \end{aligned} \tag{3.36}$$

Applying the mode expansion (3.27), these translate into relations for x^μ, p^μ and the Fourier modes. The non-zero ones are

$$[x^\mu, p_\nu] = i\delta^\mu{}_\nu \qquad [\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\,\eta^{\mu\nu}\delta_{n+m,\,0} \tag{3.37}$$

We easily notice that the relations of α^μ_n and $\tilde{\alpha}^\mu_n$ are those of harmonic oscillators. Defining

$$a_n = \frac{\alpha_n}{\sqrt{n}} \qquad a^\dagger_n = \frac{\alpha_{-n}}{\sqrt{n}} \tag{3.38}$$

gives the familiar relation $[a_m, a^\dagger_n] = \delta_{mn}$. The α^μ_n are creation operators for $n < 0$ and annihilation operators for $n < 0$. These operators act on the single string and are responsible for adding or subtracting vibrations on the string. This is different from usual creation and annihilation operators in field theory, that create and annihilate particles in spacetime.

We are ready to define string states. The vacuum state is defined as the one annihilated by each α^μ_n for $n > 0$:

$$\alpha^\mu_n\,|0\rangle = \tilde{\alpha}^\mu_n\,|0\rangle = 0 \tag{3.39}$$

As said before, note that this state is the vacuum state of a single string, not of the whole universe like in field theory. The true vacuum state of the universe will be $|0\rangle$, tensored with a spatial wavefunction $\Psi(x)$, accounting for the rest of the universe. Furthermore, in momentum space the vacuum carries another quantum number, p^μ, which is the eigenvalue of the momentum operator. We should therefore write the vacuum as $|0;p\rangle$, which still obeys (3.39), but now also

$$\hat{p}^\mu\,|0;p\rangle = p^\mu|0;p\rangle \tag{3.40}$$

Let us start building up the Fock space with creation operators α^μ_n and $\tilde{\alpha}^\mu_n$ with $n < 0$. A generic state comes from acting with any number of these creation operators on the vacuum,

$$(\alpha^{\mu_1}_{-1})^{n_{\mu_1}}(\alpha^{\mu_2}_{-2})^{n_{\mu_2}}\cdots(\tilde{\alpha}^{\nu_1}_{-1})^{n_{\nu_1}}(\tilde{\alpha}^{\nu_2}_{-2})^{n_{\nu_2}}\ldots|0;p\rangle \tag{3.41}$$

Each of these states corresponds to a different excited state, i.e. a different vibrational mode, of the string. We will see that each corresponds to a different particle; note that since there are infinite ways for a string to vibrate, there are a infinity of particles in this theory. There is a problem with this Fock space, because it does not have positive norm, since X^0 comes with the wrong sign kinetic term in the action (3.20). This may give rise to ghosts. Actually, this problem will be solved in lightcone quantization.

Although we will soon switch to lightcone quantization, we briefly report how we treat constraints in covariant quantization; recall the classical constraints can be written as a condition on the Virasoro modes

$$L_n = \tilde{L}_n = 0 \tag{3.42}$$

where L_n (and similarly $\tilde{L}_n$) are defined in (3.32). As in the Gupta-Bleuler quantization of QED, we simply require that the constraints apply only on matrix elements between two physical states $|A_{\text{phys}}\rangle$ and $|A'_{\text{phys}}\rangle$. Moreover, since $L_n^\dagger = L_{-n}$, it suffices to require

$$L_n|A_{\text{phys}}\rangle = \tilde{L}_n|A_{\text{phys}}\rangle = 0 \quad \text{for } n > 0 \tag{3.43}$$

However, a problem arises in L_0, because there is ambuiguity in the operator ordering, due to the commutation relations (3.37). Commuting the α_n^μ operators past each other in L_0 gives rise to extra constant terms. The problem is that we do not know what order to put the α_n^μ operators, and intuitively different choices lead to different theories. Suppose we want the quantum operators to be normal ordered, with the annihilation operators α_n^i, $n > 0$, moved to the right,

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \frac{1}{2}\,\alpha_0^2 \qquad \tilde{L}_0 = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m + \frac{1}{2}\,\tilde{\alpha}_0^2 \tag{3.44}$$

Then the ambiguity is apparent in the different constraint equations that we could impose, namely

$$(L_0 - a)|A_{\text{phys}}\rangle = (\tilde{L}_0 - a)|A_{\text{phys}}\rangle = 0 \tag{3.45}$$

for some constant a.

As we saw classically in (3.34), the operators L_0 and $\tilde{L}_0$ are physically important because they depend on the momentum and ultimately on the mass. Combining (3.34) with our constraint equation for L_0 and $\tilde{L}_0$, we find the spectrum of the string is given by

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m\right) = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m\right) \tag{3.46}$$

A proper treatment of this ambiguity can be read in [4]. We will now drop the covariant approach and switch to lightcone quantization, where we shall finally conclude the procedure.

3.2.2 Lightcone quantization

Classical constraints and lightcone gauge

We will now go through with *lightcone quantization*. We go back to the classical theory and find the physical degrees of freedom by imposing the classical constraints (3.35).

Even though we already chose a gauge in which the worldsheet metric is $g_{\alpha\beta} = \eta_{\alpha\beta}$, we still have some gauge freedom; for example, any coordinate transformation $\sigma \to \tilde{\sigma}(\sigma)$ which changes the metric by

$$\eta_{\alpha\beta} \to \Omega^2(\sigma)\eta_{\alpha\beta}\ , \tag{3.47}$$

can be undone by a Weyl transformation. We can see this more clearly using lightcone coordinates on the worldsheet, $\sigma^\pm = \tau \pm \sigma$, where the flat metric on the worldsheet takes the form:

$$ds^2 = -d\sigma^+ d\sigma^- \tag{3.48}$$

Then the transformation $\sigma \to \tilde{\sigma}(\sigma)$ translates into

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \qquad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \tag{3.49}$$

which evidently has the effect of (3.47) on the metric. This change can then be compensated by a suitable Weyl transformation, so that the metric remains invariant.

The remaining reparameterization invariance (3.49) has an important physical implication regarding physical degrees of freedom. The equations of motion, $X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$, contain $2D$ functions to be determined. Moreover, the contraints, which, in terms of $\sigma^\pm$, read

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \tag{3.50}$$

reduce the number down to $2(D-1)$ functions. Finally we have to apply the reparameterization invariance (3.49), which is related to how we define $\sigma^\pm$. The physical solutions of the string are therefore actually described by $2(D-2)$ functions. But this counting has a nice interpretation: the degrees of freedom describe the *transverse* fluctuations of the string.

It remains to fix the reparameterization invariance (3.49). We do this with the *lightcone gauge*. First, we define spacetime lightcone coordinates as

$$X^\pm = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}) \tag{3.51}$$

so that the spacetime metric reads

$$ds^2 = -2\, dX^+\, dX^- + \sum_{i=1}^{D-2} dX^i\, dX^i \tag{3.52}$$

Note that this choice of coordinates breaks Lorentz invariance, because it selects a particular time direction and a particular spatial direction. We have to be careful for anomalies, those classical symmetries that cease to be valid in the quantum theory.

The solution to the equation of motion for X^+ reads

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-) \tag{3.53}$$

It is now time to fix the gauge. Thanks to reparameterization invariance, we choose coordinates such that

$$X_L^+ = \frac{1}{2}\, x^+ + \frac{1}{2}\, \alpha' p^+ \sigma^+ \qquad X_R^+ = \frac{1}{2}\, x^+ + \frac{1}{2}\, \alpha' p^+ \sigma^- \tag{3.54}$$

whose sum gives

$$X^+ = x^+ + \alpha' p^+\, \tau \tag{3.55}$$

This is *lightcone gauge*. One might find perplexing the identification of a timelike worldsheet coordinate (τ) with a null spacetime coordinate (X^+). However, except particular cases, this is a valid choice of gauge.

The gauge choice (3.55) fixes the reparameterization invariance (3.49) and makes the constraint equations trivial. Note that the wave equation $\partial_\alpha \partial^\alpha X^\mu$ in the lightcone gauge translates into

$$\partial_+ \partial_- X^- = 0 \tag{3.56}$$

that we can solve by the already used ansatz, $X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-)$, so no more constraints are added other than (3.50). Moreover, note that this gauge and these contraints allows us to completely determine X^- up to an integration constant. If we write the usual mode expansion for $X^-_{L/R}$

$$X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{1}{2}\,\alpha' p^-\,\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n^-\,e^{-in\sigma^+}\ , \tag{3.57}$$

$$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{1}{2}\,\alpha' p^-\,\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^-\,e^{-in\sigma^-}\ . \tag{3.58}$$

where x^- is that integration constant, while p^-, α_n^- and $\tilde{\alpha}_n^-$ are all fixed by the constraints. For example, the oscillator modes α_n^- are given by,

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\alpha_m^i \tag{3.59}$$

Finally we can reconstruct the classical level matching conditions (3.34).

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2}p^ip^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i \tag{3.60}$$

Note here an important consequence of the lightcone gauge: the only oscillators entering the mass definition are α^i and $\tilde{\alpha}^i$ only, with $i = 1, \dots, D-2$ which we'll call *transverse*. The other modes depend on those, which in a sense means that they are the physical excitations of the string.

To sum up, the most general classical solution is described in terms of $2(D-2)$ transverse oscillator modes α_n^i and $\tilde{\alpha}_n^i$, together with a number of zero modes describing the center of mass and momentum of the string: x^i, p^i, p^+ and x^-.

Quantization

We have finally identified the physical degrees of freedom and we are ready to quantize. The commutation relations to impose are

$$[x^i, p^j] = i\delta^{ij} \qquad [x^-, p^+] = -i \qquad [\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\,\delta^{ij}\delta_{n+m,0} \tag{3.61}$$

Moreover, we impose $[x^+, p^-] = -i$ which resembles $[t, H] = -i$ from ordinary quantum mechanics. The Hilbert space of states is very similar to that described in covariant quantization: we define a vacuum state, $|0;p\rangle$ such that

$$\hat{p}^\mu|0;p\rangle = p^\mu|0;p^\mu\rangle \qquad \alpha_n^i|0;p\rangle = \tilde{\alpha}_n^i|0;p\rangle = 0 \quad \text{ for } n > 0 \tag{3.62}$$

and we build a Fock space by acting with the creation operators α_{-n}^i and $\tilde{\alpha}_{-n}^i$ with $n > 0$. The difference with the covariant quantization is that we only act with transverse oscillators which carry a spatial index $i = 1, \dots, D-2$, making the Hilbert space positive definite.

When passing to the quantum theory, in the right-hand side of (3.60) we encounter the same ordering ambiguity of covariant quantization. Normal ordering makes the ambiguity manifest itself in the form of an unfixed constant a. The final result for the mass of states in lightcone gauge is

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i - a\right) \tag{3.63}$$

It is convenient to rename those double sums as N and $\tilde{N}$, respectively, so that

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a) \tag{3.64}$$

It can be shown that a is such that [3]

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right) \tag{3.65}$$

This is the formula that we will use to characterize the mass of the string states.

3.2.3 String spectrum

It is now time to analyze the spectrum of states of a single string.

Ground state

Let's start with the ground state $|0;p\rangle$ defined in (3.62). With no oscillators excited, the mass formula (3.65) gives

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6} \ . \tag{3.66}$$

These states have negative mass squared which seems to be problematic. The corresponding particles are called *tachyons*.

The problem with tachyons is that their mass arises from an expansion around the maximum of the potential for the tachyon field. It is unknown if this potential has a stable minimum. However, the tachyon disappears when we add fermions using supersymmetry.

First excited states

We now look at the first excited states. If we act with a creation operator α^j_{-1}, then the level matching condition (3.64) tells us that we also need to act with a $\tilde{\alpha}^i_{-1}$ operator. This gives us $(D-2)^2$ particle states,

$$\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0;p\rangle \tag{3.67}$$

each of which has mass

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right) \tag{3.68}$$

But now we seem to have a problem. The modes transform in representations of $\mathrm{SO}(D-2)$ but we would like them to fit in a representation of the full Lorentz group $\mathrm{SO}(1, D-1)$. Glossing over the details (see [3]), we conclude that only massless states give representations of the Lorentz symmetry group $\mathrm{SO}(1, D-1)$; this can be achieved only if

$$D = 26 \tag{3.69}$$

that is, if spacetime has 26 dimensions. From (3.65) it follows that $a = 1$.

Therefore, we have found massless states that transform in the $\mathbf{24} \otimes \mathbf{24}$ representation of $\mathrm{SO}(24)$. They decompose into three irreducible representations:

$$\text{traceless symmetric} \oplus \text{anti-symmetric} \oplus \text{singlet (trace)} \tag{3.70}$$

Each of these vibrational modes corresponds to a quantum massless field in spacetime. The fields are:

$$G_{\mu\nu}(X) \qquad B_{\mu\nu}(X) \qquad \Phi(X) \tag{3.71}$$

where the second one is called the "Kalb-Ramond field" or the "2-form", the third is a scalar field called the *dilaton* which is proportional to the *string coupling constant* g_s as $g_\mathrm{s} \sim \mathrm{e}^{\Phi}$. The first one is the most important: it is a massless spin-2 particle, which can be identified with the *graviton*; string theory is a theory of quantum gravity.

As an aside, note that we can put Greek indices $\mu, \nu = 0, \ldots, 25$ instead of Latin (i.e. transverse) ones because actually those additional field modes are eliminated by gauge symmetries which come out in the covariant quantization.

3.3 Open strings

Open strings differ from closed strings for the presence of two endpoints. Let us understand what this difference brings to the theory. The spatial coordinate of the string is now parameterized by

$$\sigma \in [0, \pi] \tag{3.72}$$

Since the dynamics of a generic point on a string is governed by local effects, a single internal point in an open string is no different than a point in a closed string. This means we can keep using the Polyakov action to describe the string dynamics. However, we have to add *boundary conditions* for the endpoints.

3.3.1 Boundary conditions and D-branes

Let us recall the Polyakov action in conformal gauge

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \cdot \partial^\alpha X \tag{3.73}$$

As usual, we derive the equations of motion by finding the extrema of the action. This involves an integration by parts. We denote τ_i and τ_f the initial and final configuration, respectively:

$$\begin{aligned} \delta S &= -\frac{1}{2\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \, \partial_\alpha X \cdot \partial^\alpha \delta X \\ &= \frac{1}{2\pi\alpha'} \int d^2\sigma \; (\partial^\alpha \partial_\alpha X) \cdot \delta X + \text{total derivative} \end{aligned} \tag{3.74}$$

For an open string the total derivative picks up the boundary contributions

$$\frac{1}{2\pi\alpha'} \left[\int_0^\pi d\sigma \dot{X} \cdot \delta X \right]_{\tau=\tau_i}^{\tau=\tau_f} - \frac{1}{2\pi\alpha'} \left[\int_{\tau_i}^{\tau_f} d\tau \; X' \cdot \delta X \right]_{\sigma=0}^{\sigma=\pi} \tag{3.75}$$

The first term vanishes because $\delta X^\mu = 0$ at $\tau = \tau_i$ and τ_f in order to derive the equations of motion. We have to treat carefully the second term, which is exclusive for open strings. To make it vanish, we have to impose

$$\partial_\sigma X^\mu \, \delta X_\mu = 0 \qquad \text{at } \sigma = 0, \pi \tag{3.76}$$

We can do this in two different ways, leading to two different boundary conditions:

Neumann boundary conditions

$$\partial_\sigma X^\mu = 0 \qquad \text{at } \sigma = 0, \pi \tag{3.77}$$

Because there is no restriction on δX^μ, this condition allows the end of the string to move freely.

Dirichlet boundary conditions

$$\delta X^\mu = 0 \qquad \text{at } \sigma = 0, \pi \tag{3.78}$$

This means that the end points of the string lie at some constant position, $X^\mu = c^\mu$, in space.

Intuitively, it is strange that endpoints are bound to some generic point in spacetime. In fact, Joseph Polchinski figured out that the hypersurfaces on which the strings are attached are indeed a whole new type of dynamical object of the theory, that can have an energy and be charged. These hypersurfaces are called *D-branes*, where “D” stands for Dirichlet (it’s not related to the dimensions of spacetime or of the brane itself).

To see this, let’s consider Dirichlet boundary conditions for some coordinates, and Neumann for the others. This means that at both endpoints of the string, we have

$$\partial_\sigma X^a = 0 \qquad \text{for } \; a = 0, \ldots, p \tag{3.79}$$

$$X^I = c^I \qquad \text{for } \; I = p+1, \ldots, D-1 \tag{3.80}$$

This fixes the end-points of the string to lie in a $(p+1)$-dimensional hypersurface in spacetime such that the $\mathrm{SO}(1, D-1)$ Lorentz group is split into

$$\mathrm{SO}(1, D-1) \to \mathrm{SO}(1, p) \times \mathrm{SO}(D-p-1) \tag{3.81}$$

This hypersurface is the D-brane. To denote the dimensions of this brane, we write Dp-brane, so that a D0-brane is a particle, a D1-brane is itself a string, and so on. The brane sits at specific positions c^I in the transverse space. This means it extends in the directions where Neumann condtions are considered.

Note that usually Dp-branes have Neumann boundary conditions on the time direction. Choosing Dirichlet conditions for the time directiom would mean that the object is localized in time. One can define a brane that way, and it is called an D(-1)-brane or a *D-instanton.*

Let us see how these boundary conditions impose relations on the modes of the string. We take the usual mode expansion for the string, with $X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$, and

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2} x^\mu + \alpha' p^\mu \, \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \tilde{\alpha}^\mu_n \, e^{-in\sigma^+} \; , \\ X^\mu_R(\sigma^-) &= \frac{1}{2} x^\mu + \alpha' p^\mu \, \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \alpha^\mu_n \, e^{-in\sigma^-} \end{aligned} \tag{3.82}$$

Let’s see what boundary conditions impose for the modes. Neumann conditions, $\partial_\sigma X^a = 0$ at the end points, require that

$$\alpha^a_n = \tilde{\alpha}^a_n \tag{3.83}$$

while Dirichlet conditions, $X^I = c^I$ at the end points, require that

$$x^I = c^I \qquad p^I = 0 \qquad \alpha^I_n = -\tilde{\alpha}^I_n \tag{3.84}$$

So for both boundary conditions, we only have one set of oscillators and the other one is determined by the boundary conditions.

It's worth pointing out that there is a factor of 2 difference in the p^μ term between the open string (3.82) and the closed string (3.27). This is to ensure that p^μ for the open string retains the interpretation of the spacetime momentum of the string when $\sigma \in [0, \pi]$.

3.3.2 Quantization

As usual, quantization involves promoting the fields x^a and p^a and α_n^μ to operators, while the other elements are derived by the boundary conditions. It is important to point out that the position and momentum degrees of freedom, x^a and p^a, have a spacetime index that takes values $a = 0, \dots, p$, which are the coordinates of the brane. This means that quantum states of an open string are restricted to lie on the brane.

To determine the spectrum, it is again simplest to work in lightcone gauge, with spacetime lightcone coordinates chosen to lie within the brane,

$$X^\pm = \sqrt{\frac{1}{2}}\,(X^0 \pm X^p) \tag{3.85}$$

Quantization now proceeds in the same manner as for the closed string until we arrive at the mass formula for states which is a sum over the transverse modes of the string.

$$M^2 = \frac{1}{\alpha'}\left(\sum_{i=1}^{p-1}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{i=p+1}^{D-1}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a\right) \tag{3.86}$$

The first sum is over modes parallel to the brane, the second over modes perpendicular to the brane. Note that the sum is over α modes only, because the $\tilde{\alpha}$ modes are determined by boundary conditions.

As for the closed string, the Lorentz symmetry for quantum open strings is preserved if and only if $D = 26$ and $a = 1$. The fact that the same numbers come out indicates that open and closed strings are actually different states inside the same theory, rather than different theories altogether. In fact, one can show that a theory of open strings implies closed strings (because interactions can make an open string closed), while the converse is more complicated.

3.3.3 State space

Ground state

The ground state is defined by

$$\alpha_n^i|0;p\rangle = 0 \tag{3.87}$$

for $n > 0$. The spatial index now runs over $i = 1, \dots, p-1, p+1, \dots, D-1$. The ground state has mass

$$M^2 = -\frac{1}{\alpha'} \tag{3.88}$$

It is again tachyonic. The open string tachyon is confined to the brane. As for the closed string tachyon, we can dismiss problems related to this kind of particle because it does not appear in superstring theories.

First excited states

The first excited states are massless. Depending on the coordinate of the oscillator, they fall into two classes.

For $a = 1, \dots, p-1$, oscillators are longitudinal to the brane, and the resulting state

$$\alpha_{-1}^{a}|0;p\rangle \tag{3.89}$$

transforms under the $\mathrm{SO}(p-1)$ little group of a massless particle. It is a spin-1 particle on the brane or, in other words, it is a *photon*. This is remarkable because photons emerge naturally even though we did not impose any kind of electromagnetic gauge invariance in the action. Then we say that we have a gauge field A_a living on the brane.

On the other hand, $I = p+1, \dots, D-1$ oscillators are transverse to the brane, and the states are

$$\alpha_{-1}^{I}|0;p\rangle \tag{3.90}$$

These states are scalars under the $\mathrm{SO}(1,p)$ Lorentz group of the brane. They can be thought of as arising from scalar fields ϕ^I living on the brane. These scalars can be interpreted as fluctuations of the brane in the transverse directions, which is one evidence that the D-brane is a dynamical object. Note that although the ϕ^I are scalar fields under the $\mathrm{SO}(1,p)$ Lorentz group of the brane, they do transform as a vector under the $\mathrm{SO}(D-p-1)$ rotation group transverse to the brane. This appears as a global symmetry on the brane worldvolume.

3.3.4 Multiple branes

The endpoints of an open string can be both on the same D-brane or on different branes. Consider two Dp-branes and a string stretching between the two. Its endpoints' coordinates are $X^I(0,\tau) = c^I$ and $X^I(\pi,\tau) = d^I$, where c^I and d^I are the coordinates of the two branes. The coordinate of the string is

$$X^I = c^I + \frac{(d^I - c^I)\sigma}{\pi} \tag{3.91}$$

The classical constraints $\partial_+ X^I\, \partial_+ X^I = 0$ imply

$$M^2 = \frac{|\boldsymbol{d} - \boldsymbol{c}|^2}{(2\pi\alpha')^2} \tag{3.92}$$

which is interpreted as the mass due to the stretching of the string between the two branes.

This can be easily generalized to coincident N branes: depending on what brane the endpoints lie, there are N^2 different strings, so we have N^2 fields, ϕ and A_μ. They now take the form of matrices: $(\phi^I)^n_m$ and $(A_\mu)^n_m$, where m and n denote the branes on which the endpoints are placed. It is worth to analyze further the gauge field $(A_\mu)^n_m$, which looks like a $\mathrm{U}(N)$ gauge field; in fact, one can show that this is indeed the case. Therefore, N coincident branes give rise to a $\mathrm{U}(N)$ Yang-Mills gauge theory.

3.4 Superstrings

3.4.1 Worldsheet fermions

Since now, we described the position of (bosonic) strings with coordinates $X^\mu(\tau,\sigma)$, which are *commuting* variables. If we want to describe fermions, we have to introduce other dynamical coordinates on the worldsheet, which we denote $\psi_1(\tau,\sigma)$ and $\psi_2(\tau,\sigma)$, although it will be more convenient to consider just one field $\Psi(\tau,\sigma)$. For their fermionic nature, these new coordinates must be *anticommuting.* This can be understood by showing that anticommuting creation operators incorporate automatically the Pauli exclusion principle.

Boundary conditions and the equations of motion determine some constraints on the ψ's, namely, if we fix $\psi_1^I(\tau,0)=\psi_2^I(\tau,0)$, we have

$$\psi_1^I(\tau,\pi)=\pm\psi_2^I(\tau,\pi) \tag{3.93}$$

This choice of sign is physically relevant and defines two distinct types of fermions, called *sectors.* All of this is clearer if we consider a single fermion field, defined as

$$\Psi^I(\tau,\sigma)=\begin{cases}\psi_1^I(\tau,\sigma) & \sigma\in[0,\pi]\\ \psi_2^I(\tau,-\sigma) & \sigma\in[-\pi,0]\end{cases} \tag{3.94}$$

So the condition (3.93) becomes

$$\Psi^I(\tau,\pi)=\pm\Psi^I(\tau,-\pi) \tag{3.95}$$

This distinguishes two sectors: Ramond (R) sector in which the field is periodic (+) and Neveu-Schwarz (NS) sector in which the field in antiperiodic (−).

Neveu-Schwarz sector

The Neveu-Schwarz fermion is expanded as

$$\Psi^I(\tau,\sigma)\sim\sum_{r\in\mathbb{Z}+1/2} b_r^I \mathrm{e}^{-ir(\tau-\sigma)} \tag{3.96}$$

where b_r^I are fractionally moded, anticommuting operators. They obey the relations

$$\left\{b_r^I,b_s^J\right\}=\delta_{r+s,0}\,\delta^{IJ} \tag{3.97}$$

These operators act on the vacuum state $|\mathrm{NS}\rangle$. As for the bosonic operators, the b's are creation operators for $r<0$. The generic superstring state in the NS sector is a tensor product between a generic bosonic state with a generic fermionic NS state. It reads

$$|\lambda\rangle=\prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha_{-n}^I)^{\lambda_{n,I}}\prod_{J=2}^{9}\prod_{r\in\mathbb{Z}+1/2}(b_{-r}^J)^{\rho_{r,J}}|\mathrm{NS}\rangle\otimes|p^+,\boldsymbol{p}_T\rangle \tag{3.98}$$

where $\rho_{r,J}$ is forced by the anticommutation relations to be either zero or one. The mass-squared operator has the form

$$M^2=\frac{1}{\alpha'}\left(\frac{1}{2}\sum_{p\neq0}\alpha_{-p}^I\alpha_p^I+\frac{1}{2}\sum_{r\in\mathbb{Z}+1/2}r\,b_{-r}^Ib_r^I\right) \tag{3.99}$$

The ordering constant a, added when we switch to normal ordering, is to be determined. It turns out that adding fermions to the theory sets the number of spacetime dimensions to $D=10$, which in turn sets a to be $-1/2$. The final mass-squared is then

$$M^2=\frac{1}{\alpha'}\Big(-\frac{1}{2}+N\Big)\qquad\text{where}\qquad N=\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I+\sum_{r\in\mathbb{Z}+1/2}r\,b_{-r}^Ib_r^I \tag{3.100}$$

The states obtained with formula (3.98) can be classified in terms of their number eigenvalue N, or equivalently their mass, and can be either bosonic or fermionic. It can be shown that states with integer N are fermionic while states with non-integer N are bosonic. Note that this refers to the worldsheet. Their nature in spacetime is to be determined.

Ramond sector

The Ramond boundary conditions imply the fermion field is periodic and as such it can be expanded with integer modes:

$$\Psi^I(\tau,\sigma) \sim \sum_{n\in\mathbb{Z}} d_n^I \mathrm{e}^{-in(\tau-\sigma)} \tag{3.101}$$

where, again, $n<0$ denotes creation operators. The satisfied anticommutators are

$$\left\{d_m^I, d_n^J\right\} = \delta_{m+n,0}\,\delta^{IJ} \tag{3.102}$$

Ramond fermions are more complicated than NS fermions, because an integer index means we also have eight zero modes d_0^I. It can be shown that these operators can be linearly combined to form four creation operators and four annihilation operators. The former are denoted $\xi_1,\ldots,\xi_4$. Since they are zero modes, they act on vacuum state but do not contribute to the mass squared. Starting from a unique vacuum state $|0\rangle$, we construct 16 degenerate ground states. In fact, eight of these contain an even number of ξ's (0, 2 and 4) while the remaining eight contain an odd number of ξ's (1 and 3). We shall denote the first group of eight with $|\mathrm{R}_a\rangle$, the second group with $|\mathrm{R}_{\tilde{a}}\rangle$ and the full set with $|\mathrm{R}_A\rangle$. The generic Ramond state takes the form

$$|\lambda\rangle = \prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha_{-n}^I)^{\lambda_{n,I}} \prod_{J=2}^{9}\prod_{m=1}^{\infty}(d_{-m}^J)^{\rho_{m,J}}|\mathrm{R}_A\rangle \otimes |p^+,\boldsymbol{p}_T\rangle \tag{3.103}$$

All eight $|\mathrm{R}_a\rangle$ states are fermionic and all $|\mathrm{R}_{\tilde{a}}\rangle$ states are bosonic. The normal-ordered mass-squared operator for the Ramond sector is

$$M^2 = \frac{1}{\alpha'}\sum_{n\geq 1}\left(\alpha_{-n}^I\alpha_n^I + n\,d_{-n}^I d_n^I\right) \tag{3.104}$$

which implies that all 16 ground states are massless. If we listed all R states in terms of their mass, we would see that bosonic and fermionic states are present in equal number at each level. This is an instance of worldsheet supersymmetry. For future convenience, we label the set of bosonic states with R+ and the set of fermionic ones with R−.

3.4.2 Superstring theories

Open superstrings

To get a right formulation of open superstrings, we need to choose suitable subsets of the R and NS sectors to be properly identified with spacetime bosons and fermions. We use GSO projection: we truncate the R sector in R+ and R− and the NS sector similarly. We shall keep only R− and NS+, which can be shown to correspond to spacetime fermions, and spacetime bosons, respectively. It can be further shown that the result yields *supersymmetry on spacetime*: each bosonic (i.e. NS+) state is matched to a fermionic (i.e. R−) state. Hence the name "superstring theory".

Closed superstrings

Closed superstrings are roughly produced by coupling together two open superstrings, one left-moving and one right-moving. Naively we consider all four possible sectors: (NS, NS), (R, R), (NS, R) and (R, NS). However, in order to get supersymmetric closed strings we truncate and keep only some sectors. Two choices are possible and yield two different superstring theories: type IIA and type IIB.

Type IIA theory This theory is obtained by considering NS+ and R− for the left side and NS+ and R+ for the right side. This choice gives the following sectors: (NS+, NS+), (R−, R+), (NS+, R+) and (R−, NS+). The first two give rise to spacetime bosons and the second two fermions. For instance, if we study the massless states of the (NS+, NS+) sector, we recover a graviton, a Kalb-Ramond field and a dilaton; RR massless fields are instead a Maxwell field A_μ and an antisymmetric gauge field $A_{\mu\nu\rho}$; moreover, this theory contains no tachyons.

Type IIB theory The second theory differs from the first one because it has R− (or R+) in both the left and the right side. We therefore have (NS+, NS+), (NS+, R−), (R−, NS+) and (R−, R−). There are no tachyons as well. The (NS+, NS+) sector yields the same bosons of type IIA, whereas RR bosons are different. Type IIB includes a scalar field A, a Kalb-Ramond field $A_{\mu\nu}$ and a totally antisymmetric gauge field $A_{\mu\nu\rho\sigma}$. The above RR fields are deeply related with the existence of stable (and charged) D-branes in both type II theories; this is in contrast with bosonic theory, where no D-branes are stable.

Other superstring theories Other truncations of (NS, NS), (R, R), (NS, R) and (R, NS) yield different consistent theories albeit not supersymmetric. For example, NS− sector leads to tachyons. These sectors are not the only way to construct superstring theories: we have two *heterotic string theories*, which, as the name suggests ("heterotic" is Greek for "hybrid"), combine a bosonic string with a superstring. The two versions are characterized by the underlying symmetry group: SO(32) or $E_8 \times E_8$. Finally, there is also *type I theory*. Unlike any other theory we discussed, it includes *unoriented* strings, which are strings invariant under an operation that changes orientation. Nowadays, it has been discovered that all these theories are related to each other and in fact can be shown to be different limits of a single theory, called *M-theory*. In addition to strings, it also describes 2-branes and 5-branes (not D-branes).

4 | Supergravity

Now that we have laid down the principles of string theory and superstring theory, we are now ready to discuss supergravity, the low-energy limit of superstring theory. We briefly review the most important supergravity theories in 11 and 10 dimensions. We are mainly interested in type IIB supergravity, which is the theory of interest in the AdS/CFT context.

4.1 Eleven-dimensional supergravity

In $d = 11$ there is a unique supergravity theory: it has the maximum number of supersymmetries, that is 32 real supercharges. As such, all the fields are related to each other via supersymmetry and are contained in a single supermultiplet: the gravitational multiplet. The theory contains a graviton g, a 3-form A_3 and the gravitino, the fermion counterpart of the gravition. There is a U(1) gauge symmetry under which the gravitino is charged and for which A_3 is the gauge field. It transforms as $A_3 \to A_3 + d\Lambda_2$ analogously to the QED four-potential A_1, which transforms as $A_1 \to A_1 + d\Lambda$ where Λ is any scalar function.

The bosonic action contains the usual Einstein-Hilbert term and a kinetic term akin to F^2 from electrodynamics, plus a Chern-Simons term for A_3, which is a topological term required by supersymmetry:

$$S = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} R + F_4 \wedge \star F_4 - \frac{1}{12\kappa^2} \int A_3 \wedge F_4 \wedge F_4 \tag{4.1}$$

where $F_4 = dA_3$, R is the Ricci scalar and κ is proportional to the 11D Newton's gravitational constant G_{11}.

4.2 Type IIA supergravity

Type IIA supergravity is a 10-dimensional theory of supergravity that can be derived as the low-energy limit of type IIA string theory. However, there is another way to obtain this theory, which is by compactifying one dimension of 11D supergravity over a circle of radius R. To get a intuitive understanding of compactification, suppose we have a line, that is a dimension of infinite length; we take an interval $[0, 2\pi R]$, and claim that $x = x + 2\pi R$: the line is now a circumference. We can do this procedure for multiple dimensions at once; in particular, we are going to consider a 10D spacetime where four extended dimensions are compactified on a four-dimensional torus $\mathbb{T}^4$.

Let us perform the compactification. We take coordinate $y = x^{10}$ and curl it around a circle of radius R. We rewrite the 11-dimensional metric as

$$ds^2_{11} = ds^2_{10} + \mathrm{e}^{2\sigma}(dy + C_\mu \, dx^\mu) \tag{4.2}$$

where $\mu = 0, \ldots, 9$ and we have a 10D line element ds_{10}^2 corresponding to the metric $g_{\mu\nu}$. We have also a one-form $C_1 = C_\mu \, dx^\mu$ and a scalar σ related to the dilaton as $\sigma = 2/3\,\Phi$. Upon compactification, the field A_3 can be rewritten as

$$A_3 = B_2 \wedge dy + C_3 \tag{4.3}$$

where two-form B_2 and three-form C_3 are introduced. From the discussion in Sec. 3.4.2, we identify the fields C_1 and C_3 as the RR fields A_μ and $A_{\mu\nu\rho}$, while the remaining ones—$g_{\mu\nu}$, B_2 and Φ—are NSNS fields. The bosonic action of type IIA supergravity is obtained from the 11D action (4.1), plugging in (4.2) and (4.3). We have

$$\begin{aligned} S_{\text{IIA}} = \frac{1}{2\text{G}_{10}^2} \int \left(\mathrm{e}^\sigma R_{10} + \mathrm{e}^\sigma \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} \mathrm{e}^{3\sigma} |F_2|^2 \right) \sqrt{-g}\, d^{10}x - \\ \frac{1}{4\text{G}_{10}^2} \int (\mathrm{e}^{-\sigma} |H_3|^2 + \mathrm{e}^\sigma |\tilde{F}_4|^2) \sqrt{-g}\, d^{10}x - \frac{1}{4\text{G}_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \end{aligned} \tag{4.4}$$

where we defined the field strengths $F_{p+1} = dC_p$ and $H_3 = dB_2$ and $\tilde{F}_4 = F_4 - H_3 \wedge C_1$. Moreover, R_{10} is the Ricci scalar calculated from $g_{\mu\nu}$ and $\text{G}_{10} = \text{G}_{11}/(2\pi R)$ is the ten-dimensional Newton's constant.

Depending on the metric, we have different frames which correspond to different actions. For example, in (4.4), the Einstein-Hilbert action term is not in the familiar form, i.e. $\sqrt{-g}\, R$. To recover that, we move to the so-called Einstein frame, in which we redefine the metric as

$$g_{\text{E}\,\mu\nu} = \mathrm{e}^{\Phi/6} g_{\mu\nu} \tag{4.5}$$

Another useful frame is the string frame, which is the frame in which the type IIA action is written if we derive it from the underlying superstring theory. This is obtained by

$$g_{\text{s}\,\mu\nu} = \mathrm{e}^{2/3\,\Phi} g_{\mu\nu} = \mathrm{e}^{\Phi/2} g_{\text{E}\,\mu\nu} \tag{4.6}$$

In the string frame, the equations of motion for type IIA supergravity read

$$\begin{aligned} &\mathrm{e}^{-2\Phi}(R_{MN} + 2\nabla_M \nabla_N \Phi - \frac{1}{4} H_{MPQ} H_N^{PQ}) - \frac{1}{2} F_{MP} F_N^P - \frac{1}{12} \tilde{F}_{MPQR} \tilde{F}_N^{PQR} + \\ &\quad \frac{1}{4} G_{MN} \left(F_{PQ} F^{PQ} + \frac{1}{6} \tilde{F}_{PQRS} \tilde{F}^{PQRS} \right) \\ &4\, d\star d\Phi - 4\, d\Phi \wedge \star d\Phi + \star R - \frac{1}{2} H_3 \wedge \star H_3 = 0 \\ &d\star(\mathrm{e}^{-2\Phi} H_3) - F_2 \wedge \star \tilde{F}_4 - \tilde{F}_4 \wedge \star \tilde{F}_4 = 0 \\ &d\star F_2 + H_3 \wedge \star \tilde{F}_4 = 0 \\ &d\star \tilde{F}_4 + H_3 \wedge \tilde{F}_4 = 0 \end{aligned} \tag{4.7}$$

The first one is a generalization of the Einstein field equation of general relativity. The other ones are equations of motion for the other fields. Note that the general form of the equations of motion for a given field A in differential-form language is $d\star A = B$ (recall $d\star F = J$ where F is the electromagnetic field strength 2-form and J is the current 3-form).

4.3 Type IIB supergravity

Type IIB supergravity is another 10-dimensional theory that is the low-energy limit of type IIB string theory. Unlike type IIA, it cannot be derived from some higher dimensional theory via compactification; however, it is related to type IIA by a field transformation called *T-duality*, described in Sec. 4.4.

The bosonic content of type IIB supergravity consists in a graviton g_{MN}, a dilaton Φ, a NS-NS form B_2, as well as three RR forms: C_0, C_2 and C_4. Their corresponding field strengths are H_3, F_1, F_3 and F_5, respectively. Moreover, we define two modified field strengths, $\tilde{F}_3 = F_3 - H_3 \wedge C_0$ and $\tilde{F}_5 = F_5 - H_3 \wedge C_2$.

The following Bianchi identities are satisfied:

$$dH_3 = 0, \quad dF_1 = 0, \quad d\tilde{F}_3 = H_3 \wedge F_1, \quad d\tilde{F}_5 = H_3 \wedge F_3 \tag{4.8}$$

The equations of motion for type IIB supergravity are similar to the ones for type IIA (4.7). In the string frame they read

$$\begin{aligned}
&\mathrm{e}^{-2\Phi}(R_{MN} + 2\nabla_M \nabla_N \Phi - \frac{1}{4} H_{MPQ} H_N^{PQ}) + \frac{1}{4} G_{MN} \left(F_P F^P + \frac{1}{6} \tilde{F}_{PQR} \tilde{F}^{PQR} \right) - \\
&\quad \frac{1}{2} F_M F^M - \frac{1}{4} \tilde{F}_{MPQ} \tilde{F}_N^{PQ} - \frac{1}{96} \tilde{F}_{MPQRS} \tilde{F}_M^{PQRS} = 0 \\
&4\, d \star d\Phi - 4\, d\Phi \wedge \star d\Phi + \star R - \frac{1}{2} H_3 \wedge \star H_3 = 0 \\
&d \star (\mathrm{e}^{-2\Phi} H_3) - F_1 \wedge \star \tilde{F}_3 - \tilde{F}_3 \wedge \star \tilde{F}_5 = 0 \\
&d \star F_1 + H_3 \wedge \star \tilde{F}_3 = 0 \\
&d \star \tilde{F}_3 + H_3 \wedge \tilde{F}_5 = 0 \\
&\tilde{F}_5 = \star \tilde{F}_5
\end{aligned} \tag{4.9}$$

Last equation aside, which is imposed by hand, the equations (4.9) are obtained from type IIB action in the string frame,

$$\begin{aligned}
S_{\mathrm{IIB}} = \int \Big[&\mathrm{e}^{-2\Phi} \left(\sqrt{-g}\, R + 4 \star d\Phi \wedge d\Phi - \frac{1}{2} \star H_3 \wedge H_3 \right) - \frac{1}{2} \star F_1 \wedge F_1 - \\
&\frac{1}{2} \star F_3 \wedge F_3 - \frac{1}{4} \star F_5 \wedge F_5 + \frac{1}{2} H_3 \wedge F_3 \wedge C_4 \Big]
\end{aligned} \tag{4.10}$$

4.4 Dualities

As we saw, supergravity theories are not independent from one another; there are transformations, called *dualities*, which map a solution of one theory to a solution of another theory, or even another solution of the same theory.

4.4.1 T-duality

T-duality is a transformation that relates type IIA supergravity and type IIB supergravity compactified on $\mathbb{S}^1$. A string wrapping a circle $\mathbb{S}^1$ of radius R can have winding modes and momentum modes along the circle. The mass due to these modes is proportional to R and $1/R$, respectively. Therefore, T-duality, which swaps R and $1/R$, has the effect of exchanging winding and momentum modes, keeping the overall mass invariant. This means that a theory of strings wrapping a circle of radius R has the same spectrum as a theory of strings wrapping a circle of radius $1/R$. Moreover, they are equivalent at the interacting level too.

If we want to apply T-duality to type IIA supegravity, it is convenient to write the IIA fields in the following way:

$$\begin{aligned} ds^2 &= g_{yy}(dy + A_\mu\, dx^\mu)^2 + \hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu \\ B_2 &= B_{\mu y}\, dx^\mu \wedge (dy + A_\mu\, dx^\mu) + \hat{B}_2 \\ C_p &= C_{(p-1),y} \wedge (dy + A_\mu\, dx^\mu) + \hat{C}_p \end{aligned} \tag{4.11}$$

where $\hat{B}_2$, $C_{(p-1),y}$ and $\hat{C}_p$ are forms with no terms depending on y, which is the coordinate along $\mathbb{S}^1$. Then T-duality yields the following transformations

$$\begin{aligned} ds'^2 &= g_{yy}^{-1}(dy + B_{\mu y}\, dx^\mu)^2 + \hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu \\ \mathrm{e}^{2\Phi'} &= g_{yy}^{-1}\mathrm{e}^{2\Phi} \\ B_2' &= A_\mu\, dx^\mu \wedge dy + \hat{B}_2 \\ C_p' &= \hat{C}_{p-1} \wedge (dy + B_{\mu y}\, dx^\mu) + C_{p,y} \end{aligned} \tag{4.12}$$

We can make sense of these transformations: g_{yy}^{-1} replaces g_{yy} because we swapped R and $1/R$, and A_μ and $B_{\mu y}$ are swapped because the former's charge is momentum and the latter's is winding charge.

4.4.2 S-duality

S-duality is another important transformation that can be applied to supergravity theories. In particular, it relates the weakly and strongly coupled parts of type IIB theory; since the coupling is proportional to e^{Φ}, this is achieved by simply changing sign to the dilaton. Furthermore, B_2 and C_2 are swapped, and the other fields, C_0 and C_4, are left unchanged.

$$\begin{aligned} \Phi' &= -\Phi \\ g'_{\mu\nu} &= \mathrm{e}^{-\Phi} g_{\mu\nu} \\ B_2' &= C_2 \\ C_2' &= -B_2 \end{aligned} \tag{4.13}$$

This duality is useful because, in addition to allowing us to study strongly coupled IIB theory, it also allows to find a new valid solution from another one.

4.5 Branes and charges

We now take a look at how these fields interact with branes and see how these interactions can give branes electric or magnetic charges. We proceed by analogy with the usual electrodynamics case [5].

In classical electrodynamics, the interaction Lagrangian the couples the 4-potential form A with a particle along a worldline Γ is

$$\mathcal{L} = q \int_\Gamma A \tag{4.14}$$

Assuming the existence of magnetic charges, the equations of motion take the form $dF = J_m$ and $d\star F = J_e$ where $F = dA$ and J_e, J_m are 3-forms whose Hodge dual are the electric and magnetic currents, respectively. Then we define electric and magnetic charges as

$$Q_e = \int_{\mathbb{S}^2} \star F \qquad Q_m = \int_{\mathbb{S}^2} F \tag{4.15}$$

Table 4.1: Recap of all supergravity fields and their coupling with the branes.

Theory	Field	Electric brane	Magnetic brane
11D SUGRA	A_3	M2	M5
IIA SUGRA	B_2	F1	NS5
	C_1	D0	D6
	C_3	D2	D4
IIB SUGRA	B_2	F1	NS5
	C_0	–	D7
	C_2	D1	D5
	C_4	D3	D3

where $\mathbb{S}^2$ is a two-sphere enclosing the charge. We can now generalize this argument. Lagrangian (4.14) described the coupling between a 0-dimensional charged object and a 1-form. Let's now consider the coupling between a brane and a form. We take a $(p-1)$-dimensional charged object and a gauge p-form A_p, leading to

$$\mathcal{L} = q_p \int_{\Gamma_p} A_p \tag{4.16}$$

where Γ_p is the worldvolume swept by the $(p-1)$-dimensional object—the higher-dimensional analog of the worldline. We can analogously define F_{p+1} as dA_p and $\star F_{p+1}$. Now (4.15) can be easily generalized:

$$Q_e = \int_{\mathbb{S}^{D-p-1}} \star F_{p+1} \qquad Q_m = \int_{\mathbb{S}^{p+1}} F_{p+1} \tag{4.17}$$

Then we note that any p-form couples electrically to a $(p-1)$-brane and magnetically to a $(D-p-3)$-brane.[1] For example, the 3-form of 11D supergravity, A_3, couples with an electrically charged 2-brane and a magnetically charged 5-brane.

The fields we have encountered and the branes they couple with are schematized in the Table 4.1. "D" branes are the usual Dirichlet branes or D-branes, while F1 is called the fundamental string and NS5 is its magnetic dual. Finally, "M" branes are the branes of M-theory.

Since they will be relevant to construct the following brane solutions, it is worthwhile to report how string dualities act on branes. S-duality (4.13) acts on type IIB theory and swaps B_2 and C_2; by looking at Table 4.1, we notice that this implies the exchange of D1 and D5-branes with F1 and NS5-branes, respectively. T-duality (4.12) is more complicated: since it swaps momentum and winding modes, it has the effect of swapping F1 charge with P charge. Moreover, it acts on the D-branes differently depending on how the coordinate along which it is applied is related to the coordinates of the brane. More precisely, if T-duality is applied along a coordinate perpendicular to the brane, it increases the D-brane dimension by 1; otherwise (i.e. if the coordinate is parallel), the dimension is decreased by 1. All the transformations are schematized in the following:

$$\text{T-duality: F1} \leftrightarrow \text{P} \qquad \text{D}p \xrightarrow{\parallel} \text{D}(p-1) \qquad \text{D}p \xrightarrow{\perp} \text{D}(p+1)$$

$$\text{S-duality: D1} \leftrightarrow \text{F1} \qquad \text{D5} \leftrightarrow \text{NS5}$$

[1]We integrate over a 2-sphere on the transverse coordinates of the brane, where it is seen as a point. The integration space is $D-p-1$ for electric branes and $p+1$ for magnetic branes. Electric branes have thus $D-(D-p-1+2) = p-1$ dimensions and magnetic ones have $D-(p+1+2) = D-p-3$ dimensions [5].

4.6 Brane solutions

We have outlined the basic principles of supergravity theories, and we have introduced branes and gauge fields. We now want to find solutions of the supergravity equations of motion that carry D-branes or the other branes of Table 4.1; we will indifferently use solutions and geometries. Furthermore, we restrict ourselves to Bogomol'nyi–Prasad–Sommerfield states, often simply called *BPS states*. BPS states are those solutions which have the lowest possible mass given a charge. A type of BPS state can be found in the description of Reissner-Nordström black holes, which have both mass and electromagnetic charges. For those black holes gravitational energy is greater or equal than the electromagnetic energy. The BPS Reissner-Nordström black hole is the one whose gravitational energy matches exactly the electromagnetic energy.

Typically, there are two ways to find solutions, that are sometimes referred to as the *direct method* and the *indirect method*.

The direct method involves solving the equations of motion themselves. This is in general complicated, but, since we are interested in BPS solutions that are supersymmetric, the task is simplified.

The indirect method, on the other hand, involves taking a trivial (by which we mean: without charges) solution of the supergravity equations, and get to other solutions by manipulating the trivial one with transformations such as boosts, T-duality and S-duality. These transformations add or change charges. In particular, boosts add momentum charges (P), and the other dualities modify the P charge into F1, NS5 or D charges. Finally, we impose suitable conditions in order to have BPS states.

We now show how to find solutions via the indirect method.

4.6.1 One-charge solution

The trivial solution we are starting with is the generalized Schwarzschild metric in the 10D space $\mathbb{R}^{1,4} \times \mathbb{S}^1 \times \mathbb{T}^4$, where five dimensions (t, x^μ) are non-compact $(\mathbb{R}^{1,4})$, four (z^i) are the coordinates of a torus $\mathbb{T}^4$ and one (y) is compactified along a circle $\mathbb{S}^1$. The other fields, Φ, B_2 and C_p, are put to zero.

$$ds^2_{10} = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega_3^2 + dy^2 + dz_i\, dz^i \tag{4.18}$$

where G = 1. The solid angle Ω_3 is sometimes parameterized using Hopf coordinates, that are

$$\begin{cases} x^1 = r\sin\theta\cos\phi \\ x^2 = r\sin\theta\sin\phi \\ x^3 = r\cos\theta\cos\psi \\ x^4 = r\cos\theta\sin\psi \end{cases} \qquad \theta \in [0, \pi/2], \quad \phi, \psi \in [0, 2\pi] \tag{4.19}$$

We now perform a transformation in order to get a nontrivial (charged) solution. In particular, we do a boost of parameter η along $\mathbb{S}^1$, defined as

$$y \to y\cosh\eta + t\sinh\eta = y' \equiv y \qquad t \to t\cosh\eta + y\sinh\eta = t' \equiv t \tag{4.20}$$

Note that this is a boost along a periodic coordinate, so it is not globally well-defined: it implies an unphysical identification on the time coordinate. This is properly done by applying (4.20) after decompactifying and re-compactify again afterwards. So, the metric (4.18) becomes a new solution:

$$
\begin{aligned}
ds^2_{10} = &\left(1 + \frac{2M\sinh^2\eta}{r^2}\right) dy^2 + \left(-1 + \frac{2M\cosh^2\eta}{r^2}\right) dt^2 + \\
&\frac{2M}{r^2}\sinh(2\eta)\, dy\, dt + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega_3^2 + dy^2 + dz_i\, dz^i
\end{aligned}
\tag{4.21}
$$

This is new solution of type II supergravity. In particular it is a solution of type IIA theory generated by a wave carrying momentum along $\mathbb{S}^1$. Let's call this charge P_y. This is a 1-charge solution. The P charge can be read off from the $dy\, dt$ term: we have $Q = M\sinh 2\eta$. On the other hand, mass is read from the dt^2 term: $m = 2M\cosh^2\eta$, so generally $m \geq Q$. For BPS states, $m = Q$; this is achieved by taking the following limits:

$$
M \to 0 \quad \eta \to \infty \tag{4.22}
$$

so that $M\mathrm{e}^{2\eta} = 2Q$.

However, we want it to change this P charge to get a brane configuration. To this end, we perform a T-duality to get to a F1 charge. This leads to the solution

$$
\begin{aligned}
ds^2 &= \frac{1}{S_\eta}\left[dy^2 + \Big(-1 + \frac{2M}{r^2}\Big)\, dt^2\right] + \Big(1 - \frac{2M}{r^2}\Big)^{-1} dr^2 + r^2\, d\Omega_3^2 + dz_i\, dz^i \\
\mathrm{e}^{2\Phi} &= \frac{1}{S_\eta} \qquad B_2 = \frac{1}{S_\eta}\frac{M}{r^2}\sinh(2\eta)\, dt \wedge dy \qquad C_p = 0
\end{aligned}
\tag{4.23}
$$

where for brevity we introduced the factor

$$
S_\eta = \left(1 + \frac{2M\sinh^2\eta}{r^2}\right) \tag{4.24}
$$

The geometry (4.23) is the solution of type IIB supergravity carrying a F1 charge. We now impose BPS conditions (4.22), that imply

$$
S_\eta \to 1 + \frac{Q}{r^2} = Z(r) \tag{4.25}
$$

Finally, the BPS one-charge solution describing a fundamental string F1 is

$$
\begin{aligned}
ds^2 &= \frac{1}{Z(r)}(dy^2 - dt^2) + dr^2 + r^2\, d\Omega_3^2 + dz_i\, dz^i \\
\mathrm{e}^{2\Phi} &= \frac{1}{Z(r)} \qquad B_2 = -\frac{1}{Z(r)}\, dt \wedge dy \qquad C_p = 0
\end{aligned}
\tag{4.26}
$$

4.6.2 Two-charge solution

Let us now add another charge. Boosts have no effect on BPS solutions because of the term $dy^2 - dt^2$. We then go back to the non-BPS solution with F1 (4.23) on which we perform another boost, parameterized by ξ. It yields:

$$\begin{aligned}
ds^2 &= \frac{S_\xi}{S_\eta}\left(dy + \frac{M\sinh(2\xi)}{r^2+2M\sinh^2\xi}\,dt\right)^2 + \frac{1}{S_\xi S_\eta}\Big(-1+\frac{2M}{r^2}\Big)\,dt^2 + \\
&\qquad \Big(1-\frac{2M}{r^2}\Big)^{-1} dr^2 + r^2\,d\Omega_3^2 + dz_i\,dz^i \\
e^{2\Phi} &= \frac{1}{S_\eta} \\
B_2 &= \frac{1}{S_\eta}\frac{M}{r^2}\sinh(2\eta)\,dt\wedge dy \\
C_p &= 0
\end{aligned} \tag{4.27}$$

This boost produces a fundamental string with momentum along y so we have F1 and P charges. However, we are interested in a configuration with D1 and D5 branes, so we perform the following series of S and T-dualities:

$$\begin{pmatrix}\mathrm{F1}_y\\ \mathrm{P}_y\end{pmatrix} \xrightarrow{S_y} \begin{pmatrix}\mathrm{D1}_y\\ \mathrm{P}_y\end{pmatrix} \xrightarrow{T_{\mathbb{T}^4}} \begin{pmatrix}\mathrm{D5}_{y\mathbb{T}^4}\\ \mathrm{P}_y\end{pmatrix} \xrightarrow{S} \begin{pmatrix}\mathrm{NS5}_{y\mathbb{T}^4}\\ \mathrm{P}_y\end{pmatrix} \xrightarrow{T_y} \begin{pmatrix}\mathrm{NS5}_{y\mathbb{T}^4}\\ \mathrm{F1}_y\end{pmatrix} \xrightarrow{T_{z^1}+S} \begin{pmatrix}\mathrm{D5}_{y\mathbb{T}^4}\\ \mathrm{D1}_y\end{pmatrix} \tag{4.28}$$

Note that the last T-duality has the only effect of going to the type IIB theory, which is necessary in order to apply the final S-duality. We refer to the previous section for details on how the dualities act on branes. After all these transformations, we arrive at the solution

$$\begin{aligned}
ds^2 &= \sqrt{S_\xi S_\eta}\left[dy^2 - \Big(1-\frac{2M}{r^2}\Big)\,dt^2\right] + \\
&\qquad \sqrt{S_\xi S_\eta}\left[\Big(1-\frac{2M}{r^2}\Big)^{-1} dr^2 + r^2\,d\Omega_3^2\right] + \sqrt{\frac{S_\xi}{S_\eta}}\,dz_i\,dz^i \\
e^{2\Phi} &= \frac{S_\xi}{S_\eta} \\
B_2 &= 0 = C_0 = C_4 \\
C_2 &= -\frac{1}{S_\xi}\frac{M}{r^2}\sinh(2\xi)\,dt\wedge dy - f(\theta,\eta,\xi)\,d\phi\wedge d\psi
\end{aligned} \tag{4.29}$$

where $f(\theta,\eta,\xi)$ is a complicated function for non-BPS states and we report it only for BPS states. Since C_2 contains S_ξ in its $dt\wedge dy$ term, and we know from Tab. 4.1 that its electric brane is D1, we identify S_ξ with the D1 charge Q_1 and accordingly S_η with the D5 charge Q_5. Therefore, BPS limit acts as

$$S_\xi \to 1+\frac{Q_1}{r^2} = Z_1(r) \qquad S_\eta \to 1+\frac{Q_5}{r^2} = Z_5(r) \qquad f(\theta,\eta,\xi) \to -Q_5\sin^2\theta \tag{4.30}$$

The BPS two-charge solution is obtained by plugging (4.30) into the solution (4.29). For clarity we drop the dependence on r, and we have

$$\begin{aligned}
ds^2 &= \frac{1}{\sqrt{Z_1 Z_5}}(dy^2 - dt^2) + \sqrt{Z_1 Z_5}(dr^2 + r^2\,d\Omega_3^2) + \sqrt{\frac{Z_1}{Z_5}}\,dz_i\,dz^i \\
e^{2\Phi} &= \frac{Z_1}{Z_5} \\
B_2 &= 0 = C_0 = C_4 \\
C_2 &= -\Big(1-\frac{1}{Z_1}\Big)\,dt\wedge dy + Q_5\sin^2\theta\,d\phi\wedge d\psi
\end{aligned} \tag{4.31}$$

Note that Q_1 and Q_5, which are winding charges related to D1 and D5 branes, come from Q and Q_P, which are the charges related to F1 and P.

Since the solution (4.29) describes the system with D1 and D5-branes, which is the system we want to study, it is worthwhile to derive some important features.

One of the most important aspects to note is that the two charges of the solution, Q_1 and Q_5, are in fact quantized, because they depend on integer parameters, namely the number of D1 and D5-branes. Since the dt^2 coefficient is proportional to mass, and $Z_i = 1 + Q_i/r^2$, we deduce that both Q_i's depend on the mass of the branes. More precisely, since the Q_i's have to have the dimension of a length squared for the Z_i's to be dimensionless, we conclude that, up to a numerical factor,

$$Q_i \sim \mathrm{G}_5\, m_i \tag{4.32}$$

where G_5 is the 5D Newton's constant and has the dimensions of a length cubed while mass' dimension is $[\mathrm{L}]^{-1}$. The dimension of G_d is $[\mathrm{L}]^{d-2}$. The five dimensions of G_5 are the five non-compact dimensions. We now compute the masses m_i of the D-branes. It is

$$m_i = n_i\, T_i\, V_i \tag{4.33}$$

where n_i is the number of Di-branes, T_i is their tension (defined as mass per volume), and V_i is the volume they wrap. The brane tension was derived by Polchinski [6] and can be written as [7]

$$T_1 \sim \frac{1}{g_\mathrm{s}}\,(\alpha')^{-1} \qquad T_5 \sim \frac{1}{g_\mathrm{s}}\,(\alpha')^{-3} \tag{4.34}$$

up to numerical factors; note that the fundamental string has tension $T = 1/(2\pi\alpha')$. The 5-dimensional Newton constant is

$$\mathrm{G}_5 = \frac{\mathrm{G}_{10}}{\mathrm{vol}_{\mathbb{S}^1\times\mathbb{T}^4}} = \frac{\mathrm{G}_{10}}{(2\pi R)\,V_4} \qquad \mathrm{G}_{10} \sim g_\mathrm{s}^2(\alpha')^4 \tag{4.35}$$

The volume factor that relates G_5 with G_{10} comes from requiring that the five-dimensional action has the proper prefactor when it is obtained by dimensionally reducing the action in ten dimensions. As for G_{10}, it has to depend on quantities of the 10-dimensional theory, namely g_s and α'. Specifically, it must be proportional to $\ell_\mathrm{s}^8 = (\alpha')^4$ and g_s^2; the latter intuitively comes from the closed string action, which is proportional to $1/g_\mathrm{s}^2$ as the Einstein-Hilbert action is to $1/\mathrm{G}_{10}$. Putting all together, charges are

$$Q_1 \sim \mathrm{G}_5 n_1 T_1 V_1 = n_1 \frac{g_\mathrm{s}}{V_4}(\alpha')^3 \qquad Q_5 \sim \mathrm{G}_5 n_5 T_5 V_5 = n_5 g_\mathrm{s}\alpha' \tag{4.36}$$

Finally, we make a remark that will be important in the following. We take the limit in which r is smaller than $\sqrt{Q_i}$. Then the Z_i's in (4.30) become simply

$$Z_i = \frac{Q_i}{r^2} \tag{4.37}$$

and if we plug them back into the solution we get the metric of $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$.

$$ds^2 = \frac{r^2}{\sqrt{Q_1\,Q_5}}(dy^2 - dt^2) + \frac{\sqrt{Q_1\,Q_5}}{r^2}\,dr^2 + \sqrt{Q_1\,Q_5}\,d\Omega_3 + \sqrt{\frac{Q_1}{Q_5}}\,ds^2_{\text{torus}} \tag{4.38}$$

Introducing $\hat{r} = \frac{r}{\sqrt{Q_1 Q_5}}$, the metric becomes

$$ds^2 = \sqrt{Q_1\,Q_5}\left[\frac{d\hat{r}}{\hat{r}^2} + \hat{r}^2(dy^2 - dt^2)\right] + \sqrt{Q_1\,Q_5}\,d\Omega_3 + \sqrt{\frac{Q_1}{Q_5}}\,ds^2_{\text{torus}} \tag{4.39}$$

From (4.39) we can read off both the radii of AdS_3 and $\mathbb{S}^3$, which are equal. We find that they depend on the charges, namely as

$$R_{\mathbb{S}^3} = R_{\mathrm{AdS}} = (Q_1 Q_5)^{1/4} \sim (n_1 n_5 g_{\mathrm{s}}^2)^{1/4} \, (\alpha')^{1/2} \tag{4.40}$$

where $V_4 \sim (\alpha')^2$.

We shall work in a regime where $V_4 \sim O(\alpha'^2)$ and $R_{\mathbb{S}^1}^2 \gg \alpha'$, which means we take $\mathbb{T}^4$ to be small with respect to $\mathbb{S}^1$. Furthermore, in order to have a reliable supergravity description, we need a small string coupling and a curvature which is small w.r.t. the string scale. In other words, we require that the AdS radius (4.40), is greater than string scale, i.e. ℓ_{s}.

$$R_{\mathrm{AdS}} \gg \ell_{\mathrm{s}} \sim \sqrt{\alpha'} \implies n_1 n_5 \, g_{\mathrm{s}}^2 \gg 1 \tag{4.41}$$

Finally, we discuss briefly about the generic structure of these solutions, in which we can identify different regions:

- For $r \gg \sqrt{Q_i}$ the geometry becomes *asymptotically flat* and we have Minkowski spacetime.
- For $r \ll \sqrt{Q_i}$ there is the so-called *decoupling* or *near-horizon* region, in which the geometry approaches $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$. This is the region we consider, because the space is AdS and we can apply AdS/CFT.

As an aside, it is worth mentioning that these geometries were introduced as a description of black holes in terms of gravitational microstates. In these microstate geometries, if we decrease r further, we encounter a “cap” region in which there is no singularity. This is the basic idea behind the *fuzzball* conjecture proposed by Samir D. Mathur [8].

5 | AdS/CFT correspondence

Holography is a very powerful tool for studying seemingly unrelated theories. The basic idea of holography is that a theory defined on the bulk of some space is related to a theory living on the boundary of the same space. As an aside, the name “holography” comes from the fact that a hologram allows to represent 3D objects using a 2D image. The most important and useful realization of holography, and the one we have used, is the AdS/CFT correspondence.

The AdS/CFT correspondence is a conjecture introduced by Juan Martín Maldacena in 1997. It relates gravity theories on $(d+1)$-dimensional anti-de Sitter spaces (AdS) with d-dimensional conformal field theories (CFT) that live on the boundary of the AdS space. The fact that makes this correspondence both useful and difficult to prove is that it is a strong/weak duality, in the sense that when one theory is strongly coupled the other one is weakly coupled, and this allows to do calculations on one side that would be otherwise extremely difficult on the other one.

Although it lacks a rigorous mathematical proof, there is a lot of evidence for AdS/CFT and it has been successfully applied in many areas. In this chapter, we first report the most famous clues in favor of it and then what this correspondence entails.

5.1 Evidence

Juan Maldacena [9] and later Edward Witten [10] introduced AdS/CFT in the context of D-branes and black holes in string theory, but there are different ways to motivate this conjecture. We will see the way that led Leonard Susskind to formulate the *holographic principle*, using the entropy of a black hole [11], and the so-called *open/closed string duality*.

5.1.1 Holographic principle

The entropy of a black hole, also called Bekenstein-Hawking entropy, is [12]

$$S_{\text{bh}} = \frac{A}{4\text{G}} \tag{5.1}$$

where A is the area of the event horizon and G is the gravitational constant. One can show that this is the maximal entropy that can be stored in a region of space. In fact, suppose a region of space Σ has more entropy than $\partial\Sigma/4\text{G}$. Naturally, the mass inside Σ must be smaller than the one of a black hole with same area. Let us imagine to add mass to that region so to make a black hole: the entropy on the inside would decrease by hypothesis. In the same way, the entropy on the outside of the region would have decreased, because we removed matter. Then we would have an overall decrease in entropy that violates the second principle of thermodynamics. In a gravitational theory, entropy is bounded by

$$\frac{A}{4\mathrm{G}} \tag{5.2}$$

so it scales with area. This led Susskind to formulate the holographic principle: quantum gravity theories in the bulk space are dependent on physics happening on the boundary of that space.

5.1.2 Open/closed string duality

One of main motivations in favor of the correspondence comes from studying systems of D-branes, such as the ones derived in Sec. 4.6. Let us consider a D3-brane. The spectrum of a D3-brane has scalars ϕ_i, that we interpreted as the transverse fluctuations of the brane, and a U(1) gauge field A_μ. When multiple D-branes are close to each other, a non-Abelian gauge symmetry arises. If we have N close D-branes and open strings that start and end on the same brane, we have a $\mathrm{U}(1)^N$ symmetry, with field $(A_\mu)^a_a$. Open strings whose endpoints are in different branes have massive excitations proportional to the distance between the branes, denoted by r. As discussed in Section 3.3.4, if $r = 0$, that is, if the branes are on top of one another, then the fields $(A_\mu)^a_b$ are all massless and the underlying gauge symmetry becomes $\mathrm{U}(N)$. Analogously, the scalar fields become $N \times N$ matrices labeled by $(\phi_i)^a_b$, transforming in the adjoint representation of the group.

Consider $(9+1)$-dimensional spacetime in which there are N overlapping D3-branes. There are closed strings arising from empty space and open strings as excitations of the D-branes. We take the low-energy limit, in which we have only massless modes and the action can be schematically written as

$$S = S_{\mathrm{bulk}} + S_{\mathrm{brane}} + S_{\mathrm{int}} \tag{5.3}$$

where S_{bulk} is the supergravity action (due to closed string interactions) plus higher-order corrections, S_{brane} is the brane action, and S_{int} is the interaction between the two. As we said above, the branes have gauge fields and scalar fields which transform in the adjoint representation of the gauge group. Then, the action contains the Lagrangian of a Yang-Mills theory, plus higher-order derivative corrections. The Yang-Mills theory is in particular a super Yang-Mills theory with gauge group $\mathrm{U}(N)$ in $(3+1)$-dimensions. It reads [3, 13]

$$\mathcal{L} = -\frac{1}{g_{\mathrm{YM}}^2}\,\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D_\mu\phi^i D^\mu\phi_i + \sum_{i,j}\left[\phi^i,\phi^j\right]\right) \tag{5.4}$$

where the Yang-Mills coupling depends on the string coupling by $g_{\mathrm{YM}}^2 = 4\pi g_{\mathrm{s}}$. Supersymmetry causes the beta-function to vanish, making the theory conformally invariant. Moreover, the group $\mathrm{U}(N)$ can be decomposed into $\mathrm{U}(1) \times \mathrm{SU}(N)$ and we focus on the latter, because U(1) is responsible for the rigid motion of the branes' center of mass, which we do not care about.

In the low-energy limit, where $\ell_{\mathrm{s}} \to 0$ (so $\alpha' \to 0$), all interactions and higher-order terms in both S_{bulk} and S_{brane} vanish. Therefore, we are left with a free supergravity theory in the bulk and a super Yang-Mills theory on the branes.

In the perspective of the closed strings, D3-branes are massive and charged objects, acting as sources of supergravity fields. With the D3-brane, the metric reads

$$ds^2 = \frac{1}{\sqrt{H}}(-\,dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{H}(dr^2 + r^2\,d\Omega_5^2) \tag{5.5}$$

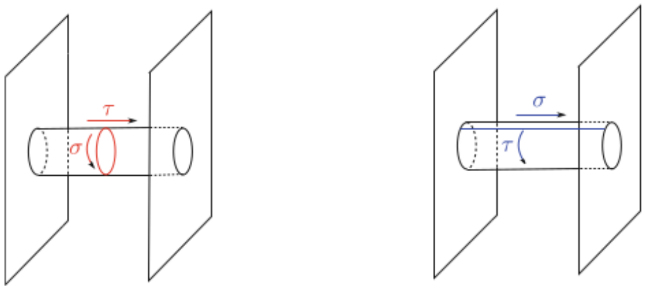

Figure 5.1: Depending on how the worldsheet coordinates σ and τ are defined, one can interpret the same process as an exchange of a closed string between two D-branes (left), or as an open string loop diagram (right).

where $H = 1 + R^4/r^4$ with $R^4 = 4\pi g_s \alpha'^2 N$ and x_1, x_2 and x_3 are the spatial coordinates along which the brane extends. Since the time component of the metric is not zero, the energy of an object changes depending on the position of the observer. The energies as seen at $r = r_0$ and at $r \to \infty$ are related by $E_\infty = H^{-1/4} E_r$. An observer at infinity notices two kinds of low-energy excitations, one from the bulk and one from the near-horizon region. It turns out that these two decouple [13]. We then have free supergravity in the bulk and the near-horizon geometry. In the near-horizon limit, $r \ll R$, we have $H \sim R^4/r^4$, and the metric becomes

$$ds^2 = \frac{r^2}{R^2}(-\,dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}(dr^2 + r^2\, d\Omega_5^2) \tag{5.6}$$

which is $\mathrm{AdS}_5 \times \mathbb{S}^5$ geometry.

We have analyzed the same system from the point of view of both open strings and closed strings. Since string theories are reparameterization invariant, we can swap σ and τ coordinates and note that we can view closed string tree-level processes as open string one-loop processes, as can be seen in Fig. 5.1. This is the basic idea behind *open/closed string duality* and basically AdS/CFT too. Both perspectives feature free bulk supergravity and another theory; since the two descriptions should be equivalent, we are led to consider equal the other two theories, which are Yang-Mills theory (for open strings) and a string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$ geometry (for closed strings).

5.2 Statement

In the previous section, we have shown how a system of D-branes features both a supergravity description from closed strings and field-theoretic description from open strings and under certain conditions they can be decoupled. In his seminal work in 1997 [9], Maldacena proposed that the two descriptions are indeed equivalent in the decoupling limit. More specifically, he proposed the following conjecture:

"$\mathcal{N} = 4$ U(N) super Yang-Mills theory in $(3+1)$ dimensions
is dual to
type IIB supergravity on $\mathrm{AdS}_5 \times \mathbb{S}^5$"

From $g_{\mathrm{YM}}^2 = 4\pi g_s$ and $R^4 = 4\pi g_s(\alpha')^2 N$ we get a relation between the parameters of the two theories:

$$\frac{R^4}{\ell_s^4} \sim g_{\mathrm{YM}}^2 N \tag{5.7}$$

where $g^2_{\rm YM} N$ is also called *'t Hooft coupling*. This relation makes clear a key feature of this correspondence, which is that strong coupling in one side is dual to a weak coupling in the other. In fact, let us study some limit cases.

If the 't Hooft coupling is big, then $R^4 \gg \ell_{\rm s}^4$. This means that the supergravity theory is reliable, but also that the conformal theory is strongly coupled. On the other hand, $g^2_{\rm YM} N \ll 1$ implies the loop expansion in the SYM theory is possible, but also that stringy effects are relevant ($\ell_{\rm s}^4 \gg R^4$) so the supergravity description is not valid.

5.2.1 States and geometries

In string theory spectrum, there are both massive fields and massless fields (or states). From the formulae (3.65) and (3.86), we see that the mass of the states is $m \sim (\alpha')^{-1/2}$. As we said in (4.41), the supergravity limit involves taking the radius to be big with respect to the string scale. This is effectively analogous to sending $\alpha' \to 0$. This implies that string states have their mass approaching infinity, meaning that they can be decoupled from the massless states. The only relevant states in supergravity are the massless ones.

We ask ourselves which operators are dual to these supergravity states. On the field-theory side, relation (5.7) tells us that the condition (4.41) implies $g^2_{\rm YM} N = \lambda \to \infty$. Typically, in an interacting field theory, dimensions of operators are functions of the coupling constant, in this case λ. However, there is a particular class of operators whose dimension is fixed regardless of the coupling constant: they are said to be "protected". In a conformal field theory, these operators are the so-called *chiral primary operators* or CPOs. CPOs are protected because their conformal dimension is $h = j$ and j is, in any conformal theory, a quantized quantum number, which cannot depend on λ. In super Yang Mills theory, it is the index labelling representations of SO(6); in the model we are going to consider, the group is $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ where j is again quantized. We conclude that supergravity fields are dual to chiral primary operators, or descendants of chiral primary operators—operators that are obtained by acting upon the CPOs with symmetry operators.

In the conformal theory, we also distinguish between *heavy states*, which have $h \sim c$ (central charge), and *light states*, which have $h \sim 1$. The heavy states are dual to nontrivial geometries and the light ones are dual to linear deformations around vacuum.

5.2.2 Correlators

Let us talk about how correlators are computed in the AdS/CFT context. Even though we will not compute correlators directly, the basic principle will be useful in the following.

In any quantum field theory, correlators can be encoded in so-called *generating functional*. For any field in the theory, a source J is introduced. We define the generating functional $\mathcal{Z}[J]$ as

$$\mathcal{Z}[J] = \int d\phi \exp\left(-S_{\rm cl}[\phi] + J\phi\right) \tag{5.8}$$

where $S_{\rm cl}$ is is the classical action and ϕ is a field in the theory. Correlators are obtained by taking derivatives of $\mathcal{Z}$ with respect to J. This means that knowing $\mathcal{Z}[J]$ for all sources J determines all the correlation functions in the quantum field theory. We have

$$\langle O_1 \cdots O_n \rangle = \frac{\delta^n \mathcal{Z}[J]}{\delta J_1 \cdots \delta J_n}\Big|_{J_i=0} \tag{5.9}$$

where J_i is the source of ϕ_i, supergravity field dual to the CFT operator O_i. Via AdS/CFT, in the supergravity approximation, which implies from (5.7) the strong coupling regime and the $N \to \infty$ limit, the generating functional can be written as

$$\mathcal{Z}[J] = \exp(-S_{\text{cl}}[\varphi_{\text{b}}]) \tag{5.10}$$

where φ are the classical fields which solve supergravity equations of motion and are fixed by some conditions at the boundary of AdS; specifically, the boundary value φ_{b} is defined to be the source J. Then, correlators are computed by calculating this classical action and differentiate $\mathcal{Z}[J]$ with respect to the boundary value of the field.

To apply this procedure, we consider a classical solution and we expand it at the boundary ($r \to \infty$):

$$\varphi \to r^{d-\Delta}\varphi_{\text{b}} + \cdots + r^{\Delta}A + \cdots \tag{5.11}$$

where Δ is the dimension of the CFT operator $\mathcal{O}$ that is dual to φ. Since the supergravity equations of motions are second-order equations in r, we have to impose another boundary condition besides (5.11), namely a regularity condition for $r \to 0$. The dominant term is the first one, which is proportional to the source J. Fixing the boundary conditions fixes the other terms; in particular the r^{Δ} term's coefficient, A, is

$$A \equiv \langle \mathcal{O} \rangle_{J \equiv \varphi_{\text{b}}} \tag{5.12}$$

that is, the expectation value of the dual operator in presence of the source with the aforementioned regularity condition. The one-point function A is important because it actually encodes the information about all correlators. Indeed, higher-point correlators are obtained by differentiating A with respect to the sources.

5.3 Other realizations of the correspondence

Actually, the relation between $\text{AdS}_5 \times \mathbb{S}^5$ and $\mathcal{N} = 4$ super Yang-Mills theory is just one of the many realizations of the correspondence: the same logic can be applied to any system of D-branes. In this work we consider the bound system made up of D1 and D5 branes, introduced as the two-charge solution in Sec. 4.6, that in a suitable limit lives in $\text{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$. This is called the *D1-D5 system.* The field theory describing this system is a two-dimensional conformal field theory, often referred to as *D1-D5 CFT*.

Even though the duality is conceptually the same for D3-branes ($\text{AdS}_5 \times \mathbb{S}^5$) and D1 and D5-branes ($\text{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$), it is important to point out a difference. In $\text{AdS}_5 \times \mathbb{S}^5$, if one compactifies the type IIB theory on $\mathbb{S}^5$, one finds all fields in AdS_5, and every one of these is dual to some conformal operators. Furthermore, since the ten-dimensional theory has the maximum number of supersymmetries (32 supercharges), all fields are related to each other via supersymmetries transformations; they are said to be in the same *supermultiplet*. A supermultiplet is a set of fields that transform into one another via supersymmetry transformations: starting from one field, e.g. the metric, one can apply any number of supersymmetries and get to any other field in the theory. In this supermultiplet, there are six 10D scalar fields; by compactifying these on $\mathbb{S}^5$, one finds all the CPOs of the dual theory. The D1-D5 theory is different; unlike super Yang-Mills, the full 10D space has a four-dimensional compact region—we took it to be the four-torus, but it can be any 4D compact space (another usual choice is the K3 surface)—so the analog of $\text{AdS}_5 \times \mathbb{S}^5$ here is the 6-dimensional $\text{AdS}_3 \times \mathbb{S}^3$. Type IIB theory on this space has half the supercharges, meaning intuitively that it contains more than one multiplet. Starting from the metric as before, we get to a number of fields that make up the so-called *gravitational multiplet*, but a lot of fields are left out. The remaining fields are organized in the so-called *tensor multiplets*. The four-torus has 5 of these multiplets (K3 has 21). If each of these multiplets are compactified on $\mathbb{S}^3$, there are CPOs. In particular, CPOs with lowest dimensions—$(1/2, 1/2)$—correspond to the compactification on the smallest spherical harmonics. There are five of these CPOs of dimension $(1/2, 1/2)$, one for each supermultiplet: these same-dimension CPOs are said to have different "flavours". The presence of multiple flavours makes the D1-D5 system more complicated than the D3-system.

6 | D1-D5 system

The D1-D5 system is the type IIB geometry on $\mathbb{R}^{1,4} \times \mathbb{S}^1 \times \mathbb{T}^4$ with n_1 D1-branes wrapping $\mathbb{S}^1$ and n_5 D5-branes wrapping $\mathbb{T}^4$; we will denote $N = n_1 n_5$. In the regime of interest, the torus is smaller than the circle; in terms of α', we assume $V_4 \sim (\alpha')^2$ and $R^2_{\mathbb{S}^1} \gg \alpha'$.

In this chapter we discuss in some detail the D1-D5 system. First we outline its corresponding conformal theory, then we report the dual supergravity description.

6.1 D1-D5 conformal field theory

In this section we describe the D1-D5 system in terms of the conformal field theory, referred to as the *D1-D5 CFT*. Since this system breaks $\frac{1}{4}$ supersymmetries, this theory will be a $\mathcal{N} = (4, 4)$ superconformal field theory (SCFT) with 8 supercharges. Moreover, since it lives on the boundary of AdS_3, it is two-dimensional. Finally, we can show that the central charge is related to AdS radius (4.40) as

$$c = \frac{3R_{\mathrm{AdS}}}{2\mathrm{G}_3} \tag{6.1}$$

where R_{AdS} is the AdS radius and G_3 is the Newton gravitational constant in three dimensions [14].

There are two main descriptions of the D1-D5 system as a field theory. We are going to briefly outline both approaches, but it turns out that the second one is more precise and less complicated to handle. An extensive description of the two ways can be found in [14, 15].

In the spirit of the open/closed string duality, one possibility is to consider the field theory of open strings originating from D-branes. Depending on what brane they attach to, open strings are of three different kinds:

- 1-1 strings. Strings that start and end on D1-branes: they give rise to a $\mathrm{U}(n_1)$ gauge theory with 16 supercharges;
- 5-5 strings. Strings that start and end on D5-branes: they give rise to a $\mathrm{U}(n_5)$ gauge theory with 16 supercharges;
- 1-5 and 5-1 strings. Strings that start on D1-branes and end on D5-branes or vice versa. They transform under the fundamental representation of $\mathrm{U}(n_5)$, and under the antifundamental representation of $\mathrm{U}(n_1)$. They break the number of supersymmetries of the theory down to 8.

Since we have taken the four-torus to be small w.r.t. the one-sphere, the theory is effectively reduced to a $(1 + 1)$-dimensional theory parameterized by time t and the $\mathbb{S}^1$ coordinate y.

Table 6.1: Symmetries of the two descriptions of the D1-D5 system.

Supergravity	CFT	Symmetry group
AdS_3	L_{-1}, L_0, L_{+1}	$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$
$\mathbb{S}^3$	R-symmetry	$\mathrm{SO}(4)_E \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$
$\mathbb{T}^4$	outer automorphism	$\mathrm{SO}(4)_I \simeq \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$

Because we are discussing the low-energy limit, we are interested in the supersymmetric minima of the theory. If we consider the potential (see [14] for an explicit expression), one can show that there are two classes of minima, selecting two different regions of the moduli space of the theory. These classes are

Coulomb branch The adjoint scalars of the 1-1 and 5-5 strings acquire a non-zero vacuum expectation value, causing the separation of the branes and the breaking of the gauge symmetry.

Higgs branch The gauge fields of the 1-1 and 5-5 strings that parameterize the displacement of D1-branes inside D5-branes acquire a non-zero VEV. In this case, there is no separation of the branes which instead form a bound state.

The Higgs branch is what we want, because we are interested in bound states. However, as said before, this approach makes it complicated to study the theory. Let us now see the other way.

The second approach consists in considering D1-branes as instantonic solutions of the $\mathrm{U}(n_5)$ gauge theory of the D5-branes. That is, D1-branes (strings) wrapping $\mathbb{S}^1$ but localized in $\mathbb{T}^4$. We are interested in describing n_1 instantons in the D5-brane theory. They give rise to a family of solutions whose parameters form the instanton moduli space, since the D1-branes have a $(1+1)$-dimensional worldvolume. This space will be the target space of the theory we want, that is a $(1+1)$-dimensional sigma model. The structure of this moduli space is generally complicated but we are going to pick a particular choice of parameters that allows to consider the theory in the so-called *orbifold point*, where the target space is

$$\frac{(\mathbb{T}^4)^N}{S_N} \tag{6.2}$$

where S_N is the permutation group of order $N = n_1 n_5$. In the orbifold point, the CFT is the theory of a collection of $N = n_1 n_5$ strings wrapping $\mathbb{S}^1$ with $\mathbb{T}^4$ as the target space. The permutation group is there to keep account of the fact that two configurations with some strings exchanged are in fact equivalent.

Note that strings can wrap the circle more than once: when going around the circle $\mathbb{S}^1$, one can end in a different copy of $\mathbb{T}^4$. We will identify two sectors: the *untwisted sector*, where all N strings wrap the circle only once, and the *twisted sector*, where some strings wrap $\mathbb{S}^1$ multiple times; sometimes multi-wound strings are called “strands”. Regardless, considering m_i strings with winding w_i, the sum

$$\sum_i m_i w_i = N \tag{6.3}$$

must hold.

Finally, it remains to discuss the symmetries of the two theories, and verify that they match. The supergravity near horizon geometry lives on $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$, corresponding to the following isometry groups: $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ for AdS_3, $\mathrm{SO}(4)_E$ for $\mathbb{S}^3$ and $\mathrm{SO}(4)_I$ for

$\mathbb{T}^4$, which is broken due to compactification. On the CFT side, the theory is generated by the Virasoro generators L_n and $\bar{L}_n$, whose global subalgebra matches $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Furthermore, it has a R-symmetry based on SO(4) which we can identify with $\mathrm{SO}(4)_E$ for $\mathbb{S}^3$. R-symmetry is a symmetry that relates supercharges with each other. Finally the torus $\mathrm{SO}(4)_I$ is associated to another SO(4) symmetry—giving an outer automorphism of the superconformal algebra [16, 17]. The symmetries are schematized in Table 6.1.

6.1.1 Field content

We now list the fields present in this theory; in the following we consider for simplicity the holomorphic or left-moving sector, because the generalization to the antiholomorphic or right sector is straightforward. As usual we have the stress-energy tensor $T(z)$; then there are four (fermionic) currents $G^{\alpha A}(z)$ due to supersymmetry, and three (bosonic) currents $J^a(z)$ because of R-symmetry.

It is more convenient to express both SO(4) groups as $\mathrm{SU}(2) \times \mathrm{SU}(2)$, namely $\mathrm{SO}(4)_E \sim \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and $\mathrm{SO}(4)_I \sim \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$. We introduce the following conventions for the indices:

$$\begin{aligned} &\alpha, \beta \leftrightarrow \mathrm{SU}(2)_L \qquad \dot{\alpha}, \dot{\beta} \leftrightarrow \mathrm{SU}(2)_L \\ &A, B \leftrightarrow \mathrm{SU}(2)_1 \qquad \dot{A}, \dot{B} \leftrightarrow \mathrm{SU}(2)_2 \end{aligned} \tag{6.4}$$

where $\alpha, \dot{\alpha} = \pm$ and $A, \dot{A} = 1, 2$.

We can view the CFT states as $N = n_1 n_5$ copies of the theory on $\mathbb{T}^4$, or strands, each of which contains four free bosons and four free fermions. Since each boson contribute 1 to the central charge and each fermion contribute 1/2, the central charge of one copy is 6 and the total one is

$$c = 6 n_1 n_5 = 6N \tag{6.5}$$

The boson fields are denoted $X_r^{A\dot{A}}(z)$ and the fermions are $\psi_r^{\alpha\dot{A}}(z)$ where $r = 1, \ldots, N$ labels the copy. It will be better to consider ∂X's as the bosonic fields. The OPEs involving these fields are

$$\begin{aligned} \psi_r^{1\dot{A}}(z)\, \psi_s^{2\dot{B}}(w) &\sim \frac{\varepsilon^{AB}\delta_{rs}}{z-w} \\ \partial X_r^{A\dot{A}}(z)\, \partial X_s^{B\dot{B}}(w) &\sim \frac{\varepsilon^{AB}\varepsilon^{\dot{A}\dot{B}}\delta_{rs}}{(z-w)^2} \end{aligned} \tag{6.6}$$

where $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{12} = -\varepsilon^{\dot{1}\dot{2}} = +1$ and $\sim$ restricts to the singular part of the OPE.

From these fields we can write the currents that generate the SCFT. First, the total stress-energy tensor is the sum of the contributions from every field in one copy summed again to consider all N copies. For one copy we have

$$T_r(z) = T_r^{\mathrm{bos}}(z) + T_r^{\mathrm{fer}}(z) = \frac{1}{2}\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}} :\!\partial X_r^{A\dot{A}} \partial X_r^{B\dot{B}}\!: + \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot{A}\dot{B}} :\!\psi_r^{A\dot{A}} \psi_r^{B\dot{B}}\!: \tag{6.7}$$

The R-symmetry currents are

$$J_r^+ = \frac{1}{2}\varepsilon_{\dot{A}\dot{B}} :\!\psi_r^{1\dot{A}}\, \psi_r^{1\dot{B}}\!: \tag{6.8}$$

$$J_r^- = -\frac{1}{2}\varepsilon_{\dot{A}\dot{B}} :\!\psi_r^{2\dot{A}}\, \psi_r^{2\dot{B}}\!: \tag{6.9}$$

$$J_r^3 = -\frac{1}{2}\left(\varepsilon_{\dot{A}\dot{B}} :\!\psi_r^{1\dot{A}} \psi_r^{1\dot{B}}\!: - 1\right) \tag{6.10}$$

which we collectively denote with J^a_r. Finally we have the supercurrents

$$G^{\alpha A}(z) = \sum_{r=1}^{N} \psi_r^{\alpha\dot{A}} \partial X_r^{\dot{B}A} \varepsilon_{\dot{A}\dot{B}} \tag{6.11}$$

The OPEs for bosons and fermions (6.6) help us figure out the OPEs involving these currents.

$$\begin{aligned}
&T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\
&G^{\alpha A}(z)G^{\beta B}(z) \sim -\frac{c}{3}\frac{\varepsilon^{\alpha\beta}\varepsilon^{AB}}{(z-w)^3} + \varepsilon^{AB}\varepsilon^{\beta\gamma}(\sigma^{*\,a})^{\alpha}_{\gamma}\left[\frac{2J^a(w)}{(z-w)^3} + \frac{\partial J^a(w)}{z-w}\right] - \\
&\qquad\qquad \varepsilon^{\alpha\beta}\varepsilon^{AB}\,\frac{T(w)}{z-w} \\
&J^a(z)J^b(w) \sim \frac{c}{12}\frac{\delta^{ab}}{(z-w)^2} + i\varepsilon^{ab}_{\ \ c}\frac{J^c(w)}{z-w} \\
&T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \\
&T(z)G^{\alpha A}(w) \sim \frac{3}{2}\frac{G^{\alpha A}(w)}{(z-w)^2} + \frac{\partial G^{\alpha A}(w)}{z-w} \\
&J^a(z)G^{\alpha A}(w) \sim \frac{1}{2}(\sigma^{*\,a})^{\alpha}_{\beta}\,\frac{G^{\beta A}(w)}{z-w}
\end{aligned} \tag{6.12}$$

Finally, we can write the commutation relations that define the superconformal algebra. Let us denote the mode coefficients of the currents with L_n, $G^{\alpha A}_n$ and J^a_n and we get

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
&[J^a_m, J^a_n] = \frac{c}{12}\, m\, \delta^{ab}_{m+n,0} + i\varepsilon^{abc}J^c_{m+n} \\
&\left\{G^{\alpha A}_m, G^{\beta B}_n\right\} = -\frac{c}{6}\Big(m^2 - \frac{1}{4}\Big)\varepsilon^{\alpha\beta}\varepsilon^{AB}\delta_{m+n,0} + \\
&\qquad\qquad (m-n)\varepsilon^{AB}\varepsilon^{\beta\gamma}(\sigma^{*\,a})^{\alpha}_{\gamma}J^a_{m+n} - \varepsilon^{AB}\varepsilon^{\alpha\beta}L_{m+n} \\
&\left[J^a_m, G^{\alpha A}_n\right] = \frac{1}{2}(\sigma^{*\,a})^{\alpha}_{\beta}G^{\beta A}_{m+n} \\
&[L_m, J^a_n] = -nJ^a_{m+n} \\
&\left[L_m, G^{\alpha A}_n\right] = -\Big(\frac{m}{2} - n\Big)G^{\alpha A}_{m+n}
\end{aligned} \tag{6.13}$$

This algebra has a well-defined finite global subalgebra, which is generated by the subset

$$\{L_0, L_\pm, J^a_0, G^{\alpha A}_{\pm 1/2}\} \tag{6.14}$$

This global algebra does not depend on c. Its Cartan subalgebra is spanned by L_0 and J^3_0, so the states can be classified in terms of their eigenvalues h and m.

6.1.2 States of the untwisted sector

In this work, we are going to focus mainly on the untwisted sector, where strings wrap $\mathbb{S}^1$ just once. It is possible to switch to the twisted sector, where strings are wound around $\mathbb{S}^1$ multiple times, via so-called *twist operators*, one of which is denoted Σ_2 and will be introduced in the following.

Irreducible representations of the algebra (6.13) are constructed by choosing a highest-weight state, called primary, and acting upon it with negative modes in order to build a set of descendants. Depending on the generator, we identify three kinds of primary states. A state $|P\rangle$ is Virasoro primary if it is annihilated by L_n $(n > 0)$, affine primary if it is annihilated by J^a_n $(n > 0)$, and chiral if it is annihilated by G^{+A}_r $(r \geq -1/2)$ and chiral primary if it is both Virasoro primary and chiral.

Vacuum states

Since strings are singly wound, we have to impose boundary conditions in order to make sure that for $\sigma \to \sigma + 2\pi$ we get back to the same copy. On the plane it corresponds to $z \to \mathrm{e}^{i(\tau+r)}z$. For the bosons we have

$$\partial X^{A\dot{A}}_r(\mathrm{e}^{2\pi i}z) = \partial X^{A\dot{A}}_r(z) \tag{6.15}$$

This implies the bosonic field can be mode-expanded as

$$\partial X^{A\dot{A}}_r(z) = \sum_{n\in\mathbb{Z}} \alpha^{A\dot{A}}_{r\,n}\, z^{-n-1} \tag{6.16}$$

Fermions, as in superstring theory, can have periodic or antiperiodic boundary conditions on the cylinder. This distinguishes two different types, or sectors, of fermions: Ramond and Neveu-Schwarz, respectively. On the plane the periodicity is reversed and we have:

$$\psi^{\alpha\dot{A}}_r(\mathrm{e}^{2\pi i}z) = -\psi^{\alpha\dot{A}}_r(z) \qquad \text{R sector} \tag{6.17}$$

$$\psi^{\alpha\dot{A}}_r(\mathrm{e}^{2\pi i}z) = \psi^{\alpha\dot{A}}_r(z) \qquad \text{NS sector} \tag{6.18}$$

Let's consider NS sector first. The mode expansion yields

$$\psi^{\alpha\dot{A}}_r(z) = \sum_{n\in\mathbb{Z}+1/2} \psi^{\alpha\dot{A}}_{r\,n}\, z^{-n-1/2} \tag{6.19}$$

We now build the NS vacuum state. One can show that the state with lower energy lies in the NS sector. For a given copy, the vacuum state is made up of a tensor product between the bosonic vacuum state and the fermionic vacuum state. Both have a holomorphic and an antiholomorphic part, but they commute with each other so we can consider just the holomorphic part. The bosonic vacuum state is the one annihilated by the boson modes:

$$\alpha^{A\dot{A}}_{r\,n}|0\rangle_r = 0 \qquad n \geq 0 \tag{6.20}$$

while fermion vacuum is determined by

$$\psi^{\alpha\dot{A}}_{r\,m}|0\rangle_r \qquad m \geq \frac{1}{2} \tag{6.21}$$

These are states with $(h, m) = (0, 0)$. In contrast, the R sector vacuum states are more complicated; this is because the mode expansion,

$$\psi^{\alpha\dot{A}}_r(z) = \sum_{n\in\mathbb{Z}} \psi^{\alpha\dot{A}}_{r\,n}\, z^{-n-1/2} \tag{6.22}$$

includes zero modes $\psi^{\alpha\dot{A}}_{r\,0}$ that yield multiple degenerate vacuum states, sixteen in total. These states come from all possible non-zero combinations of $(\alpha, \dot{A})$ indices in the left and right sector, and can be represented as

$$|\alpha\dot{\alpha}\rangle \qquad |\alpha\dot{A}\rangle \qquad |\dot{A}\dot{\alpha}\rangle \qquad |\dot{A}\dot{B}\rangle \tag{6.23}$$

We can interpret the states (6.23) as two-spin states; among these, the highest-weight state, sometimes referred to as the *maximally spinning state*, is $|{+}{+}\rangle_r$ and has weights $(h, m) = (1/4, 1/2)$ and satisfies the conditions

$$\psi^{\alpha\dot{A}}_{r\,n}|{+}{+}\rangle_r = 0 \ (n > 0) \qquad \psi^{+\dot{A}}_{r\,0}|{+}{+}\rangle_r = 0 \qquad \psi^{2\dot{A}}_{r\,0}|{+}{+}\rangle_r \neq 0 \tag{6.24}$$

This means that if we take all $N = n_1 n_5$ copies to be in a Ramond ground state, the overall conformal dimension is going to be $h = N/4 = c/24$. The other states in (6.22) come from acting on $|{+}{+}\rangle_r$ with $\psi^{-\dot{A}}_{r\,0}$ and $\tilde{\psi}^{-\dot{B}}_{r\,0}$.

Chiral primary states

The above discussion is about the conformal theory in the free orbifold point in the moduli space, where the theory is weakly coupled and contains free bosons and fermions. However, due to the strong/weak nature of AdS/CFT, the point in the moduli space that is dual to weakly-coupled supergravity is different from the free orbifold point. We would like to have quantities, like vacuum expectation values and correlators, that are protected (i.e. do not vary) when we move in the moduli space. These are obtained using chiral primary states, that are both chiral and Virasoro primary. Let us now determine the properties of these states. If we consider the anticommutator of the G's in (6.13) for $G^{\alpha A}_{\pm 1/2}$, we have

$$\left\{G^{-A}_{+1/2}, G^{+B}_{-1/2}\right\} = \varepsilon^{AB}(J^3_0 - L_0) \tag{6.25}$$

$$\left\{G^{+A}_{+1/2}, G^{-B}_{-1/2}\right\} = \varepsilon^{AB}(J^3_0 + L_0) \tag{6.26}$$

If we sandwich these with a state $|\psi\rangle$ with eigenvalues j and m for L_0 and for J^3_0 we have

$$\sum_B \left|G^{+B}_{-1/2}|\psi\rangle\right|^2 + \sum_B \left|G^{-B}_{+1/2}|\psi\rangle\right|^2 = 2(h - m) \tag{6.27}$$

$$\sum_B \left|G^{-B}_{-1/2}|\psi\rangle\right|^2 + \sum_B \left|G^{+B}_{+1/2}|\psi\rangle\right|^2 = 2(h + m) \tag{6.28}$$

Since in any unitary CFT the left-hand sides are non-negative, from (6.27) we derive a bound on all physical states:

$$h \geq m \implies h \geq j \tag{6.29}$$

Chiral primary states are the ones and only ones that saturate this bound: $h = m$. From (6.27), one sees that chiral states $|\chi\rangle$ are such that

$$G^{+A}_{-1/2}|\chi\rangle = 0 \tag{6.30}$$

as anticipated. Since $h = m = j$, chiral primaries are also the highest-weight states of the $SU(2)_L$ multiplet generated by J's. Note that m is quantized, so it cannot depend on moduli. This means that $h = m$ is moduli-independent for CPOs.

As we have already discussed, chiral primary states/operators are important because they are related to the supergravity fields. In particular, these are identified as the global subalgebra descendants of chiral primaries, which are obtained by acting upon chiral primary operators with L_{-1}, J^-_0 or $G^{-A}_{-1/2}$. There are four chiral primary states in the untwisted sector:

$$|0\rangle_{\mathrm{NS}} \qquad \psi^{+\dot{A}}_{-1/2}|0\rangle_{\mathrm{NS}} \qquad \psi^{+\dot{1}}_{-1/2}\psi^{+\dot{2}}_{-1/2}|0\rangle_{\mathrm{NS}} \propto J^+_{-1}|0\rangle_{\mathrm{NS}} \tag{6.31}$$

where the first one has $h = m = 0$, the second two have $h = m = 1/2$, and the last one has $h = m = 1$.

6.1.3 Moduli of the theory

Moduli are marginal deformations of the conformal theory. As we seen in Chapter 2 and more precisely in Section 2.2.6, marginal operators deform the CFT leaving the conformal invariance untouched. They are operators of dimension $(1,1)$; moreover, they should belong to a so-called *short multiplet*.[1] These two requirements imply moduli are descendants of chiral primary operators of dimension $(\frac{1}{2},\frac{1}{2})$. The only chiral primary operators with conformal dimension $(h,\bar{h})=(\frac{1}{2},\frac{1}{2})$ are [14]

$$\psi^{+\dot{A}}\bar{\psi}^{+\dot{B}} \qquad \Sigma_2^{++} \tag{6.32}$$

where Σ_2^{++} is a twist operator. These five operators, if are acted upon with $G^{-A}_{-1/2}G^{-B}_{-1/2}$, give rise to all the 20 moduli of this conformal theory. Let's write them explicitly. We have from the untwisted sector

$$G^{-A}_{-1/2}G^{-B}_{-1/2}\,\psi^{+\dot{A}}\bar{\psi}^{+\dot{B}}=\partial X^{A\dot{A}}\bar{\partial}X^{B\dot{B}} \tag{6.33}$$

It is convenient to write it in the vector representation of $\mathrm{SO}(4)_I$, using indices i and j: $\partial X^i\bar{\partial}X^j$. Since each index A or B takes two values, there are 16 of these marginal operators. As expected, the remaining 4 come from the twisted sector:

$$G^{-A}_{-1/2}G^{-B}_{-1/2}\,\Sigma_2^{++}=\mathcal{T}^{AB} \tag{6.34}$$

We can split both (6.33) and (6.34) in their irreducible representations with respect to $\mathrm{SO}(4)_I$

$$\begin{aligned}\partial X^i\bar{\partial}X^j&=\left[\partial X^{(i}\bar{\partial}X^{j)}\right]+\left[\partial X^{[i}\bar{\partial}X^{j]}\right]+\left[\frac{1}{4}\delta_{kl}\partial X^k\bar{\partial}X^l\right]\delta^{ij}\\ \mathcal{T}^{AB}&=\mathcal{T}^{[AB]}+\mathcal{T}^{AA}\equiv\mathcal{T}^1+\mathcal{T}^0\end{aligned} \tag{6.35}$$

So we have written all twenty moduli of D1-D5 CFT. We will see in the following that these 20 moduli corresponds to just as many moduli in the supergravity side.

6.1.4 Spectral flow

Spectral flow is a map that sends states and operators to other states and operators. Under spectral flow, the states' left conformal dimension h and spin m are changed as

$$h'=h+\alpha\,m+\frac{c\alpha^2}{24} \qquad m'=m+\frac{c\alpha}{12} \tag{6.36}$$

where α is a parameter which characterizes a particular spectral flow transformation; we are going to restrict to integer values of α. Right sector quantities are changed similarly with $\bar{\alpha}$.

The importance of spectral flow is that transformations parameterized by an odd α exchange R and NS boundary conditions, effectively sending NS states to R states. In particular, spectral flow with $\alpha=-1$ sends chiral primary states (that have $h=m$) to states with $h=c/24$, which is the dimension of Ramond ground states. In fact, the four chiral primary states listed in (6.31) are related to four Ramond ground states, and, moreover, since the right-moving sector has four more chiral primaries, if one joins left and right sectors in all possible ways, one finds that 16 Neveu-Schwarz chiral primary states map into all 16 Ramond vacuum states (6.23); among these, the NS vacuum state is mapped to the maximally spinning R state, $|{+}{+}\rangle_r$.

[1]While supermultiplets are obtained by acting on a global primary state with $G^{\alpha A}_{-1/2}$, short multiplets are multiplets for which some $G^{\alpha A}_{-1/2}$ annihilate the highest-weight state, and this means the highest-weight state is chiral primary [14].

6.1.5 Heavy and light operators

We can take a particular combination of the operators in (6.32), to get

$$O^{\alpha\dot{\alpha}}_{\text{fer}} = \sum_{r=1}^{N} O^{\alpha\dot{\alpha}}_r(z,\bar{z}) = \sum_{r=1}^{N} \frac{-i}{\sqrt{2N}}\, \varepsilon_{\dot{A}\dot{B}} \psi^{\alpha\dot{A}}_r(z)\bar{\psi}^{\dot{\alpha}\dot{A}}_r(\bar{z}) \tag{6.37}$$

which is $\mathbb{T}^4$ invariant. These are also called "fermion operators" because they contain ψ's, but there also similar operator that are "bosonic", such as

$$O^{AB}_{\text{bos}} = \sum_{r=1}^{N} \frac{1}{\sqrt{2N}}\, \varepsilon_{\dot{A}\dot{B}} \partial X^{A\dot{A}}_r(z)\bar{\partial} X^{B\dot{B}}_r(\bar{z}) \tag{6.38}$$

which contains some of the moduli. The above operators are examples of so-called *light operators*, which are operators that possess a conformal dimension of the order of unity, i.e. small w.r.t. $c \sim N = n_1 n_5$. They are generally written as a sum of single-strand operators:

$$O_L(z,\bar{z}) = \sum_{r=1}^{N} \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes O_r(z,\bar{z}) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \tag{6.39}$$

so that the total conformal dimension of the full operator does not depend on the number of strands N. The opposite kind of operators is important as well: they are called *heavy operators* and their dimension scales with $c \sim N$. They are schematically written as a sum of products of operators acting nontrivially on every strand:

$$O_H = \sum_{r=1}^{N} \bigotimes_{r=1}^{N} O_r \tag{6.40}$$

Chiral primary operators with $h \sim 1$ are an example of light operators. Via spectral flow, they are related to Ramond ground states. By looking at how the conformal dimension are changed by spectral flow (6.36), we conclude that Ramond ground states have dimensions that scale with c, and are therefore heavy states.

In the AdS/CFT context, light operators are dual to linear deformations around the vacuum (in our case, $\text{AdS}_3 \times \mathbb{S}^3$), while heavy operators are dual to nontrivial geometries. In particular, Ramond ground states are dual to geometries that carry D1 and D5-branes. We see in the following section how these dual geometries are constructed.

6.2 Supergravity description

6.2.1 Moduli space

The moduli space of this supergravity solution is given by the following six-dimensional scalar fields [14]

$$h_{ij}\ (10) \qquad B_{ij}\ (6) \qquad C_{ij}\ (6) \qquad C_0\ (1) \qquad \hat{C}_4\ (1) \qquad \Phi\ (1) \tag{6.41}$$

where i, j run on the four-torus, $\hat{C}_4$ denotes the term in C_4 which lives on torus, and the number inside brackets denotes the number of degrees of freedom of each field, which is in total 25. In the near horizon limit, there is an "attractor mechanism" which imposes some constraints that fix 5 of these 25 moduli, arriving to 20. These five scalars get fixed to

Table 6.2: The CFT moduli and their corresponding supergravity moduli, along with their representations under the symmetry groups and the number of degrees of freedom. [14, 20]

SUGRA	CFT	$SO(4)_I \sim SU(2)_1 \times SU(2)_2$	$SO(4)_E \sim SU(2)_L \times SU(2)_R$	dof
$h_{ij} - \frac{1}{4}\delta_{ij} h^k{}_k$	$\partial X^{(i}\bar{\partial}X^{j)} - \frac{1}{4}\partial X^i \bar{\partial} X_i$	$(\mathbf{3}, \mathbf{3})$	$(\mathbf{1}, \mathbf{1})$	9
C_{ij}	$\partial X^{[i}\bar{\partial}X^{j]}$	$(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$	$(\mathbf{1}, \mathbf{1})$	6
B^+_{ij}	$\mathcal{T}^1$	$(\mathbf{3}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1})$	3
V_4	$\partial X^i \bar{\partial} X_i$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1})$	1
Ξ	$\mathcal{T}^0$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1})$	1

values that depend on the charges; the simplest example to see this is the dilaton, which in the near horizon limit, $r \ll \sqrt{Q_i}$, becomes

$$e^{2\Phi} = \frac{Q_1}{Q_5} \sim \frac{n_1}{n_5} \tag{6.42}$$

More details about this mechanism can be found in [18, 19].

A consistency check of the AdS/CFT duality comes from matching moduli from the two sides. A matching is presented in [15] which we report in Table 6.2, where the correspondence between the CFT side and the gravity side is done by associating each term in (6.35) with the corresponding gravity modulus that shares the same symmetry under $\mathbb{S}^3$ and $\mathbb{T}^4$. However, there is a two-fold ambiguity for the symmetries $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{3}, \mathbf{1})$. To solve this ambiguity, some indirect arguments can be provided [15]. In particular, in Chapter 7 we will explain an approach that justifies the holographic map for B_2 and C_2 via a direct calculation.

Furthermore, recall we have lost five moduli by going to the near horizon limit in supergravity side. It can be shown that these five moduli can be associated to five irrelevant perturbations of the CFT, making the correspondence even more precise [14].

6.2.2 Dual geometries

In the context of the AdS/CFT correspondence, one is generally interested in studying geometries that are dual to "coherent states" of chiral primary operators (CPO) of the CFT, that we have introduced in the previous chapter. This is because these coherent states are expected to have classical description in terms of geometries; as an aside, this is analogous to ordinary quantum mechanics, where coherent states are those quantum states that resemble classical states the most. CPOs are states with $(h, \bar{h}) = (j, \bar{j})$. Coherent states of these operators are heavy states of the schematic form

$$\sum_n b^n O^n \tag{6.43}$$

where O is a CPO.[2] Chiral primary operators are operators in the NS sector. Thanks to spectral flow (Sec. 6.1.4), CPOs are related to Ramond ground states which are heavy states corresponding to geometries that possess D1 and D5 charges. The six-dimensional metric dual to Ramond ground states that are invariant under rotations in the four compact dimensions is [21–23]

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv + \beta)(du + \omega) + \sqrt{\mathcal{P}}\, ds_4^2 \tag{6.44}$$

[2]The nomenclature of "coherent states" comes from the ordinary quantum states of the same name. A standard example of coherent state is $e^{\alpha a^\dagger}|0\rangle$ which can be expanded in the form $\sum_n \frac{(\alpha a^\dagger)^n}{n!}|0\rangle$, that resembles (6.43).

This ansatz is the most general supergravity solution that carries D1 and D5 charges and preserves the necessary supersymmetries [24]. The full 10D metric includes the torus metric:

$$ds^2_{10} = \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\, ds^2_6 + \sqrt{\frac{Z_1}{Z_2}}\, dz_i\, dz^i \tag{6.45}$$

Here $\mathcal{P} = Z_1 Z_2 - Z_4^2$ and we have used light-cone coordinates

$$u = \frac{t-y}{\sqrt{2}} \qquad v = \frac{t+y}{\sqrt{2}} \tag{6.46}$$

to parameterize time t and the $\mathbb{S}^1$ coordinate y. The metric ds^4 is the flat metric on $\mathbb{R}^4$, while Z_1, Z_2, Z_4 are harmonic scalar functions on $\mathbb{R}^4$, sometimes referred to as "warp factors"; the functions Z_1 and Z_2 are analogous to Z_1 and Z_5 from the two-charge solution (4.31). Finally, β and ω are one-forms with self-dual and anti-self-dual 2-form field strengths. The space $\mathbb{R}^4$ is parameterized by coordinates x_i defined such that

$$x_1 + i x_2 = \hat{r} \mathrm{e}^{i\phi} \sin\hat{\theta} \qquad x_3 + i x_4 = \hat{r} \mathrm{e}^{i\psi} \cos\hat{\theta} \tag{6.47}$$

where $\hat{r}^2 = r^2 + a^2 \sin^2\theta$ and

$$\cos^2\hat{\theta} = \frac{r^2 \cos^2\theta}{r^2 + a^2 \sin^2\theta} \tag{6.48}$$

so that the flat $\mathbb{R}^4 \simeq \mathbb{R} \times \mathbb{S}^3$ reads

$$ds^2_4 = \Sigma\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta\, d\phi^2 + r^2\cos^2\theta\, d\psi^2 \tag{6.49}$$

In this ansatz, for the presence of Z_4, the other fields of type IIB supergravity are nontrivial too. The type IIB fields are the dilaton Φ, the NSNS 2-form B_2 and the RR forms C_0, C_2 and C_4. They are given by:

$$\begin{aligned}
C_0 &= \frac{Z_4}{Z_1} \qquad \mathrm{e}^{2\Phi} = \frac{Z_1^2}{\mathcal{P}} \\
B_2 &= -\frac{Z_4}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_4 \wedge (dv+\beta) + \delta_2 \\
C_2 &= -\frac{Z_2}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_1 \wedge (dv+\beta) + \gamma_2 \\
C_4 &= -\frac{Z_4}{\mathcal{P}}\gamma_2 \wedge (du+\omega)\wedge(dv+\beta) + x_3 \wedge (dv+\beta) + \frac{Z_4}{Z_2}\, dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4
\end{aligned} \tag{6.50}$$

where we introduced the one-forms a_1, a_4, the two-forms δ_2, γ_2 and the three-form x_3. In the following, we will consider the field strengths: F_1, H_3, F_3 and F_5. Actually we will pick $\tilde{F}_3 = F_3 - H_3 \wedge C_0$ and $\tilde{F}_5 = F_5 - H_3 \wedge C_2$. They are defined as

$$\begin{aligned}
F_1 &= d\Big(\frac{Z_4}{Z_1}\Big) \qquad \mathrm{e}^{2\Phi} = \frac{Z_1^2}{\mathcal{P}} \\
\tilde{F}_3 &= \frac{d\hat{u}\wedge d\hat{v}}{\mathcal{P}} \wedge \left(\frac{Z_2}{Z_1}\, dZ_1 - \frac{Z_4}{Z_1}\, dZ_4\right) - \\
&\qquad \frac{1}{Z_1}(d\hat{v}\wedge d\omega - d\hat{u}\wedge d\beta) + \star_4\, dZ_2 - \frac{Z_4}{Z_1}\star_4 dZ_4 \\
H_3 &= -\, d\hat{u}\wedge d\hat{v}\wedge d\left(\frac{Z_4}{\mathcal{P}}\right) - \frac{Z_4}{\mathcal{P}}(d\hat{v}\wedge d\omega - d\hat{u}\wedge d\beta) + \star_4\, dZ_4 \\
\tilde{F}_5 &= -\frac{d\hat{u}\wedge d\hat{v}}{\mathcal{P}} \wedge \star_4 (Z_4\, dZ_2 - Z_2\, dZ_4) + d\Big(\frac{Z_4}{Z_2}\Big) \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4
\end{aligned} \tag{6.51}$$

where for brevity we defined $d\hat{u} = du + \omega$ and $d\hat{v} = dv + \beta$. The form of the warp factors and the one-forms β and ω is in general complicated and depends on the specific ground state [25]. We can give a general formula for the warp factors and the two one-forms. We have seen in Sec. 4.6 that, via a series of string dualities (T-duality and S-duality), these D1-D5 geometries are related to geometries containing a fundamental string carrying momentum (F1P). These F1-P states are well known: they are described by eight functions $g_A(v)$, that live in $\mathbb{R}^4 \times \mathbb{T}^4$ and represent the profile of left-moving oscillations of the fundamental string [26, 27]. Each choice of the profile functions describes a different geometry and thus a different coherent state (6.43). The detailed procedure with the precise holographic relations is described in [28–30].

We ask for these solutions to be invariant with respect to the torus, so we consider only non-zero profiles $g_i(v)$ for $i = 1, \dots, 4$. If we do again the series of dualities (4.28) to go back to the D1D5 frame, which is the frame of interest, it turns out that the torus invariance survives if $g_\mu(v)$ is non-zero for $\mu = 1, \dots, 5$. Therefore, the warp factors and the forms ω and β are determined by the following relations:

$$\begin{aligned}
&Z_1 = \frac{Q_5}{L}\int_0^L \frac{\left|\dot{g}_i(v)\right|^2 + \left|\dot{g}_5(v)\right|^2}{\left|x_i - g_i(v)\right|^2}\,dv \qquad Z_2 = \frac{Q_5}{L}\int_0^L \frac{1}{\left|x_i - g_i(v)\right|^2}\,dv \\
&Z_4 = -\frac{Q_5}{L}\int_0^L \frac{\dot{g}_5(v)}{\left|x_i - g_i(v)\right|^2}\,dv \qquad A = -\frac{Q_5}{L}\int_0^L \frac{\dot{g}_j(v)\,dx_j}{\left|x_i - g_i(v)\right|^2}\,dv \\
&dB = -\star_4 dA \qquad \beta = \frac{-A + B}{\sqrt{2}} \qquad \omega = -\frac{A + B}{\sqrt{2}}
\end{aligned} \tag{6.52}$$

where the dot indicates the derivative with respect to v and $\star_4$ is the Hodge dual with respect to ds^4. We further notice that $g_5 \neq 0$ implies $Z_4 \neq 0$. The integration bound $L = 2\pi Q_5/R$ can be interpreted as the length of the fundamental string. Note that the physical interpretation of $g(v)$'s is lost in the D1-D5 frame.

We now pick a particular selection of the profile functions and derive the corresponding geometry [20]. The non-zero functions are:

$$g_1(v) = a\cos\left(\frac{2\pi v}{L}\right) \qquad g_2(v) = a\sin\left(\frac{2\pi v}{L}\right) \qquad g_5(v) = g(v) = -\frac{b_k}{k}\cos\left(\frac{2\pi k v}{L}\right) \tag{6.53}$$

which represent a circular oscillation on two directions of $\mathbb{R}^4$ and an oscillation in one direction along the torus. First, we compute the denominator of the integrands of the warp factors:

$$\begin{aligned}
\left|x_i - g_i(v)\right|^2 &= (x_1 - g_1(v))^2 + (x_2 - g_2(v))^2 + x_3^2 + x_4^2 \\
&= \left|(x_1 - g_1(v)) + i(x_2 - g_2(v))\right|^2 + |x_3 + ix_4|^2 \\
&= \left|(x_1 + ix_2) - a\mathrm{e}^{iw}\right|^2 + |x_3 + ix_4|^2 \\
&= \left|\hat{r}\sin\hat{\theta}\mathrm{e}^{i\phi} - a\mathrm{e}^{iw}\right|^2 + \left|\hat{r}\cos\hat{\theta}\mathrm{e}^{i\psi}\right|^2 \\
&= \hat{r}^2 + a^2 - a\hat{r}\sin\hat{\theta}\left(\mathrm{e}^{i(\phi-w)} + \mathrm{e}^{-i(\phi-w)}\right) \\
&= r^2 + a^2 + a^2\sin^2\theta - a\sqrt{r^2 + a^2}\sin\theta\left(\mathrm{e}^{i(\phi-w)} + \mathrm{e}^{-i(\phi-w)}\right) \\
&= A - B\left(\mathrm{e}^{i(\phi-w)} + \mathrm{e}^{-i(\phi-w)}\right)
\end{aligned} \tag{6.54}$$

where we have used the change of coordinate (6.47) and redefined $w = (2\pi k/L)v$. We further introduce the complex coordinate $\mathrm{z} = \mathrm{e}^{i(\phi-w)}$, which implies

$$i\,\frac{dz}{z} = dw = \frac{L}{2\pi}\,dv \tag{6.55}$$

Also we notice that

$$\dot{g}(v) = -b_k \frac{2\pi}{L} \cos\left(\frac{2\pi k v}{L}\right) = -b_k \frac{2\pi}{L} \cos w = -\frac{\pi b_k}{L} \left[\mathrm{e}^{-ik\phi} z^k + \mathrm{e}^{ik\phi} \bar{z}^k\right] \tag{6.56}$$

The poles in the denominator are

$$z_\pm = \frac{1}{2B} \left[A \pm \sqrt{A^2 - 4B^2}\right] \tag{6.57}$$

where

$$\sqrt{A^2 - 4B^2} = r^2 + a^2 \cos^2\theta \qquad z_- = \frac{a \sin\theta}{\sqrt{r^2 + a^2}} \tag{6.58}$$

The integral for Z_4 becomes complex:

$$Z_4 = -\frac{Q_5}{L} \frac{\pi b_k}{L} \frac{L}{2\pi} \oint_C \frac{dz}{z} i \, \frac{\mathrm{e}^{-ik\phi} z^k + \mathrm{e}^{ik\phi} \bar{z}^k}{A - B(z + 1/z)} \tag{6.59}$$

$$= \frac{\pi b_k Q_5}{L} \left\{ \operatorname*{Res}_{\Omega} \left[\frac{\mathrm{e}^{-ik\phi} z^k}{Bz^2 - Az + B}\right] + \operatorname*{Res}_{\bar{\Omega}} \left[\frac{\mathrm{e}^{ik\phi} \bar{z}^k}{Bz^2 - Az + B}\right] \right\} \tag{6.60}$$

$$= R\, b_k\, a^k \, \frac{\sin^k\theta \cos k\phi}{(r^2 + a^2)^{k/2} (r^2 + a^2 \cos^2\theta)} \tag{6.61}$$

where $C = \partial\Omega$ is the circle with $|z|^2 = 1$ and $\bar{\Omega} = \mathbb{C} \setminus \Omega$. The other two warp factors are computed similarly; we are going to simply report the result. Finally, we obtained the warp factors derived from the profile (6.53):

$$\begin{aligned} Z_1 &= 1 + \frac{R^2}{Q_5} \frac{a^2 + b_k^2/2}{r^2 + a^2 \cos^2\theta} + \frac{R^2 b_k^2}{2Q_5} \frac{a^{2k} \sin^{2k}\theta \cos 2k\phi}{(r^2 + a^2)^k (r^2 + a^2 \cos^2\theta)} \\ Z_2 &= 1 + \frac{Q_5}{r^2 + a^2 \cos^2\theta} \\ Z_4 &= R\, b_k\, a^k \, \frac{\sin^k\theta \cos k\phi}{(r^2 + a^2)^{k/2} (r^2 + a^2 \cos^2\theta)} \end{aligned} \tag{6.62}$$

We note that Z_4 depends linearly on b_k, Z_1 depends quadratically on it, while Z_2 does not depend on it at all. We may generalize it and consider multiple k's. The geometry is defined in [25]. The warp factors are

$$\begin{aligned} Z_1 = \frac{R^2}{Q_5\,\Sigma} \Bigg[& a_0^2 + \sum_{k,k'} \frac{b_k b_{k'}}{2} \frac{a^{k+k'}}{(r^2 + a^2)^{(k+k')/2}} \sin^{k+k'}\theta \cos[(k + k')\phi] + \\ & \sum_{k>k'} b_k b_{k'} \frac{a^{k-k'}}{(r^2 + a^2)^{(k+k')/2}} \sin^{k-k'}\theta \cos[(k - k')\phi] \Bigg] \\ Z_2 = \frac{Q_5}{\Sigma} \qquad & Z_4 = \frac{R}{\Sigma} \sum_k b_k \frac{a^k}{(r^2 + a^2)^{k/2}} \sin^k\theta \cos k\phi \end{aligned} \tag{6.63}$$

while the 1-forms β and ω are in both cases equal to

$$\beta = \frac{Ra^2}{\sqrt{2}\Sigma} (\sin^2\theta\, d\phi - \cos^2\theta\, d\psi) \qquad \omega = \frac{Ra^2}{\sqrt{2}\Sigma} (\sin^2\theta\, d\phi + \cos^2\theta\, d\psi) \tag{6.64}$$

The warp factors and the one-forms depend on a and b_k which are related to some parameters characterizing the dual CFT state. For generic values of b_k, the geometry is complicated, but it can be shown to be regular and without horizon for any values of the parameters, as far as the constraint

$$a^2 + \sum_k \frac{b_k^2}{2} = a_0^2 \tag{6.65}$$

is satisfied. Here a_0 is defined by

$$a_0^2 = \frac{Q_1 Q_5}{R^2} \tag{6.66}$$

where R is the radius of the one-sphere $\mathbb{S}^1$ and the charges are

$$Q_1 = \frac{(2\pi)^4 n_1 g_s}{V_4}(\alpha')^3 \qquad Q_5 = n_5 g_s \alpha' \tag{6.67}$$

where we included in (4.36) the numerical factors. We shall consider the geometry with a single b_k where k is generic. It is worthwhile to study this geometry in orders of b_k. At order $O(b_k^0)$, the metric becomes simply that of vacuum: $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$—in the following we will drop the torus part and focus on the 6D geometry. Indeed, as $b_k = 0$, the warp factors (6.62) become

$$Z_1 = \frac{R^2}{Q_5\,\Sigma}\, a_0^2 \qquad Z_2 = \frac{Q_5}{\Sigma} \qquad Z_4 = 0 \qquad \mathcal{P} = Z_1 Z_2 \tag{6.68}$$

therefore the 10D metric (6.45) is changed into

$$ds_{10}^2 = ds_6^2 + \sqrt{\frac{Q_1}{Q_5}}\, ds_{\text{torus}}^2 \tag{6.69}$$

where ds_6^2 is now the metric of $\mathrm{AdS}_3 \times \mathbb{S}^3$, as in (4.39). From (6.50) follows that the fields C_0, B_2 and C_4 are zero, and the dilaton is constant:

$$e^{2\Phi} = \frac{Z_1^2}{Z_1 Z_2} = \frac{Z_1}{Z_2} = \frac{R^2\, a_0^2}{Q_5^2} \tag{6.70}$$

Since it does not depend on b_k, the only non-vanishing, non-constant form is C_2, whose (modified) field strength can now be written in the simplified form:

$$\tilde{F}_3 = 2Q_5(-\mathrm{vol}_{\mathrm{AdS}_3} + \mathrm{vol}_{\mathbb{S}^3}) \tag{6.71}$$

where we defined the volumes of AdS_3 and $\mathbb{S}^3$ as

$$\mathrm{vol}_{\mathrm{AdS}_3} = \frac{r}{Q_1 Q_5}\, dr \wedge dt \wedge dy \qquad \mathrm{vol}_{\mathbb{S}^3} = \sin\theta\cos\theta\, d\theta \wedge d\phi \wedge d\psi \tag{6.72}$$

In six dimensions, this makes it Hodge anti-self-dual: $\tilde{F}_3 = - \star_6 \tilde{F}_3$.

At linear order in b_k, the metric, the dilaton and $\tilde{F}_3$ are left unchanged. We have a deformation of the vacuum caused by the fields B_2 and C_0 (or, equivalently, the part of C_4 on the torus). These fields satisfy a coupled set of equations [31] which is

$$dB_2 - \star_6\, dB_2 = 2w\tilde{F}_3 \qquad d\star_6 dw = \frac{Q_1}{Q_5}\, dB_2 \wedge \tilde{F}_3 \tag{6.73}$$

where w is C_0 or the torus part of C_4. Let us consider the particular case where $k = 1$; the conformal (light) operator that is dual to this linear deformation must be a chiral primary operator of dimensions $(h, \bar{h}) = (1/2, 1/2)$. Since we have five tensor multiplets, for any given k we have five operators of dimension $(k/2, k/2)$. It turns out that this is

$$O^{\alpha\dot{\alpha}} \sim \varepsilon_{AB} \psi^{\alpha A} \tilde{\psi}^{\dot{\alpha} B} \tag{6.74}$$

Let us give a intuitive understanding of why is this the right operator. We introduced in (6.4) that capital-letter indices denote the $\mathrm{SO}(4)_I$ symmetry, whose gravity equivalent is the $\mathbb{T}^4$ symmetry. Then, the operator (6.74) is invariant under the torus SO(4), just like the profile (6.53) is.

The exact, all-orders geometry in b_1 is the nonlinear completion of this first-order deformation, so we are led to conclude that it is dual to the coherent state (6.43) with $O = O^{\alpha\dot{\alpha}}$:

$$\sum_n b^n \left(O^{\alpha\dot{\alpha}}\right)^n \tag{6.75}$$

It is customary to denote the operators of same dimension ($h = k/2$) but belonging to different tensor multiplets collectively as $S_k^{(i)}$ where k denotes the dimension and $i = 1, \ldots, 5$ denotes the multiplet, and is sometimes called the "flavor" index. In the case of $O^{\alpha\dot{\alpha}}$, we have $k = 1$ and, by convention, $i = 1$, so

$$O^{\alpha\dot{\alpha}} = S_1^{(1)} \tag{6.76}$$

As the coherent state composed by $S_1^{(1)}$ is dual to geometry with b_1, analogous sums of $S_k^{(1)}$ are dual to the same geometry with b_k.

7 | Holographic map

In this chapter we report the original contribution of the thesis. We have said that unlike the popular $\mathcal{N}=4$ super Yang-Mills theory, the D1-D5 system features multiple flavours, which make it more difficult to construct the entire holographic map. This means there are multiple operators that are "similar", in that they have the same dimensions but different flavours. Our objective is to establish the holographic map between operators and fields of different flavours. In particular, given a chiral primary $S^{(i)}$ of flavour i, we want to find what are the fields dual to the descendants $G\tilde{G}S^{(j)}$ with $j=i$ and $j\neq i$. To do so, we exploit some properties of three-point correlators in their holographic supergravity description.

7.1 Correlators

Correlators are important objects in quantum and conformal field theories. When the theory is strongly coupled, the computation becomes involved. Holographic dualities provide a powerful tool for treating these strongly coupled correlators by studying their supergravity counterpart [32, 33]. In the most famous realization of the holographic duality, that is $\mathrm{AdS}_5\times\mathbb{S}^5$ and super Yang-Mills, there is a vast literature on these holographic correlators (see [34] and references therein). We are going to discuss correlators in $\mathrm{AdS}_3/\mathrm{CFT}_2$ which are less understood.

Let us outline the types of operators that enter into correlators. In type IIB supergravity, when one compactifies over the four torus dimensions, fields organize into five tensor multiplets, and the gravitational multiplet, containing the metric. In the dual CFT we also have five operators of dimension $(k/2,k/2)$ for any $k=1,\dots,\infty$. For example, for $k=1$, the five operators are (6.32). We denote these operators as $S_k^{(i)}$, where $i=1,\dots,5$ denotes the multiplet (also called "flavour"). The operators dual to the fields in the gravitational multiplet are denoted as σ_k and have dimensions $(k/2,k/2)$ where $k=2,\dots,\infty$.[1] Among the five $S_k^{(i)}$'s for a fixed k, there are two that are torus invariant; for $k=1$, they are $O^{\alpha\dot{\alpha}}$ and the twist operator Σ_2. A conventional notation identifies $O^{\alpha\dot{\alpha}}$, defined in (6.37), as $S_1^{(1)}$ and Σ_2 as $S_1^{(2)}$.

In supergravity, the five flavours generate a SO(5) symmetry, which means we can pass from one flavour to another, leaving the Lagrangian invariant. This means that if we take correlators of flavoured states, the result must preserve that SO(5) symmetry. For instance, let us consider a three-point function like

$$\left\langle S_{k_1}^{(i)}\, S_{k_2}^{(j)}\, S_{k_3}^{(k)}\right\rangle \tag{7.1}$$

[1] There is no σ_k for $k=1$ because the five $S_1^{(i)}$'s in (6.32) exhaust all conformal operators of dimensions $(1/2,1/2)$.

We need to have as result a tensor that is invariant under SO(5). However, the only SO(5)-invariant tensors are Kronecker delta and the Levi-Civita symbol, and there is no way they can be put together to get an invariant three-index tensor. So, we conclude that the correlator (7.1) is zero. The only non-vanishing three-point correlators are the ones which have only two flavours, such as

$$\left\langle S_{k_1}^{(i)}\, S_{k_2}^{(j)}\, \sigma_{k_3} \right\rangle \sim \delta^{ij} \tag{7.2}$$

Therefore, the tensor structure of these correlators is helpful in determining in advance if a correlator is going to be non-vanishing, even though there can be values of the k's that make it zero anyway.

Let us take correlators involving chiral primary descendants, such as

$$\left\langle S_{k_1}^{(1)}\, G^+ \tilde{G}^+ S_{k_2}^{(i)}\, G^- \tilde{G}^- \sigma_{k_3} \right\rangle \tag{7.3}$$

Even if it involves chiral primary descendants, it can be shown that a Ward identity relates (7.3) with the correlator involving the original chiral primaries [25], so we have

$$\left\langle S_{k_1}^{(1)}\, G^+ \tilde{G}^+ S_{k_2}^{(i)}\, G^- \tilde{G}^- \sigma_{k_3} \right\rangle \xleftrightarrow{\text{Ward identity}} \left\langle S_{k_1}^{(1)}\, S_{k_2}^{(i)}\, \sigma_{k_3} \right\rangle \tag{7.4}$$

The conformal theory is in the strong coupling regime, so it is necessary to compute this correlator holographically. The prescription for computing correlators using holography is detailed in Sec. 5.2.2. One can use the SO(5) symmetry and some information of the holographic correlator to pin down the field/operator map. Following (7.2), correlator (7.3) is proportional to δ^{1i} where we restrict to $i = 1$ or 2 to consider torus-invariant operators. By determining if the result of the supergravity calculation is vanishing or not, we will be able to associate the descendant $G^+\tilde{G}^+S_{k_2}^{(1)}$ to its dual supergravity field and accordingly $G^+\tilde{G}^+S_{k_2}^{(2)}$.

Let us outline the holographic calculation of (7.3). The basic prescription follows what we reported in Sec. 5.2.2. Importantly, holographic correlators are usually computed on vacuum states, that are dual to anti-de Sitter spaces. Instead, we are going to apply the ordinary procedure for correlators on a heavy state, dual to a nontrivial background geometry. Namely the heavy state of interest is the coherent state built out of $S_{k_1}^{(1)}$, defined as

$$\mathcal{O}_{k_1}^{(1)} = \sum_n b_{k_1}^n\, (S_{k_1}^{(1)})^n \tag{7.5}$$

that is dual to the geometry specified by the data in (6.62) with the switch $b_k \to b_{k_1}$. Correlators of this kind have the form of a four-point correlator or equivalently of a two-point correlator on nontrivial states:

$$\left\langle \mathcal{O}_{k_1}^{(1)}\, G^+ \tilde{G}^+ S_{k_2}^{(i)}\, G^- \tilde{G}^- \sigma_{k_3} \mathcal{O}_{k_1}^{(1)} \right\rangle \longleftrightarrow \left\langle \mathcal{O}_{k_1}^{(1)} \middle| G^+ \tilde{G}^+ S_{k_2}^{(i)}\, G^- \tilde{G}^- \sigma_{k_3} \middle| \mathcal{O}_{k_1}^{(1)} \right\rangle \tag{7.6}$$

The full correlator (7.6) is complicated, but for our purpose it suffices to stop at linear order in b_{k_1} in the sum (7.5) and extract only one $S_{k_1}^{(1)}$ from one of the two heavy states, and nothing from the other one. The upshot is that we recover a three-point correlator, namely (7.3). As seen in the previous chapter, $S_{k_1}^{(1)}$, being light, corresponds to the linear deformation of the vacuum caused by B_2 and C_0, which satisfy the equations (6.73). The field dual to $G^-\tilde{G}^-\sigma_{k_3}$ can be shown to be a certain two-form in six-dimensional space, that we shall denote λ and introduce shortly: details are discussed in [35] and other unpublished work by the same authors.

The dual fields to the other operator in (7.6) for every i have never been derived directly in the literature. The goal of this thesis is to determine them. We shall call

$$\mathcal{B}^{(i)} = G^{-\dot{A}}\tilde{G}^{-\dot{B}}\, S_k^{(i)} \tag{7.7}$$

where $i = 1, 2$. Both operators $\mathcal{B}^{(i)}$ are descendants of chiral primaries—namely $O^{\alpha\dot{\alpha}}$ and Σ_2—so they are moduli; as such they holographically correspond to six-dimensional scalar fields. As $\dot{A}$ and $\dot{B}$ vary, we have four possible $\mathcal{B}^{(i)}$'s for every i, one of which is torus invariant and the other three are not; we shall consider the latter three. Since $\mathcal{B}^{(i)}$ are both light, the supergravity dual fields are going to be linear deformations of the background geometry, rather than a whole different geometry. In [25] the deformations of the torus metric were discussed, but they are not dual to $\mathcal{B}^{(1)}$ nor $\mathcal{B}^{(2)}$. We then claim that the supergravity dual are the six-dimensional scalar deformations of C_2 and B_2; however, even though their conformal counterparts are structurally different (Σ_2 is a twist operator, while $O^{\alpha\dot{\alpha}}$ contains simply ψ's) these two supergravity fields are similar, making the distinction not immediate. We will compute the correlator in supergravity using both deformations and, based on the result of the calculation, we will be able to distinguish which is dual to which: since (7.3) vanishes for $i = 2$, we associate the field that makes the supergravity calculation vanish with $\mathcal{B}^{(2)}$, and accordingly the other field will be dual to $\mathcal{B}^{(1)}$. We start the holographic supergravity calculation from the correlator (7.6), so we consider the background geometry that is dual to $\mathcal{O}^{(1)}_{k_1}$, which is the one characterized by warp factors (6.62). The fields are then defined by (6.50). The aforementioned deformations perturb this background. We then restrict ourselves to first order in b_{k_1}, where in the CFT side (7.6) becomes (7.3). Mathematically, the deformations are treated as an additional term of the fields: for example C_2 becomes $C_2 + \delta C_2$. This deformation takes the form

$$\delta C_2 = \tau\, c \tag{7.8}$$

where c is a 0-form on 6D space, while τ is a constant 2-form on the torus, thus satisfying $d\tau = 0$; moreover, we are going to assume τ to be Hodge anti-self-dual with respect to the torus, so $\tau + \hat{\star}_4\tau = 0$ where $\hat{\star}_4$ denotes indeed the Hodge operator on $\mathbb{T}^4$. Besides, a Hodge anti-self-dual 2-form has three parameters that can be associated to the three non torus-invariant $\mathcal{B}^{(i)}$'s. An analogous deformation regards B_2, that is

$$\delta B_2 = \tau\, b \tag{7.9}$$

where b is 0-form on 6D space and τ is the same 2-form defined above.

However, if one turns on these two deformations, they find that the equations of motion (4.9) cannot be satisfied, and it is necessary to involve another deformation on C_4, which is

$$\delta C_4 = \tau \wedge \lambda \tag{7.10}$$

where λ is a 2-form on 6D space. Note that λ, being a 2-form and not a scalar, cannot be dual to one of the moduli. In fact, it is the λ that is dual to $G^{-}\tilde{G}^{-}\sigma_{k_3}$.

To sum up, our claim is that $\mathcal{B}^{(i)}$ for $i = 1, 2$ is dual to the fields c and b. The basic idea of holographic computation of a correlator involves turning on a source of a field in the background geometry, and calculate the vacuum expectation value (vev) of the other fields in presence of this source. Suppose a source for λ is introduced; the other 6D field, which is either b or c, can become excited or not. if the other field is activated, then the associated vev is non-zero, implying a non-zero correlator as well. If instead there is no excitation, its vev is zero, and so is the correlator. We need to know which of the two fields, b and c, is coupled with λ. To do so, we have to derive their equations of motion. As we said above, we write the equations for the six-dimensional fields first in the full background geometry, then we take the zeroth and first orders of these equations to finally pin down the holographic map.

7.2 Equations of motion

In this section, we will write the equations of motion for the deformations. First, we derive the deformations for the field strengths induced by the deformations (7.8), (7.9) and (7.10), then we plug the deformed field strengths into the equations of motion of type IIB supergravity, in the background geometry encoded by the warp factors (6.62). The gauge potentials and the field strengths that satisfy the equations of motion in this background are (6.50) and (6.51), respectively.

7.2.1 Deformations of the field strengths

We will consider the geometry (6.62), dual to the coherent state (7.5) and study the linear deformations (7.8), (7.9) and (7.10). We will later impose that they satisfy the equations of motion at first order in b_{k_1}. The equations of motion will help us in determining which field between b and c is coupled with λ and thus yields a non-zero expectation value. After having obtained their equations under the exact geometry, we require that the deformations satisfy the equations at the zeroth and linear order in b_{k_1}, because, as we already said the correlator (7.3) involves a single $S^{(1)}_{k_1}$, which corresponds to a linear deformation. Since we perturbed the gauge potentials, Bianchi identities (4.8) are automatically satisfied, so we focus on the equations (4.9). In this case, the fields C_0, Φ and the metric are left unchanged, so the equations involving the deformed fields are

$$d\star(\mathrm{e}^{-2\Phi}H_3) - F_1\wedge\star\tilde{F}_3 - \tilde{F}_3\wedge\star\tilde{F}_5 = 0 \tag{7.11a}$$

$$d\star\tilde{F}_3 + H_3\wedge\tilde{F}_5 = 0 \tag{7.11b}$$

$$\tilde{F}_5 = \star\tilde{F}_5 \tag{7.11c}$$

The deformations on the gauge potentials C_2, B_2 and C_4 obviously induce deformations of their respective field strengths. For simplicity, from now on we drop the tildes on the modified field strengths $\tilde{F}_3$ and $\tilde{F}_5$. Moreover, $\star$ is used to denote the Hodge operator on the 10-dimensional metric ds^2_{10} (6.45), while the associated metrics of $\star_6$, $\star_4$ and $\hat{\star}_4$ are ds^2_6, ds^2_4 and $ds^2_{\text{torus}}\equiv d\hat{s}^2_4$. We have

$$\begin{aligned}\delta H_3 &= d(\delta B_2) = db\wedge\tau\\ \delta F_3 &= d(\delta C_2) - C_0\,\delta H_3 = (dc - C_0\,db)\wedge\tau\\ \delta F_5 &= d(\delta C_4) - \delta H_3\wedge C_2 - H_3\wedge\delta C_2\\ &= (d\lambda - db\wedge C_2 - H_3\wedge c)\wedge\tau\end{aligned} \tag{7.12}$$

It is actually better to work with field strengths only; we start from δF_5, where we do a sort of "integration by parts"

$$\begin{aligned}\delta F_5 &= \Big(d\lambda - d(b\wedge C_2) + b\,dC_2 - H_3\,c\Big)\wedge\tau\\ &= \Big(d(\lambda - b\wedge C_2) + b(F_3 + C_0H_3) - c\,H_3\Big)\wedge\tau\\ &= \Big(d\hat{\lambda} + bF_3 + bC_0H_3 - c\,H_3\Big)\wedge\tau\\ &= \Big(d\hat{\lambda} + bF_3 - (c - bC_0)H_3\Big)\wedge\tau\\ &= \Big(d\hat{\lambda} + bF_3 - \hat{c}\,H_3\Big)\wedge\tau\end{aligned} \tag{7.13}$$

where we defined $\hat{\lambda} = \lambda - b\,C_2$ and $\hat{c} = c - b\,C_0$ for convenience. Using $\hat{c}$, we can rewrite also δF_3. Since $c = \hat{c} + bC_0$, we get

$$\delta F_3 = (d\hat{c} + d(bC_0) - C_0\,db) \wedge \tau = (d\hat{c} + b\,dC_0) \wedge \tau \tag{7.14}$$

Finally, these are the deformations we will plug into the equations (7.11):

$$\delta H_3 = db \wedge \tau \tag{7.15a}$$

$$\delta F_3 = (d\hat{c} + b\,dC_0) \wedge \tau \tag{7.15b}$$

$$\delta F_5 = (d\hat{\lambda} + bF_3 - \hat{c}\,H_3) \wedge \tau \tag{7.15c}$$

7.2.2 Computation of the equations

The deformations (7.15) are valid for every background geometry. We now specialize to the full geometry encoded by the warp factors (6.62), whereas the field strengths are defined by (6.51), and compute the equations of motion for the deformations (7.15). Actually, since we are ultimately interested in the six-dimensional fields b, $\hat{c}$ and $\hat{\lambda}$, we will drop τ from all equations and restrict to the six-dimensional equations describing those fields. Since the resulting equations will be difficult to study at first sight, in the following section we will simplify them by taking simple limits at zeroth and linear order in b_k.

We start from equation (7.11c), which is the simplest. We plug $F_5 + \delta F_5$ into (7.11c) and keep only the deformed terms, so we have

$$\delta F_5 - \star\,\delta F_5 = 0 \tag{7.16}$$

The Hodge star is easily taken care of; since we want six-dimensional equations at the end, we simply write the ten-dimensional $\star$ in terms of the six-dimensional $\star_6$. This yields

$$\begin{aligned}\star\,\delta F_5 &= \star\left[(d\hat{\lambda} + bF_3 - \hat{c}H_3) \wedge \tau\right] \\ &= -\frac{Z_2}{Z_1}\tau \wedge \frac{Z_1}{Z_2}\star_6(d\hat{\lambda} + bF_3 - \hat{c}H_3) \\ &= -\tau \wedge \star_6(d\hat{\lambda} + bF_3 - \hat{c}H_3)\end{aligned} \tag{7.17}$$

where we used the anti-self-duality of τ. Therefore, we can write the equation that governs the six-dimensional field $\hat{\lambda}$. It is

$$(d\hat{\lambda} + bF_3 - \hat{c}H_3) + \star_6(d\hat{\lambda} + bF_3 - \hat{c}H_3) = 0 \tag{7.18}$$

where, as announced, we dropped τ.

Let us now consider the equation for δF_3 (7.11b). The calculations that have to be done are similar to the previous case, but they are slightly more involved. The equation for the deformation is

$$d\star\delta F_3 + \delta H_3 \wedge F_5 + H_3 \wedge \delta F_5 + \delta H_3 \wedge \delta F_5 = 0 \tag{7.19}$$

The latter term is zero, because both deformations contain τ. Then we do the Hodge-star term. Computing the Hodge star, we get

$$\begin{aligned}\star\,\delta F_3 &= \star\left[(d\hat{c} + b\,dC_0) \wedge \tau\right] \\ &= -\frac{Z_2}{Z_1}\tau \wedge \frac{Z_1^2}{\mathcal{P}}\star_6(d\hat{c} + bF_1) \\ &= -\frac{Z_1 Z_2}{\mathcal{P}}\tau \wedge \star_6(d\hat{c} + bF_1)\end{aligned} \tag{7.20}$$

and its derivative is

$$\begin{aligned} d\star\delta F_3 &= -\tau\wedge d\left[\frac{Z_1Z_2}{\mathcal{P}}\tau\wedge\star_6(d\hat{c}+bF_1)\right] \\ &= -\tau\wedge\left[d\Big(\frac{Z_1Z_2}{\mathcal{P}}\Big)\wedge\star_6(d\hat{c}+bF_1)+\frac{Z_1Z_2}{\mathcal{P}}(\Box_6\,\hat{c}+d\star_6(bF_1))\right] \end{aligned} \tag{7.21}$$

where we used the notation $\Box_6\equiv d\star_6 d$. Now the two wedge products are left. The first one is

$$\delta H_3\wedge F_5=(db\wedge\tau)\wedge F_5=db\wedge\tau\wedge\bar{F}_5 \tag{7.22}$$

where $\bar{F}_5$ is the six-dimensional term of F_5, because the other one vanishes upon the multiplication with τ. The other product yields

$$\begin{aligned} H_3\wedge\delta F_5 &= H_3\wedge\left[(d\hat{\lambda}+bF_3-\hat{c}H_3)\wedge\tau\right] \\ &= H_3\wedge(d\hat{\lambda}+bF_3)\wedge\tau \end{aligned} \tag{7.23}$$

Putting all together, we get the six-dimensional equation for $\hat{c}$, that is

$$-\frac{Z_1Z_2}{\mathcal{P}}(\Box_6\,\hat{c}+d\star_6\,bF_1)-d\Big(\frac{Z_1Z_2}{\mathcal{P}}\Big)\wedge\star_6(d\hat{c}+bF_1)+db\wedge\bar{F}_5+H_3\wedge(d\hat{\lambda}+bF_3)=0 \tag{7.24}$$

It remains to treat the equation for δH_3, which will give the equation for b. Equation (7.11a) for the deformations becomes

$$d\star(\mathrm{e}^{-2\Phi}\delta H_3)-F_1\wedge\star\delta F_3-\delta F_3\wedge\star F_5-F_3\wedge\star\delta F_5-\delta F_3\wedge\star\delta F_5=0 \tag{7.25}$$

Again, the latter term is zero, because $\tau\wedge\star\tau\sim\tau\wedge\hat{\star}_4\tau=-\tau\wedge\tau=0$. We start from the first term as usual. The Hodge star is computed in the same way as the other two cases:

$$\begin{aligned} \mathrm{e}^{-2\Phi}\star\delta H_3 &= \mathrm{e}^{-2\Phi}\star(db\wedge\tau) \\ &= -\mathrm{e}^{-2\Phi}\frac{Z_2}{Z_1}\tau\wedge\frac{Z_1^2}{\mathcal{P}}\star_6\,db \\ &= -\frac{Z_2}{Z_1}\tau\wedge\star_6\,db \end{aligned} \tag{7.26}$$

Its derivative is:

$$\begin{aligned} d(\mathrm{e}^{-2\Phi}\star\delta H_3) &= -\,d\left(\frac{Z_2}{Z_1}\tau\wedge\star_6\,db\right) \\ &= -\tau\wedge d\Big(\frac{Z_2}{Z_1}\star_6\,db\Big) \\ &= -\tau\wedge\left[d\Big(\frac{Z_2}{Z_1}\Big)\wedge\star_6\,db+\frac{Z_2}{Z_1}\Box_6\,db\right] \end{aligned} \tag{7.27}$$

Now the three wedge products remain. They yield

$$F_1\wedge\star\delta F_3=-\frac{Z_1Z_2}{\mathcal{P}}F_1\wedge\tau\wedge\star_6(d\hat{c}+bF_1) \tag{7.28}$$

and

$$\delta F_3\wedge\star F_5=\delta F_3\wedge F_5=(d\hat{c}+bF_1)\wedge\tau\wedge\bar{F}_5 \tag{7.29}$$

and finally

$$F_3\wedge\star\delta F_5=-F_3\wedge\tau\wedge\star_6(d\hat{\lambda}+bF_3-\hat{c}H_3) \tag{7.30}$$

Therefore, the equation for b is

$$-\frac{Z_2}{Z_1}\Box_6\, b - d\Big(\frac{Z_2}{Z_1}\Big)\wedge\star_6\, db + \frac{Z_1Z_2}{\mathcal{P}}F_1\wedge\star_6(d\hat{c}+bF_1) - \\ (d\hat{c}+bF_1)\wedge\bar{F}_5 + F_3\wedge\star_6(d\hat{\lambda}+bF_3-\hat{c}H_3) = 0 \tag{7.31}$$

The equations together are

$$(d\hat{\lambda}+bF_3-\hat{c}H_3)+\star_6(d\hat{\lambda}+bF_3-\hat{c}H_3)=0 \tag{7.32a}$$

$$-\frac{Z_1Z_2}{\mathcal{P}}(\Box_6\,\hat{c}+d\star_6\, bF_1) - d\Big(\frac{Z_1Z_2}{\mathcal{P}}\Big)\wedge\star_6(d\hat{c}+bF_1) + \\ db\wedge\bar{F}_5 + H_3\wedge(d\hat{\lambda}+bF_3) = 0 \tag{7.32b}$$

$$-\frac{Z_2}{Z_1}\Box_6\, b - d\Big(\frac{Z_2}{Z_1}\Big)\wedge\star_6\, db + \frac{Z_1Z_2}{\mathcal{P}}F_1\wedge\star_6(d\hat{c}+bF_1) - \\ (d\hat{c}+bF_1)\wedge\bar{F}_5 + F_3\wedge\star_6(d\hat{\lambda}+bF_3-\hat{c}H_3) = 0 \tag{7.32c}$$

The equations (7.32) are the equation for the six-dimensional fields computed in the full background geometry with warp factors (6.62). As we already said, it is useful to to study these equations at two simpler limits, namely $O(b_{k_1}^0)$ and $O(b_{k_1})$.

7.2.3 Limits

We now take the equations (7.32) at zeroth and linear order in b_{k_1}. The zeroth order corresponds to deformations around $\mathrm{AdS}_3\times\mathbb{S}^3$, while the linear-order geometry is dual to $S_{k_1}^{(1)}$ which is included in the three-point function (7.3).

Order zero

For $b_{k_1}=0$, the warp factors become

$$Z_1=\frac{R^2}{Q_5\,\Sigma}\,a_0^2 \qquad Z_2=\frac{Q_5}{\Sigma} \qquad Z_4=0 \qquad \mathcal{P}=Z_1Z_2 \tag{7.33}$$

therefore the field strengths which depend on Z_4, i.e. F_1, H_3 and F_5, vanish; the dilaton becomes constant and F_3 becomes Hodge anti-self-dual.

$$\mathrm{e}^{2\Phi}=\frac{Z_1^2}{Z_1Z_2}=\frac{Z_1}{Z_2}=\frac{R^2\,a_0^2}{Q_5^2} \qquad F_1=0 \qquad H_3=0 \qquad F_5=0 \qquad F_3=-\star_6\,F_3 \tag{7.34}$$

Since most fields vanish, the equations simplify greatly. The equation for δF_5, which is $\delta F_5-\star\,\delta F_5=0$, becomes

$$(d\hat{\lambda}+bF_3-\hat{c}H_3)+\star_6(d\hat{\lambda}+bF_3-\hat{c}H_3)=0 \implies \boxed{d\hat{\lambda}+\star_6\, d\hat{\lambda}=0} \tag{7.35}$$

where we used $F_3=-\star_6\,F_3$ and $H_3=0$. The equation for δF_3 keeps only the first term: $d\star\delta F_3=0$, which is

$$-\frac{Z_1Z_2}{\mathcal{P}}(\Box_6\,\hat{c}+d\star_6\, bF_1) - d\Big(\frac{Z_1Z_2}{\mathcal{P}}\Big)\wedge\star_6(d\hat{c}+bF_1)=0 \implies \boxed{\Box_6\,\hat{c}=0} \tag{7.36}$$

where we used $\mathcal{P}=Z_1Z_2$ and $F_1=0$. Finally, the equation for δH_3 is $d\star(\mathrm{e}^{-2\Phi}\delta H_3)-F_3\wedge\star\,\delta F_5=0$, which is

$$-\frac{Z_2}{Z_1}\Box_6\, b - d\Big(\frac{Z_2}{Z_1}\Big)\wedge\star_6\, db + \frac{Z_1Z_2}{\mathcal{P}}F_1\wedge\star_6(d\hat{c}+bF_1)=0 \implies \boxed{\Box_6\, b=0} \tag{7.37}$$

where we used that Z_2/Z_1 is constant (so its derivative is zero) and $F_1 = 0$.

Note that all equations are decoupled at order zero. This means that all fields are independent from one another if one makes a perturbation of the vacuum, and, as expected, they are dual to three distinct CFT operators. At zeroth order, the spacetime is simply the product $\mathrm{AdS}_3 \times \mathbb{S}^3$, so these six-dimensional fields can be written in factorized form with as factors a spherical harmonic in $\mathbb{S}^3$ and a term on AdS_3. The various spherical harmonics correspond to the different possible k's and the functions on AdS_3 denote the dimension. One can further demonstrate that the equation for $\hat{\lambda}$ (7.35) admits solutions with spherical harmonics with $k > 1$ only, which goes to show that $\hat{\lambda}$ is related to a field in the gravity multiplet, as expected. Instead, the harmonics satisfying equations (7.36) and (7.37) can have $k = 1$, since the corresponding fields are in tensor multiplets.

First order

At first order in b_{k_1}, Z_4 is no longer zero. We have

$$Z_1 = \frac{R^2}{Q_5\,\Sigma}\,a_0^2 \qquad Z_2 = \frac{Q_5}{\Sigma} \qquad Z_4 = \frac{R}{\Sigma}\frac{a}{\sqrt{r^2+a^2}}\sin\theta\cos\phi \qquad \frac{Z_1}{Z_2} = \frac{R^2\,a_0^2}{Q_5^2}$$

The fields F_1, F_5 and H_3 are no longer zero, while Φ is still constant and F_3 is still Hodge anti-self-dual. In formulae, we have

$$F_1 = d\Big(\frac{Z_4}{Z_1}\Big) \qquad \mathrm{e}^{2\Phi} = \frac{Z_1^2}{\mathcal{P}} = \frac{Z_1^2}{Z_1 Z_2 - O(b^2)} = \frac{Z_1}{Z_2} = \frac{R^2\,a_0^2}{Q_5^2}$$

$$\begin{aligned}
F_5 &= -\frac{d\hat{u}\wedge d\hat{v}}{\mathcal{P}}\wedge\star_4(Z_4\,dZ_2 - Z_2\,dZ_4) + d\Big(\frac{Z_4}{Z_2}\Big)\wedge dz^1\wedge dz^2\wedge dz^3\wedge dz^4 \\
F_3 &= \frac{d\hat{u}\wedge d\hat{v}}{\mathcal{P}}\wedge\left(\frac{Z_2}{Z_1}\,dZ_1\right) - \frac{1}{Z_1}(d\hat{v}\wedge d\tau - d\hat{u}\wedge d\beta) + \star_4\,dZ_2 = -\star_6\,F_3 \\
H_3 &= -\,d\hat{u}\wedge d\hat{v}\wedge d\left(\frac{Z_4}{\mathcal{P}}\right) - \frac{Z_4}{\mathcal{P}}(d\hat{v}\wedge d\tau - d\hat{u}\wedge d\beta) + \star_4\,dZ_4
\end{aligned} \tag{7.38}$$

Let us first consider the simplest of the three equations, (7.32a). Thanks to the anti-self-duality of F_3, the two F_3's cancel out and we are left with

$$d\hat{\lambda} + \star_6\,d\lambda - \hat{c}(H_3 + \star_6\,H_3) = 0 \tag{7.39}$$

There is nothing left to simplify, so (7.39) is the equation for $\hat{\lambda}$ at linear order. Now we take care of the other two equations, (7.32b) and (7.32c). First, in (7.32b) we use that $Z_1 Z_2/\mathcal{P} = 1$, while in (7.32c) we use the constancy of Z_1/Z_2 and the Hodge-anti-self-duality of $(d\hat{\lambda} + bF_3 - \hat{c}H_3)$ and F_3,[2] which we have deduced from equation (7.32a). We have

$$\Box_6\,\hat{c} + d\star_6\,bF_1 - db\wedge F_5 - H_3\wedge(d\hat{\lambda} + bF_3) = 0 \tag{7.40a}$$

$$-\frac{Z_2}{Z_1}\Box_6\,b + F_1\wedge\star_6(d\hat{c} + bF_1) - (d\hat{c} + bF_1)\wedge\bar{F}_5 = 0 \tag{7.40b}$$

Then in (7.40b) we cancel out $F_1\wedge b\star_6\,F_1$ and $F_1\wedge\bar{F}_5$ because they are higher-order terms in b_{k_1}. In (7.40a) we expand the derivative $d\star_6\,bF_1$ as $db\wedge\star_6\,F_1 + b\,d\star_6\,F_1$:

$$\Box_6\,\hat{c} + b\,d\star_6\,F_1 + db\wedge(\star_6\,F_1 - F_5) - H_3\wedge(d\hat{\lambda} + bF_3) = 0 \tag{7.41a}$$

$$-\frac{Z_2}{Z_1}\Box_6\,b + F_1\wedge\star_6\,d\hat{c} - d\hat{c}\wedge F_5 = 0 \tag{7.41b}$$

[2]Note that if the forms v and w are Hodge anti-self-dual (or self-dual), then $v\wedge w = \pm v\wedge\star_6\,w = \pm w\wedge\star_6\,v = w\wedge v = -v\wedge w = 0$.

In (7.41a), we apply the equation for F_1, for which $d \star_6 F_1 = -b\, H_3 \wedge \star_6 F_3$ which in this case, it is also equal to $b\, H_3 \wedge F_3$. In (7.41b), we rewrite $F_1 \wedge \star_6 d\hat{c}$ as $d\hat{c} \wedge \star_6 F_1$, so:

$$\Box_6\, \hat{c} - H_3 \wedge d\hat{\lambda} + db \wedge (\star_6 F_1 - F_5) = 0 \tag{7.42a}$$

$$-\frac{Z_2}{Z_1} \Box_6\, b + d\hat{c} \wedge (\star_6 F_1 - F_5) = 0 \tag{7.42b}$$

It can be shown that $\star_6 F_1 = \bar{F}_5$ at first order. Indeed, as $F_5 = \star F_5$, $\bar{F}_5$ is the Hodge dual of the torus term of F_5, that is,

$$\bar{F}_5 = \star \left[d\Big(\frac{Z_4}{Z_2}\Big) \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \right] = \frac{Z_1 Z_2}{\mathcal{P}} \frac{Z_2}{Z_1} \star_6 d\Big(\frac{Z_4}{Z_2}\Big) = \frac{Z_2}{Z_1} \star_6 d\Big(\frac{Z_4}{Z_2}\Big) \tag{7.43}$$

where we used $Z_1 Z_2/\mathcal{P} = 1$. Since Z_2/Z_1 is constant, we can put it inside the derivative and obtain

$$\star_6\, d\Big(\frac{Z_2}{Z_1}\frac{Z_4}{Z_2}\Big) = \star_6\, d\Big(\frac{Z_4}{Z_1}\Big) \tag{7.44}$$

which is precisely $\star_6 F_1$. Then the terms with $\star_6 F_1 - F_5$ are zero. Finally, the three equations together are

$$d\hat{\lambda} + \star_6 d\lambda - \hat{c}(H_3 + \star_6 H_3) = 0 \tag{7.45a}$$

$$\Box_6\, \hat{c} - H_3 \wedge d\hat{\lambda} = 0 \tag{7.45b}$$

$$\Box_6\, b = 0 \tag{7.45c}$$

The equations (7.45) yield what we were looking for: they show that $\hat{c}$ is coupled to $\hat{\lambda}$ while b is an independent field. Let us now resume the discussion regarding the holographic computation of the correlator (7.3),

$$\Big\langle S^{(1)}_{k_1}\, G^+ \tilde{G}^+ S^{(i)}_{k_2}\, G^- \tilde{G}^- \sigma_{k_3} \Big\rangle \tag{7.46}$$

We have found that λ is coupled with c. This means that turning on a source for λ yields a non-vanishing c: therefore, the supergravity calculation with c produces a non-vanishing result, which implies a non-vanishing correlator. This in turn implies, due to the SO(5) tensor structure, that the conformal operator dual to c has flavour equal to 1, specifically it is $G^+\tilde{G}^+ S^{(1)}_{k_1}$. Accordingly, b is dual to the operator $G^+\tilde{G}^+ S^{(2)}_{k_2}$.

8 Conclusions

8.1 Review

Our work focused on analyzing the holographic map between the D1-D5 supergravity description and the D1-D5 conformal theory. In particular, the main goal was to establish the supergravity fields that are dual to some chiral primary operators belonging in different tensor multiplets in the CFT.

In order to apply and understand AdS/CFT correspondence, we needed to introduce many concepts. We first described the latter side of the duality, introducing the basic principles of any conformal field theory, both classical and quantum. Then, we presented the former side, starting from string theory which, even though it has not been our primary framework, it paved the way to supergravity, being its low-energy limit and the gravitational theory we worked with. In the context of supergravity, we described its fields and their coupling with branes. We then provided two solutions that carry charges associated to these branes. Afterwards, we introduced and gave some motivations for the AdS/CFT correspondence, and we illustrated the D1-D5 system, starting from the conformal side and working our way to constructing the dual geometries of heavy states. Finally, we explained the main problem that was approached in the thesis. We wanted to find the supergravity dual to the descendants defined by

$$\mathcal{B}^{(i)} = G^+\tilde{G}^+\, S_k^{(i)} \tag{8.1}$$

where $S_k^{(i)}$ is a chiral primary operator of dimensions $(k/2, k/2)$ belonging in the i-th tensor multiplet, where $i = 1, \ldots, 5$. For a fixed k, there are therefore five fields belonging to different multiplets, that enjoy a SO(5) symmetry that rotates between the multiplets, also known as flavours. Among those fields, three transform nontrivially under the torus $\mathbb{T}^4$, while the other two are torus invariant. We focused on the latter two, whose flavours are conventionally denoted by $i = 1$ and $i = 2$. We claimed that the dual fields to $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ were going to be B_2 and C_2, but we needed to distinguish which is dual to which. The approach consists in performing an holographic computation of a correlator exploiting the SO(5) flavour symmetry. Specifically, the correlator

$$\left\langle S_{k_1}^{(1)}\, G^+\tilde{G}^+ S_{k_2}^{(i)}\, G^-\tilde{G}^-\sigma_{k_3} \right\rangle \tag{8.2}$$

is proportional to δ^{1i}: this means that (8.2) is zero for $i = 2$ and non-vanishing for $i = 1$. Therefore, we have done the supergravity computation with both fields, and we have identified the field that makes the computation vanish with $\mathcal{B}^{(2)}$ and, accordingly, the other field with $\mathcal{B}^{(1)}$. The complete holographic map regarding these fields is summarized in Table 8.1.

Table 8.1: Holographic map in summary

State	Geometry/Field
$S_k^{(1)}$	deformation by C_0 and B_2 (6.73)
$G^+\tilde{G}^+S_k^{(1)}$	$\hat{c}$
$G^+\tilde{G}^+S_k^{(2)}$	b
$G^+\tilde{G}^+\sigma_k$	$\hat{\lambda}$

8.2 Possible developments

Now that we derived the dual field to the descendant $G^+\tilde{G}^+S_{k_3}^{(1)}$ it would be natural to go on and holographically compute the four-point function with two of those descendants in the background given by two heavy states, namely

$$\left\langle \mathcal{O}_{k_1}^{(1)}\mathcal{O}_{k_2}^{(1)}G^+\tilde{G}^+S_{k_3}^{(1)}G^-\tilde{G}^-S_{k_4}^{(1)}\right\rangle \tag{8.3}$$

where $\mathcal{O}_k^{(1)}$ is the coherent state defined as a sum of multiple $S_k^{(1)}$ introduced in (7.5). One can show that a Ward identity relates (8.3) with the simpler

$$\left\langle \mathcal{O}_{k_1}^{(1)}\mathcal{O}_{k_2}^{(1)}S_{k_3}^{(1)}S_{k_4}^{(1)}\right\rangle \tag{8.4}$$

The correlator (8.3) has never been computed directly. The function computed in [25] can be written as

$$\left\langle \mathcal{O}_{k_1}^{(i)}\mathcal{O}_{k_2}^{(i)}S_{k_3}^{(j)}S_{k_4}^{(j)}\right\rangle \tag{8.5}$$

which is slightly simpler, in that it has two flavours. An argument to explain how this is simpler is provided intuitively by considering the four-point function

$$\left\langle S_1^{(i)}S_1^{(j)}S_1^{(k)}S_1^{(l)}\right\rangle \tag{8.6}$$

whose result is bound to be of the form

$$\delta^{ij}\delta^{kl}(\cdots) + \delta^{ik}\delta^{jl}(\cdots) + \delta^{il}\delta^{jk}(\cdots) \tag{8.7}$$

implying that the correlator (8.6) with $i = j$ and $k = l$ has fewer terms than the one with $i = j = k = l$. Note that the result of (8.3) will not have the structure (8.7), but the same idea applies.

In the supergravity context, the correlator (8.5) correspond to computing a simple wave equation involving a dual field, namely $\Box_6\, h_{ij} = 0$, because h_{ij} is independent. Instead, the situation is more complicated in the case of (8.3), because we found in our work that $\hat{c}$ is coupled with $\hat{\lambda}$. This means that the equations to be solved are actually the coupled set of (7.45a) and (7.45b).

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