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**Absolute integral closure and applications
towards some homological conjectures**

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Introduzione

In questa tesi studieremo la chiusura integrale assoluta di un dominio. Esiste un chiaro parallelo tra estensioni di campi ed estensioni integrali, la chiusura integrale assoluta di un dominio è l'analogo integrale della chiusura algebrica di un campo; successivamente la indicheremo con R^+ quando R è un dominio d'integrità. Questo anello, in caratteristica positiva e con opportune ipotesi su R , è un esempio di algebra Big Cohen-Macaulay. L'esistenza di moduli Big Cohen-Macaulay si è rivelata fondamentale per dimostrare diverse congetture omologiche; si veda, ad esempio, [1] o [2]. Lo scopo principale di questa tesi è dimostrare che quando R è un dominio locale in caratteristica positiva che è l'immagine di un anello Cohen-Macaulay, R^+ è un'algebra Big Cohen-Macaulay. È importante notare che la chiusura integrale assoluta di un dominio possiede proprietà interessanti al di là delle sue coomologie locali, e alcune di esse saranno spiegate nel capitolo uno. Nella prima sezione del capitolo uno richiamiamo alcune nozioni e proprietà di base delle estensioni integrali e dimostriamo che per ogni dominio R , $\overline{R[x]} = \overline{R}[x]$. Nella seconda sezione introduciamo i concetti di chiusura integrale assoluta e di domini assolutamente integralmente chiusi. Dimostreremo inoltre alcune proprietà di base di R^+ e forniremo altre utili caratterizzazioni per i domini assolutamente integralmente chiusi. Nella terza sezione forniamo un'introduzione agli anelli completi con l'obiettivo di dimostrare una versione del Lemma di Hensel e discuteremo il Teorema di Struttura di Cohen. Infine, nella quarta sezione del capitolo uno, forniamo un'introduzione ai concetti fondamentali dell'algebra commutativa quali anelli regolari, moduli Cohen-Macaulay, involucri iniettivi ed anelli Gorenstein. Concludiamo il capitolo con alcuni teoremi utili, riguardanti la relazione tra moduli e anelli completi.

Il capitolo due è dedicato a fornire una caratterizzazione esplicita dell'anello $K[[t]]^+$ quando K è un campo algebricamente chiuso. Vedremo che la costruzione di una chiusura algebrica di $K((t))$ dipende fortemente dalla caratteristica del campo. Nella prima sezione del capitolo due, dimostreremo che il campo di Newton-Puiseux è una chiusura algebrica di $K((t))$ in caratteristica zero. Per dimostrarlo, sarà sufficiente trovare un modo per applicare il Lemma di Hensel (si veda [3]). Dopo aver trovato la chiusura algebrica, saremo in grado di descrivere $K[[t]]^+$ in caratteristica zero. Nella seconda sezione del capitolo due, studieremo la questione in caratteristica positiva. Seguiremo [4] per definire quelle che verranno chiamate "twist recurrent series" e dimostreremo che l'insieme di tutte le twist recurrent series è esattamente la chiusura algebrica che stiamo cercando. Anche in questo caso, forniremo una caratterizzazione di $K[[t]]^+$ in caratteristica positiva. Nel capitolo tre dimostreremo il teorema principale di questa tesi. Inizieremo con necessari richiami di coomologia locale e dualità di Matlis. Nella seconda sezione forniamo un'introduzione alla rappresentazione secondaria e ai primi attaccati. Nella terza sezione presentiamo una dimostrazione completa della proprietà Big Cohen-Macaulay di R^+ sotto le nostre ipotesi. Vedremo che la proprietà di Big Cohen-Macaulay su R^+ è equivalente all'annullamento di tutte le coomologie locali con supporto nell'ideale massimale di R , per tutti gli indici diversi dalla dimensione di R . La dimostrazione presentata è tratta da [5], che costituisce un leggero miglioramento rispetto a quanto fatto in [6]. In generale, tutte le dimostrazioni di questo fatto argomentano che qualsiasi elemento nella coomologia locale deve essere annullato in un'estensione finita dell'anello sufficientemente grande.

Nel Capitolo quattro dimostriamo alcune delle congetture omologiche presentate in [1], seguendo la trattazione di [6]. Per essere espliciti, saremo in grado di utilizzare la proprietà di Big Cohen-Macaulay di R^+ per dimostrare, in caratteristica positiva, la congettura monomiale, una congettura d'intersezione, la congettura dei divisori dello zero e un'affermazione di Grothendieck riguardante lo spettro bucato di un anello locale.

Nell'appendice enunciamo e dimostriamo alcuni fatti utili che dovrebbero già essere noti al lettore o che risultano semplicemente troppo fuori contesto per essere inseriti nelle sezioni principali.

Introduction

In this thesis, we will study the absolute integral closure of a domain. There is a clear parallel between field extensions and integral extensions. The absolute integral closure of a domain is the integral analog of the algebraic closure of a field. We will later denote it with R^+ when R is an integral domain. Indeed, R^+ is the maximal integral extension domain over R . This ring, in positive characteristic and with appropriate hypotheses on R , is an example of a Big Cohen-Macaulay algebra. The existence of Big Cohen-Macaulay modules has proved to be central to proving various homological conjectures; see, for example, [1] or [2]. The main goal of this thesis is to prove that when R is a local positive characteristic domain that is the homomorphic image of a Cohen-Macaulay ring, R^+ is a Big Cohen-Macaulay algebra.

It is important to notice that the absolute integral closure of a domain has interesting properties beyond its local cohomologies, and some of them will be explained in chapter one. In the first section of chapter one, we recall some basic notions and properties of integral extensions, and we prove that for any domain R , $\overline{R[x]} = \overline{R}[x]$. In the second section, we introduce the concepts of absolute integral closure and absolutely integrally closed domains. We will also prove some basic properties of R^+ and give other useful characterizations for absolutely integrally closed domains. In the third section, we give an introduction to complete rings with the goal of proving the factorization version of Hensel's Lemma and we will discuss Cohen's Structure Theorem. Finally, in the fourth section of chapter one, we give an introduction to basic commutative algebra concepts such as regular rings, Cohen-Macaulay modules, injective hulls, and Gorenstein rings. We end the chapter with some useful theorems regarding the relation between modules and complete rings.

Chapter two is dedicated to giving an explicit characterization of the ring $K[[t]]^+$ when K is an algebraically closed field. It turns out that the construction of an algebraic closure of $K((t))$ strongly relies on the characteristic of the field. In the first section of chapter two, we shall prove that the Newton-Puiseux field is an algebraic closure of $K((t))$ in characteristic zero. To prove this, it will be sufficient to find a way to use Hensel's Lemma (see [3]). After finding the algebraic closure, we will be able to describe $K[[t]]^+$ in characteristic zero. In the second section of chapter two, we will study the case of positive characteristic. We will follow [4] to define what will be called "twist recurrent series", and we will prove that the set of all twist recurrent series is exactly the algebraic closure we are looking for. Again, we will provide a characterization of $K[[t]]^+$ in positive characteristic.

In chapter three, we shall move towards, and prove, the main theorem of this thesis. We begin with some needed background on local cohomology and Matlis duality. In the second section, we give an introduction to secondary representation and attached primes. In the third section, we present a complete proof of the Big Cohen-Macaulayness of R^+ in our hypotheses. It turns out that the Big Cohen-Macaulay property of R^+ is equivalent to the annihilation of all the local cohomologies supported on the maximal ideal of R , for all the indexes different from the dimension of R . The proof presented is taken from [5], which is a slight improvement upon what was done in [6]. Generally, all the proofs of this fact argue that any element in local cohomology needs to be annihilated in a sufficiently large finite integral extension of the ring.

In Chapter four, we prove some of the homological conjectures presented in [1], following the discussion from [6]. To be explicit, we will be able to use the Big Cohen-Macaulayness of R^+ to prove, in positive characteristic, the Monomial conjecture, an intersection conjecture, the zero divisor conjecture and a statement from Grothendieck regarding the punctured spectrum of a local ring.

In the appendix, we state and prove some useful facts that either should already be known by the reader or are simply too out of place to be added to the main sections.

1 Preliminaries

Before proceeding, the reader is expected to have familiarity with the definitions of the basic objects and ideas of commutative algebra, such as groups, rings, ideals, modules, Noetherianity, Krull dimension and some category theory.

Every ring will always be commutative and have a multiplicative identity (which is not zero); every morphism and inclusion of rings will map the identity of the first ring to the identity of the second ring, unless specified otherwise. (For example, if we say that a ring contains a field, we also mean that they both have the same identity).

In this chapter we first give a brief introduction to integral theory. We will recall the basic properties that are necessary to work with integral extension, for example the classic characterization of integral finite type extensions and the transitivity of integral extensions. With this tools we will prove that whenever R is a domain, $\overline{R[x]} = \overline{R}[x]$; the proof will subtly make use of what we will later call R^+ .

In the second section, we shall introduce R^+ as done by C. Huneke in [6], its basic property and some of its characterizations. We will prove that localization and quotient by an ideal are, in some sense, well behaved with absolute integral closure. We will also prove two property discovered by M. Artin [7] about sums of ideal inside R^+ , mainly the fact that sums of prime ideals are prime and sums of primary ideals are primary.

In the third section of this chapter we follow sporadically chapter 7 of Eisenbud's book [8]. We shall give an introduction to complete rings and prove some useful properties. We then shall follow [8] to prove the "factorization version" of Hensel's Lemma, which we will use in the next chapter to find $K[[t]]^+$ in the case when K is an algebraically closed field of characteristic zero. We end the chapter by stating Cohen's Structure Theorem for rings containing a field. We shall prove it only in the specific easy case where the ring is a domain of characteristic zero. For a complete proof, see [8], pag. 201-203.

In the fourth section of this chapter we will recall some basic definitions and theorems from commutative algebra and we will prove some results that will later be used in chapter 3.

1.1 Integral extensions

Definition 1.1. Let R, S be two rings such that $R \subseteq S$.

An element $s \in S$ is said to be integral over R if there exists a monic polynomial $p \in R[x]$ such that $p(s) = 0$.

The integral closure of R inside S is the set of elements in S that are integral over R , we shall denote it with $\overline{R}^S$. If R is a domain, we will use $\overline{R}$ to denote $\overline{R}^{\text{frac}(R)}$.

A ring R is said to be integrally closed in S if $R \subseteq S$ and $R = \overline{R}^S$.

A domain R is integrally closed (or normal) if $R = \overline{R}$.

A rings extension $R \subseteq S$ is integral if $S = \overline{R}^S$.

Definition 1.2. Let $R \subseteq S$ be a ring extension.

We say that the extension is finite if S is a finitely generated R -module.

We say that the extension is of finite type if S is a finitely generated R -algebra.

Remark 1.3. (Transitivity) Let $R \subseteq S \subseteq T$ be ring inclusions, it is easy to verify the following statements:

1. If $R \subseteq S$ and $S \subseteq T$ are finite then $R \subseteq T$ is finite.
2. If $R \subseteq S$ and $S \subseteq T$ are of finite type then $R \subseteq T$ is of finite type.
3. If $R \subseteq T$ is integral then $R \subseteq S$ and $S \subseteq T$ are both integral.
4. $\overline{R}^T \subseteq \overline{S}^T$.

For a more in depth discussion on the converse of statements 1 and 2, see 5.1. The transitivity of integral extensions is also true, but to prove it we need more than the definition; we will make use of the following classical fact.

Theorem 1.4. Let $R \subseteq S$ be a finite type rings extension, so that we can write $S = R[s_1, \dots, s_n]$, then the following statements are equivalent:

1. all the generators s_i are integral over R
2. $R \subseteq S$ is finite
3. $R \subseteq S$ is integral.

Corollary 1.5. Let $R \subseteq S$ be a ring extension, then $\overline{R}^S$ is a ring and $\overline{\overline{R}^S}^S = \overline{R}^S$; furthermore if $R \subseteq S$ and $S \subseteq T$ are integral then $R \subseteq T$ is integral.

Proof. For every pair of elements $s_1, s_2 \in \overline{R}^S$, we have, by the previous theorem, that $R \subseteq R[s_1, s_2]$ is integral; in particular $R[s_1, s_2] \subseteq \overline{R}^S$, so $\overline{R}^S$ is a ring.

Now to prove the transitivity, assume that $R \subseteq S$ and $S \subseteq T$ are both integral, then if $t \in T$ there exists a monic polynomial $p \in S[x]$ such that $p(t) = 0$.

Let us write $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$, then $t \in \overline{R[a_0, \dots, a_{n-1}]}^T$ so we have that $R \subseteq R[a_0, \dots, a_{n-1}] \subseteq R[a_0, \dots, a_{n-1}, t]$ are all finite extensions, and thus integral, in particular t is integral over R .

To prove that $\overline{\overline{R}^S}^S = \overline{R}^S$, we shall use the just proven transitivity.

Since $R \subseteq \overline{R}^S$ and $\overline{R}^S \subseteq \overline{\overline{R}^S}^S$ are by definition integral extension, we have that $R \subseteq \overline{\overline{R}^S}^S$ is also integral and so $\overline{\overline{R}^S}^S \subseteq \overline{R}^S$ which concludes the proof. $\square$

Remark 1.6 (Closure property). The intersection of integrally closed rings is integrally closed. Formally, if I is a non empty set and $R_i \subseteq S$ are subrings of S for all $i \in I$, then $\bigcap_{i \in I} \overline{R_i}^S$ is integrally closed over S , or equivalently:

$$\bigcap_{i \in I} \overline{R_i}^S = \overline{\bigcap_{i \in I} \overline{R_i}^S}^S.$$

Indeed, if $s \in \overline{\bigcap_{i \in I} \overline{R_i}^S}^S$ then there exists a monic polynomial $p \in \bigcap_{i \in I} \overline{R_i}^S[x]$ such that $p(s) = 0$, in particular for every $i \in I$, $p \in \overline{R_i}^S[x]$, so $s \in \overline{\overline{R_i}^S}^S = \overline{R_i}^S$ and so $s \in \bigcap_{i \in I} \overline{R_i}^S$. The other inclusion is tautological.

Remark 1.7. In general if $R \subseteq S$ is a ring extension, then $\overline{R}^S$ doesn't invert any element of R , that is to say if $r \in R$ is not a unit then $r \in \overline{R}^S$ is also not a unit. Indeed if $r^{-1} \in \overline{R}^S$, then this means that there exists a monic polynomial $p \in R[x]$ such that $p(r^{-1}) = 0$. If we write $p(x) = x^n + \sum_{i=0}^{n-1} p_i x^i$, then $r^n p(r) = 0$ implies that $1 + \sum_{i=0}^{n-1} p_i r^{n-i} = 0$. In particular $1 = -r(\sum_{i=0}^{n-1} p_i r^{n-1-i})$ and so r is a unit.

Lemma 1.8. Let $R \subseteq S$ be an integral extension of rings, $s \in S$.

If $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in R[x]$ is the minimal monic polynomial that annihilates s . Then if a_0 is a unit, s is a unit.

The converse holds if R is a UFD (Unique Factorization Domain).

Proof. If a_0 is a unit, then clearly $s(s^{n-1} + a_{n-1}s^{n-2} + \dots + a_1) = a_0$ so s is a unit. If s is a unit, then, by the previous remark, s is a unit in $R[s]$, thus there is a polynomial $q(x) \in R[x]$ such that $sq(s) - 1 = 0$. Since R is an UFD, Gauss's Lemma applies, so $p(x)$ is actually the minimal polynomial that annihilates s (considering also non monic polynomials). Hence, $xq(x) - 1 = p(x)g(x)$ for some $g(x) \in R[x]$. So $-1 = a_0 g_0$ where g_0 is the zero'th coefficient of g , therefore a_0 is a unit. $\square$

Theorem 1.9. Every UFD is integrally closed.

Proof. Let R be an UFD, let $s = \frac{b}{c} \in \overline{R}$ where $b, c \in R$ are such that they don't share any irreducible element in their factorization.

Let $p \in R[x]$ be a monic polynomial such that $p(s) = 0$, $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ then by first evaluating $s = \frac{b}{c}$ and then multiplying by c^n , we get:

$$b^n + a_{n-1}b^{n-1}c + \dots + a_0c^n = 0$$

so $b^n = -c(a_{n-1}b^{n-1} + \dots + a_0c^{n-1})$. If c' is an irreducible in the factorization of c , then $c' | b^n$ and $c' | b$. This contradicts the choices of b and c , so c must be a unit of R and $s \in R$. $\square$

Lemma 1.10. Let K be an algebraically closed field, let us denote with μ_n the set of n 'th roots of the identity inside K . Let $n, k \in \mathbb{N}$, then:

$$\sum_{\eta \in \mu_n} \eta^k = \begin{cases} n' & \text{if } n' | k \\ 0 & \text{otherwise} \end{cases} \quad \text{where } n' = |\mu_n|$$

Proof. The exponential map $\wedge k : \mu_n \rightarrow \mu_n$ is an homomorphism of multiplicative groups, so its fibers all have the same cardinality as $\ker(\wedge k)$; furthermore, the image of $\wedge k$ is equal to μ_l for some $l \in \mathbb{N}$, since the image of a group homomorphism is a group and every subgroup of a cyclic group is cyclic. Note that

$$\sum_{\eta \in \mu_n} \eta^k = |\ker(\wedge k)| \sum \text{Im}(\wedge k) = |\ker(\wedge k)| \sum \mu_l.$$

We want to evaluate $\sum \mu_l$.

Recall that the polynomial $x^t - 1$ is always separable in characteristic zero, while in characteristic p it is separable if and only if p does not divide t .

Moreover recall that $n' = |\mu_n| = n$ in characteristic zero and $n' = |\mu_n| = \frac{n}{\gcd(n, p)}$ in characteristic p .

Let us assume that $n = n'$. Now, since l divides n , (and in characteristic p , p does

not divide $|\mu_n| = n$) it is true that the polynomial $x^l - 1$ does not have multiple roots, and so $\sum \mu_l$ is exactly the $(l-1)$ 'st coefficient of $-\prod_{\theta \in \mu_l}(x - \theta) = 1 - x^l$ and so it is zero whenever $l \neq 1$.

If $l = 1$, then $Im(\wedge k) = \{1\}$ and so $|ker(\wedge k)| = |\mu_n| = n$; moreover this means that all the n 'th roots of the identity are also k 'th roots of the identity, so $\mu_n \subseteq \mu_k$, which means that $n = |\mu_n|$ divides $|\mu_k|$ which divides k .

For the converse: if $n = |\mu_n|$ divides k , then clearly $\eta^k = 1$ for any $\eta \in \mu_n$, and so $l = 1$.

Now let us prove the statement in general.

As said before, in characteristic zero it is always true that $n = n'$. So let us assume $char(K) = p$, now if $n = p^k m$ where $m = \frac{n}{gcd(n,p)} = |\mu_n|$, then $x^n - 1 = (x^m - 1)^{p^k}$ and so $\mu_n = \mu_m$.

We deduce that $m = |\mu_n| = |\mu_m|$, this equality tells us that $m = m'$ so we can use what we proved before and get:

$$\sum_{\eta \in \mu_n} \eta^k = \sum_{\eta \in \mu_m} \eta^k = \begin{cases} m & \text{if } m|k \\ 0 & \text{otherwise} \end{cases} \quad \text{where } m = |\mu_n|$$

□

Proposition 1.11. If R is a domain, $\overline{R[x]} = \overline{R}[x]$

Proof. The inclusion $\overline{R[x]} \subseteq \overline{R[x]}$ is given by the fact that $\overline{R[x]}$ is a ring containing $\overline{R}$ and x , moreover, using 1.9, we have that $\overline{R[x]} \subseteq \overline{k[x]} = k[x]$, where $k = frac(R)$. We shall now prove the other inclusion, let $r \in \overline{R[x]} \subseteq k[x]$ so that we can think r as a polynomial in $k[x]$, $r(x) = \sum_{i=0}^{n-1} r_i x^i$; our goal is to prove that $r_i \in \overline{R}$ for all i . In case the characteristic of R is p we can assume (by multiplying if necessary r times x) that p does not divide n and so $n = n'$ in the notation of the previous lemma. Let $q(x, y) \in R[x][y]$ be a monic polynomial in y such that $q(x, r(x)) = 0$, which we know it exists because $r \in \overline{R[x]}$.

We first observe that every evaluation of $r(x)$ in an integral element produces an integral element. Indeed if s is integral, the polynomial $q(s, y)$ is monic in y with integral coefficients and it annihilates when $y = r(s)$, thus $r(s) \in \overline{R}$.

Let K be an algebraic closure of k so that we can define μ_n as the set of the n 'th roots of the identity inside K .

$R \subseteq R[\mu_n]$ is an integral extension, since all the generators annihilate the monic polynomial $x^n - 1$.

For every $\eta \in \mu_n$ we have that $r(\eta) \in K$ is integral over R , similarly to before we have that $q(\eta, y)$ is monic in y and it has coefficients in $R[\mu_n]$ which are integral, thus $r(\eta) \in \overline{R}^K$. For every $j \in \{0, \dots, n-1\}$, $\sum_{\eta \in \mu_n} \eta^{-j} r(\eta) \in K$ is integral over R because it is an algebraic combination of integral elements over R , thus nr_j is integral:

$$\sum_{\eta \in \mu_n} \eta^{-j} r(\eta) = \sum_{\eta \in \mu_n} \eta^{-j} \sum_{i=0}^{n-1} r_i \eta^i = \sum_{i=0}^{n-1} r_i \sum_{\eta \in \mu_n} \eta^{i-j} = nr_j$$

To justify the last equality:

we want $\sum_{\eta \in \mu_n} \eta^{i-j} = n\delta_{0, i-j}$, but since we have fixed $i, j \in \{0, \dots, n-1\}$ the difference can only stay in $\{-n+1, \dots, 0, \dots, n-1\}$ which contains only one multiple

of n namely, $i - j = 0$, so by the previous lemma we get what we need.

We have proved, for every $j \in \{0, \dots, n-1\}$, that $nr_j \in \overline{R}^K$, but since $r_j \in k$ we have that $nr_j \in \overline{R}^K \cap k = \overline{R}$. In characteristic p , since n is a unit, we conclude that $r_j \in \overline{R}$.

In characteristic zero, consider $xr(x) \in \overline{R[x]}$; for the same reasons $(n+1)r_j \in \overline{R}$ for every $j \in \{0, \dots, n-1\}$, so $r_j = (n+1)r_j - nr_j \in \overline{R}$ for every $j \in \{0, \dots, n-1\}$. $\square$

Corollary 1.12. If R is an integrally closed domain then $R[x_1, \dots, x_n]$ is integrally closed for any $n \in \mathbb{N}$.

Question. If R is a domain, is there an easy characterization of $\overline{R[[x]]}$?

Remark 1.13. If k is a field, $k[[x]]$ is integrally closed.

In fact $\text{frac}(k[[x]]) = k((x)) = k[[x]][x^{-1}]$, and since any element of $k((x))$ is of the form $x^n \lambda$, with $\lambda \in k[[x]]$ a unit and $n \in \mathbb{Z}$, then $x^n \lambda \in k[[x]]$ if and only if $x^n \in k[[x]]$.

Using Remark 1.7, we know that $x^{-1} \notin k[[x]]$, and so $x^n \in k[[x]]$ if and only if $n \geq 0$. Thus $x^n \lambda \in k[[x]]$ if $x^n \lambda \in k[[x]]$ if $x^n \lambda \in k[[x]]$.

Another classical result from the theory of integral extensions is the following Proposition.

Proposition 1.14. If $R \subseteq S$ is an integral extension of domains, then R is a field if and only if S is a field.

As a direct consequence, without necessarily assuming that R and S are domains, if $Q \in \text{Spec}(S)$ then Q is a maximal ideal if and only if $Q \cap R$ is a maximal ideal.

Proof. Assume that R is a field, then if $s \in S \setminus \{0\}$, it is sufficient to show that it is a unit. Let $p(x) = x^n + \sum_{i=0}^{n-1} p_i x^i$ be a monic polynomial in $R[x]$ such that $p_0 \neq 0$ and $p(s) = 0$. The existence of such polynomial with the condition $p_0 \neq 0$ is guaranteed by the hypothesis that S is a domain.

We have that $0 = p(s) = s(s^{n-1} + \sum_{i=1}^{n-1} p_i s^{i-1}) + p_0$, but since p_0 is a unit, it follows that also s is a unit.

Assume that S is a field, then if $r \in R \setminus \{0\}$, $r^{-1} \in S$, let $p(x) = x^n + \sum_{i=0}^{n-1} p_i x^i \in R[x]$ be a monic polynomial such that $0 = r^{n-1} p(r^{-1}) = r^{-1} + \sum_{i=0}^{n-1} p_i r^{n-1-i}$, and so $r^{-1} \in R$.

Now if $Q \in \text{Spec}(S)$, it is easy to see that the natural map $R/Q \cap R \hookrightarrow S/Q$ is an integral extension between domain. And so Q is maximal if and only if $Q \cap R$ is maximal. $\square$

1.2 Absolute integral closure

Remark 1.15. Let $R \subseteq S$ and $R \subseteq S'$ be two rings extensions.

If $\phi : S \rightarrow S'$ is an R -algebra homomorphism, then

1. $\phi(\overline{R}^S) \subseteq \overline{R}^{S'}$ or equivalently $\overline{R}^S \subseteq \phi^{-1}(\overline{R}^{S'})$
2. if ϕ is injective then $\overline{R}^S = \phi^{-1}(\overline{R}^{S'})$
3. if ϕ is an isomorphism then $\phi(\overline{R}^S) = \overline{R}^{S'}$ and $\overline{R}^S \cong \overline{R}^{S'}$

All this properties follows from the definitions.

Remark 1.16. If $R \subseteq S$ is an integral extension between domains, then the fields extension $\text{frac}(R) \subseteq \text{frac}(S)$ is algebraic, indeed if $\lambda \in \text{frac}(S)$ then $\lambda = \frac{a}{b}$, with $a, b \in S$, $b \neq 0$, since a is integral over R , $\frac{a}{1}$ is an algebraic element of $\text{frac}(S)$ over $\text{frac}(R)$, moreover b is also integral over R so there exists a monic polynomial $p \in R[x]$ such that $p(b) = 0$, so $p(b)b^{-\deg(p)} = 0$ but this is a non zero polynomial evaluated in b^{-1} with coefficients in R so b^{-1} is algebraic over $\text{frac}(R)$.

In the end, λ is algebraic, for it is a product of algebraic elements.

Remark 1.17. Note that the converse does not hold: if $R \subseteq S$ is a domain extension such that $\text{frac}(R) \subseteq \text{frac}(S)$ is algebraic, we cannot deduce that $S \subseteq R$ is integral. For example the domain extension $\mathbb{Z} \subseteq \mathbb{Q}$ is not integral but it generates the identity. This said, it is true that if $R \subseteq S$ is a domain extensions and $\text{frac}(R) \subseteq \text{frac}(S)$ is algebraic, then $\overline{R}^S \subseteq S$ is an essential extension of R -modules, this means that for every $s \in S \setminus \{0\}$ there exists $r \in R$ such that $rs \in \overline{R}^S \setminus \{0\}$, in fact if $s \in S$ then it is algebraic over $\text{frac}(R)$ so there exists a positive degree polynomial $q \in \text{frac}(R)[x]$ such that $q(s) = 0$. We can assume, without loss of generality, that $q \in R[x]$ (for example we can multiply q with the product of the denominators of its coefficients). Let $q(x) = \sum_{i=0}^n a_i x^i$, then

$$0 = a_n^{n-1} q(s) = \sum_{i=0}^n a_n^{n-1} a_i s^i = \sum_{i=0}^n a_n^{n-1-i} a_i (a_n s)^i = \tilde{q}(a_n s)$$

We observe that $\tilde{q} \in R[x]$ and it is monic, so $a_n s \in \overline{R}^S \setminus \{0\}$.

Definition 1.18. If R is a domain, we shall denote with R^+ the integral closure of R inside an arbitrary algebraic closure of its quotient field.

A domain is said to be absolutely integrally closed, if it does not admit any proper integral extensions into a domain.

Remark 1.19. Let R be a domain.

1. R^+ is well defined up to isomorphism of R -algebras, in fact given two different algebraic closure of $\text{frac}(R)$, there exists a $\text{frac}(R)$ -linear ring isomorphism between the two and thus we can use 3 of 1.15.
2. If K is an algebraic closure of $\text{frac}(R)$ and $R^+ = \overline{R}^K$, then $\text{frac}(R^+) = K$. To show this, notice that $\text{frac}(R) \subseteq K$ is an algebraic extension, so by Remark 1.17 we have that $R^+ \subseteq K$ is an essential extension of R -modules, so we have that for every $\lambda \in K$ there exists $r \in R$ such that $r\lambda \in R^+ \setminus \{0\}$, so $\lambda = \frac{r\lambda}{r} \in \text{frac}(R^+)$.

Remark 1.20. Is there a converse to point 2 of Remark 1.19? Is it true that if a domain R has a quotient field that is algebraically closed, then it must be equal to R^+ ? The answer is no, see 5.2 for a counterexample.

Corollary 1.21. A domain R is absolutely integrally closed if and only if $R = R^+$.

Proof.

- If R is a domain, R^+ is absolutely integrally closed:
We want to show that if T is a domain such that $R^+ \subseteq T$ is an integral extension, then T is equal to R^+ . Let $K = \text{frac}(R^+)$ viewed inside $\text{frac}(T)$; we know that K is algebraically closed by Remark 1.19, but we have, since $R^+ \subseteq T$ is integral, that $K \subseteq \text{frac}(T)$ is an algebraic extension, and so $K = \text{frac}(T)$, in particular $T \subseteq K$. Since the extension $R^+ \subseteq T$ is integral inside K , we have that $T \subseteq R^+$.
- If R is an absolutely integrally closed domain, then it is isomorphic to R^+ :
if K is an algebraic closure of $\text{frac}(R)$, then $R \subseteq \overline{R}^K = R^+$ is an integral extension into a domain and so it is the identity, that is to say, R is integrally closed over any such K .

□

Proposition 1.22. Let R be a domain, the following definitions of R^+ are equivalent:

1. R^+ is a ring isomorphic to the integral closure of R inside any algebraic closure of the quotient ring of R .
2. R^+ is an absolutely integrally closed domain containing R such that $R \subseteq R^+$ is an integral extension.

Proof. We already know that $1 \implies 2$;

Let us show that $2 \implies 1$, we will denote with R^+ a ring that satisfies 2.

If K is an algebraic closure of $\text{frac}(R)$, then $\text{frac}(R) \subseteq \text{frac}(R^+)$ is algebraic, so there exists an inclusion $\text{frac}(R^+) \subseteq K$ that preserves $\text{frac}(R)$; so $R^+ \subseteq K$ in particular $R^+ \subseteq \overline{R}^K$ but this is an integral extension of R^+ into a domain and since R^+ is absolutely integrally closed, we have that $R^+ = \overline{R}^K$ □

Remark 1.23. From the second definition, it follows that whenever $R \subseteq S$ is an integral extension of domains, then $R^+ = S^+$, since S^+ is an absolutely integrally closed domain containing R such that $R \subseteq S \subseteq S^+$ is integral.

Knowing this, it will be sufficient to study the absolute integral closure of integrally closed domain.

Remark 1.24. Let R be a domain, R^+ is the direct limits of the diagram generated by all the finite integral extensions of rings $S \subseteq T$ such that $R \subseteq S \subseteq T \subseteq R^+$ and $R \subseteq S$ is finite. This is true since taking the direct limit, in this situation, is equivalent to taking the union. (see appendix 5.22)

Proposition 1.25. A domain R is absolutely integrally closed if and only if every monic polynomial in $R[x]$ of positive degree is a product of monic polynomials of degree 1 in $R[x]$.

Proof. It is a fact that given two monic polynomials $p, q \in R[x]$, it is always possible to do the euclidean division: this means one can find $r, t \in R[x]$ such that $p = rq + t$, with $\deg(t) < \deg(q)$ and r a monic polynomial.

Let R be an absolutely integrally closed domain, then, for the fact above, it is sufficient to show that any monic polynomial $p \in R[x]$ has a root in R . We know that $R = R^+$ so if $p \in R[x]$ is monic of positive degree, it admits a root $\eta \in \text{frac}(R^+)$ since it is algebraically closed, and so $p(\eta) = 0$, which implies that $\eta \in R^+ = R$.

For the converse, if $t \in R^+$ then there exists a monic polynomial $p \in R[x]$ such that $p(t) = 0$ but $p(x) = \prod_i (x - r_i)$ so since R is a domain, $t = r_i$ for some i and so $t \in R$, it follows that $R = R^+$ and so R is absolutely integrally closed. $\square$

Corollary 1.26. Let $R \subseteq S$ be an integral domain extension.

$S \cong R^+$ if and only if every monic polynomial in $R[x]$ factors completely into monic polynomials of $S[x]$.

Proof. We already know the first implication. Let us assume that every monic polynomial in $R[x]$ factors completely in $S[x]$, it is sufficient to show that $S = S^+$. Let $s \in S^+$, since $R \subseteq S \subseteq S^+$ are integral extension, there exists a monic polynomial $p \in R[x]$ such that $p(s) = 0$, but since $p(x) = \prod_i (x - s_i)$ and we are working with domains, it follows that $s = s_i$ for some i and so $s \in S$. $\square$

Recall that by Remark 1.7, integral extensions do not invert elements. This means that our ring R^+ is not a field unless R is, and so it is significant to ask questions about its ideals:

Proposition 1.27. (Artin) Let R be a domain, if $\mathfrak{p}, \mathfrak{q} \subseteq R^+$ are prime ideals then $\mathfrak{p} + \mathfrak{q}$ is either a prime ideal or the entire ring.

Proof. Let $a, b \in R^+$ such that $ab \in \mathfrak{p} + \mathfrak{q}$, let us set $z = b - a$. We have that $a^2 + za = a^2 + (b - a)a = ab = u + v$ for some $u \in \mathfrak{p}$ and $v \in \mathfrak{q}$.

The polynomial equation $X^2 + zX = u$ must have a solution $x \in R^+$, and so we have that $x(x + z) = u \in \mathfrak{p}$, so either $x \in \mathfrak{p}$ or $x + z \in \mathfrak{p}$.

Now consider $(x^2 + zx) - (a^2 + za) = u - (u + v) = -v \in \mathfrak{q}$ so we have that $(x^2 - a^2) + z(x - a) = (x - a)(x + a + z) \in \mathfrak{q}$, it follows that either $x - a$ or $x + a + z$ belongs to $\mathfrak{q}$.

All this cases gives us that either a or b belongs to $\mathfrak{p} + \mathfrak{q}$, indeed:

1. if $x \in \mathfrak{p}$ and $x - a \in \mathfrak{q}$ then $a \in \mathfrak{p} + \mathfrak{q}$
2. if $x \in \mathfrak{p}$ and $x + a + z \in \mathfrak{q}$ then $a + z = b \in \mathfrak{p} + \mathfrak{q}$
3. if $x + z \in \mathfrak{p}$ and $x - a \in \mathfrak{q}$ then $a + z = b \in \mathfrak{p} + \mathfrak{q}$
4. if $x + z \in \mathfrak{p}$ and $x + a + z \in \mathfrak{q}$ then $a \in \mathfrak{p} + \mathfrak{q}$

$\square$

Using the same technique, we can also prove the statement for primary ideals.

Proposition 1.28. Let R be a domain, if $I, J \subseteq R^+$ are primary ideals then $I + J$ is either a primary ideal or the entire ring.

Proof. Recall that $\sqrt{I} + \sqrt{J} \subseteq \sqrt{I + J}$

Let $a, b \in R^+$ such that $ab \in I + J$, assume that $a \notin \sqrt{I + J}$, we need to show that $b \in I + J$. Set $z = b - a$, we have that $ab = u + v$ where $u \in I$ and $v \in J$.

Let $x \in R^+$ such that $x(x + z) = u \in I$.

We have that $u + v = ab = a(a + z)$, and so

$$v = a^2 + za - u = a^2 + za - x^2 - zx = (a - x)(a + x + z) \in J$$

Since I is a primary ideal, and $u = x(x + z) \in I$, at least one of the factors of u must belong to the radical of I .

1. Let us assume $x \in \sqrt{I}$. If $a - x \in \sqrt{J}$ then $a \in \sqrt{I} + \sqrt{J}$ which we have assumed to be false. So $a - x \notin \sqrt{J}$ which means that $a + x + z \in J$. Using again the assumption on a , we have that $x + z \notin \sqrt{I}$ and so $x \in I$. Concluding we have $b + x = a + x + z \in J \implies b \in I + J$.

2. Let us assume $x + z \in \sqrt{I}$. If $x + a + z \in \sqrt{J}$ then we would have $a \in \sqrt{I} + \sqrt{J}$ which we have assumed to be false, so $a - x \in J$, which again means that $x \notin \sqrt{I}$ and so $x + z \in I$.

Finally we can write $b = a + z = (a - x) + (x + z) \in I + J$.

□

Proposition 1.29. Let R be a domain,

1. If $\mathfrak{q} \in \text{Spec}(R)$, then $(R_{\mathfrak{q}})^+ \cong (R^+)_{\mathfrak{q}}$ where the localization is always referred to the multiplicative set $R \setminus \mathfrak{q}$.
2. If $Q \in \text{Spec}(R^+)$, then $R^+/Q \cong (R/Q \cap R)^+$

Proof. To prove both this statements, we will make use of the characterization given in Proposition 1.26: recall that given an integral domain extension $R \subseteq S$, $S \cong R^+$ if and only if every monic polynomial in $R[x]$ completely factors into monic polynomial of S .

1. Let $\mathfrak{q} \in \text{Spec}(R)$, let us fix an algebraic closure $K = \text{frac}(R^+)$, then $R_{\mathfrak{q}} \subseteq K$ and $(R^+)_{\mathfrak{q}} \subseteq K$, in particular we have that $R_{\mathfrak{q}} \subseteq (R^+)_{\mathfrak{q}}$. It is easy to see that localization preserves integral extensions so that $R_{\mathfrak{q}} \subseteq (R^+)_{\mathfrak{q}}$ is integral. Let $p \in R_{\mathfrak{q}}[x]$ be monic of positive degree. We can write $p(x) = x^n + \sum_{i=0}^{n-1} \frac{p_i}{s} x^i$, for some $s \notin \mathfrak{q}$. So we have that $s^n p(x) = (sx)^n + \sum_{i=0}^{n-1} p_i s^{n-1-i} (sx)^i = q(sx)$, where $q \in R[x]$ is a monic polynomial of positive degree, so it admits a linear factorization in R^+ , $q(x) = \prod_{i=0}^n (x - a_i)$, therefore $s^n p(x) = q(sx) = \prod_{i=0}^n (sx - a_i)$, and so $p(x) = \prod_{i=0}^n (x - \frac{a_i}{s})$.

2. Let $Q \in \text{Spec}(R^+)$, as previously observed in Proposition 1.14, we have that $R/Q \cap R \hookrightarrow R^+/Q$ is an integral extension of domains, thus we only need to show that given a monic polynomial $p \in (R/Q \cap R)[x]$, it factors into linear monic polynomials in R^+/Q . But this is clear given the fact that if we consider a lifting of p in $R[x]$, we can factor it into monic linear polynomials in $R^+[x]$ and then take the quotients.

□

Proposition 1.30. Let R be a domain, then R^+ is Noetherian if and only if R is a field. In such case R^+ is also a field.

Proof. Assume that R is not a field, then there exists $r \in R$ such that r is neither a unit nor zero.

By Remark 1.7, we know that $r \in R^+$ is not a unit, moreover since for every $n \in \mathbb{N}_{>0}$ the polynomial $x^n - r$ is monic in $R[x]$ we know that it admits all roots in R^+ . For every $n \in \mathbb{N}$, let us denote as $r^{1/n}$ one of the roots of the polynomials $x^n - r$.

Now let us consider the following chain of ideals:

$$(r) \subseteq (r^{1/2}) \subseteq (r^{1/4}) \subseteq \dots \subseteq (r^{1/2^n}) \subseteq \dots$$

If we had for some $n \in \mathbb{N}$ that $(r^{1/2^n}) = (r^{1/2^{n+1}})$ then there would be an element $v \in R^+$ such that $vr^{1/2^n} = r^{1/2^{n+1}} \implies v^{2^{n+1}}r^2 = r \implies v^{2^{n+1}}r = 1$ which contradicts the fact that r is not a unit, therefore R^+ is not Noetherian.

By Proposition 1.14 we deduce the other implication. □

1.3 Complete rings and Hensel's Lemma

Definition 1.31. Let R be a ring, if $\mathfrak{m} \subseteq R$ is an ideal, there exists a natural diagram induced by the projections:

$$R/\mathfrak{m} \leftarrow R/\mathfrak{m}^2 \leftarrow \dots \leftarrow R/\mathfrak{m}^n \leftarrow \dots$$

we define $\widehat{R}^{\mathfrak{m}} := \lim_{\leftarrow} R/\mathfrak{m}^n$ the completion of R with respect to $\mathfrak{m}$.

Clearly for all $n \in \mathbb{N}$ there exists the projections $\pi_n : R \rightarrow R/\mathfrak{m}^n$ and these projections commute with the diagram above. So by the universal property of the inverse limit, there exists a unique map of rings $R \rightarrow \widehat{R}^{\mathfrak{m}}$ which commutes with the maps π_n and the projections of $\widehat{R}^{\mathfrak{m}}$.

We say that R is complete with respect to $\mathfrak{m}$ if this map is an isomorphism.

Remark 1.32. Let R be a ring and $\mathfrak{m}$ be some ideal of R and $\pi_n^k : R/\mathfrak{m}^k \rightarrow R/\mathfrak{m}^n$ the projections for any $n \leq k$. We can describe $\widehat{R}^{\mathfrak{m}}$ explicitly as this subring:

$$\widehat{R}^{\mathfrak{m}} := \{(r_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} R/\mathfrak{m}^n : \pi_n^k(r_k) = r_n \forall k \geq n\} \subseteq \prod_{n \in \mathbb{N}} R/\mathfrak{m}^n$$

where the sum and multiplication are pointwise.

From this description, we can also find the maps of the limit, we have that the projections from $\widehat{R}^{\mathfrak{m}}$ to $R/\mathfrak{m}^n$ are exactly the restrictions of the projections from $\prod_{n \in \mathbb{N}} R/\mathfrak{m}^n$ to $R/\mathfrak{m}^n$. Moreover we can also describe the natural map $R \rightarrow \widehat{R}^{\mathfrak{m}}$, in fact this map turns out to be the most natural possible choice: $r \mapsto (\pi_n(r))_{n \in \mathbb{N}}$.

Remark 1.33. Let R be a ring and $\mathfrak{m}$ be some ideal of R .

If R is complete with respect to $\mathfrak{m}$ and $\mathfrak{m}^k$ for any $k \in \mathbb{N}$. It suffices to show that R is the inverse limit of the diagram induced by $\mathfrak{m}^k$. But clearly for any ring L with projections $l_n : L \rightarrow R/\mathfrak{m}^{nk}$ we can construct the projections $p_n = \pi_n^{nk} \circ l_n : L \rightarrow R/\mathfrak{m}^n$ and thus there must be a unique ring homomorphism $L \rightarrow R$ that commutes with all the projections.

Remark 1.34. Using the previous description, we define the ideals:

$$\widehat{\mathfrak{m}}_n = \{(r_k)_{k \in \mathbb{N}} \in \widehat{R}^{\mathfrak{m}} : r_k = 0\}$$

Clearly these ideals are the kernels of the projections $\widehat{R}^{\mathfrak{m}} \rightarrow R/\mathfrak{m}^n$ which are surjective. So we have that $\widehat{R}^{\mathfrak{m}}/\widehat{\mathfrak{m}}_n \cong R/\mathfrak{m}^n$; if R is Noetherian it is true that $\widehat{\mathfrak{m}}_n = \widehat{\mathfrak{m}}_1^n$. Using these two facts, we deduce that when R is Noetherian, $\widehat{R}^{\mathfrak{m}}$ is complete with respect to $\widehat{\mathfrak{m}}_1$, since it is the inverse limit of the diagram generated by the projections of $\widehat{\mathfrak{m}}_1$.

Definition 1.35. Let R be a ring, $\mathfrak{m} \subseteq R$ an ideal, $r \in R$

$$\text{ord}_{\mathfrak{m}}(r) := \begin{cases} \max\{n \in \mathbb{N} : r \in \mathfrak{m}^n\} & \text{if } r \notin \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n \\ \infty & \text{if } r \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n \end{cases}$$

We define the following pseudo metric, $d : R \times R \rightarrow [0, 1] : d(v, w) = 2^{-\text{ord}_{\mathfrak{m}}(v-w)}$ (where $2^{-\infty} := 0$)

- It is easy to verify that d is symmetric since $v - w \in \mathfrak{m}^n \iff w - v \in \mathfrak{m}^n$

- Let $v, w, z \in R$ we want $d(v, z) \leq d(v, w) + d(w, z)$.
If $d(v, z) = 0$ then the inequality holds, so let us assume $\text{ord}_{\mathfrak{m}}(v - z) = n < \infty$.
Now $v - z = (v - w) + (w - z) \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$, this means that at least one between $v - w$ and $w - z$ does not stay in $\mathfrak{m}^{n+1}$ and so its order must be less or equal than n . Without loss of generality let us assume $v - w \notin \mathfrak{m}^{n+1}$, so we have that

$$d(v, w) + d(w, z) \geq 2^{-n} + d(w, z) \geq d(v, z)$$

In general d is a metric if and only if $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$.

Remark 1.36. With the previous notations, a Cauchy sequence in (R, d) is a sequence $(c_n)_{n \in \mathbb{N}}$ in R such that for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that for any $n, m \geq N$, $d(c_n, c_m) < \varepsilon$.

This means that $\text{ord}_{\mathfrak{m}}(c_n - c_m) > -\log_2(\varepsilon)$. Let $e := \max(0, \lceil -\log_2(\varepsilon) \rceil)$, then the previous equation is equivalent to $\text{ord}_{\mathfrak{m}}(c_n - c_m) \geq e$ and so it is also equivalent to $c_n - c_m \in \mathfrak{m}^e$.

In conclusion the Cauchy condition on a sequence $(c_n)_{n \in \mathbb{N}}$ is equivalent to: for any $e \in \mathbb{N}$ there exist $N_e \in \mathbb{N}$ such that for any $n, m \in \mathbb{N}$, $n, m \geq N_e$, $c_n - c_m \in \mathfrak{m}^e$.

Theorem 1.37. Let R be a ring and $\mathfrak{m} \subseteq R$ some ideal of R .

R is complete with respect to $\mathfrak{m}$ if and only if (R, d) is a complete metric space.

Proof.

- Let us prove the first implication.

First of all, we know that the map $r \mapsto (\pi_n(r))_{n \in \mathbb{N}}$ is bijective, in particular it is injective and so $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$ and d is a metric.

Let $(c_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in R , using the characterization given in the previous remark, we have that for any $e \in \mathbb{N}$ there exist $N_e \in \mathbb{N}$ such that for any $n, m \in \mathbb{N}$, $n, m \geq N_e$, $c_n - c_m \in \mathfrak{m}^e$. (Let us assume for the sake of simplicity that $N_e \leq N_{e+1}$ for all $e \in \mathbb{N}$)

We have that $\pi_e(c_n) = \pi_e(c_{N_e})$ for any $n \geq N_e$. We define $r_n := \pi_n(c_{N_n})$, using the previous statement it is clear that r_n does not depend on the choices of the N_e .

We claim that $(r_n)_{n \in \mathbb{N}} \in \widehat{R}^{\mathfrak{m}}$, to prove this let $n, m \in \mathbb{N}$ be such that $m \leq n$, then $\pi_m^n(r_n) = \pi_m^n(\pi_n(c_{N_n})) = \pi_m(c_{N_n}) = \pi_m(c_{N_m}) = r_m$.

Finally using the surjectivity of the natural map $R \longrightarrow \widehat{R}^{\mathfrak{m}}$, we find $r \in R$ such that $\pi_n(r) = r_n$ for all $n \in \mathbb{N}$.

We now only need to prove that $c_n \rightarrow r$, but this is clear since for any $n \in \mathbb{N}$, $\pi_n(r - c_m) = 0$ for all $m \geq N_n$ which implies that $d(r, c_m) \leq 2^{-n}$ for all $m \geq N_n$.

- Let us now prove the second implication.

The injectivity of the natural map $R \longrightarrow \widehat{R}^{\mathfrak{m}}$ is given by the fact that d is a metric and so $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$. To prove the surjectivity, let $(r_n)_{n \in \mathbb{N}} \in \widehat{R}^{\mathfrak{m}}$.

For all $n \in \mathbb{N}$ let us define $c_n \in R$ such that $\pi_n(c_n) = r_n$. Clearly if n, m, e are all natural numbers such that $n, m \geq e$ then $\pi_e(c_n - c_m) = \pi_e^n(\pi_n(c_n)) - \pi_e^m(\pi_m(c_m)) = \pi_e^n(r_n) - \pi_e^m(r_m) = r_e - r_e = 0$. And so $(c_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in R . Let $r \in R$ be the element such that $c_n \rightarrow r$. We have that for every $n \in \mathbb{N}$, $\pi_n(c_m - r) = 0$ for sufficiently large m . But this means that $\pi_n(r) = \pi_n(c_m) = \pi_n^m(\pi_m(c_m)) = r_n$. And so r maps to $(r_n)_{n \in \mathbb{N}}$.

□

Proposition 1.38 (Continuity).

Let R be a ring and $\mathfrak{m} \subseteq R$ an ideal, the topological space generated by the basis $\{r + \mathfrak{m}^n : r \in R, n \in \mathbb{N}\}$ is the same topological space induced by the pseudo metric d . Moreover, using this topology, the sum and the product of the ring are continuous.

Proof. The canonical basis of the topology induced by d is $\{B_d(r, \varepsilon) : r \in R, \varepsilon > 0\}$, we notice that $B_d(r, \varepsilon) = \{r' \in R : d(r, r') < \varepsilon\} = \{r' \in R : r - r' \in \mathfrak{m}^e\} = r + \mathfrak{m}^e$, where e is chosen as in Remark 1.33.

Let us now prove the continuity of the operations. Recall that to prove continuity it is sufficient to prove it pointwise. Let $(x, y) \in R \times R$ and $n \in \mathbb{N}$, we want to find a neighborhood U of (x, y) such that $*(U) \subseteq (x * y) + \mathfrak{m}^n$ for $*$ in $\{+, \cdot\}$, but this is easy since the choice $U = (x + \mathfrak{m}^n) \times (y + \mathfrak{m}^n)$ works for both operations:

- $+(x + \mathfrak{m}^n) \times (y + \mathfrak{m}^n) \subseteq (x + y) + \mathfrak{m}^n$
- $\cdot((x + \mathfrak{m}^n) \times (y + \mathfrak{m}^n)) \subseteq xy + \mathfrak{m}^n$

□

Proposition 1.39. Let R and S be rings, $I \subseteq R$ and $J \subseteq S$ ideals. A ring homomorphism $\phi : R \rightarrow S$ is continuous in the topologies induced by I and J respectively, if and only if $\phi(I^n) \subseteq J$ for some $n \in \mathbb{N}$.

(For example, local homomorphisms between local rings are continuous)

Proof. Assume that $\phi(I^n) \subseteq J$, let $x \in R$ and $m \in \mathbb{N}$, we want to find a neighborhood U of x , such that $\phi(U) \subseteq \phi(x) + J^m$.

Indeed $U := x + I^{nm}$ works, since $\phi(I^{nm}) = \phi(I^n)^m \subseteq J^m$.

Assume that ϕ is continuous, then continuity in zero implies that there exists $n \in \mathbb{N}$ such that $\phi(I^n) \subseteq J$. □

Lemma 1.40. Let R be a ring and $\mathfrak{m} \subseteq R$ an ideal such that R is complete with respect to $\mathfrak{m}$. The elements of the multiplicative set $1 + \mathfrak{m}$ are all units.

Moreover, whenever $r \in R$ is such that $\pi_1(r) \in R/\mathfrak{m}$ is a unit, then r must also be a unit.

Proof. Let us prove the first statement. Let $r = 1 + a$ with $a \in \mathfrak{m}$. We can define $b = \sum_{n \in \mathbb{N}} (-a)^n \in R$ (in the sense of a convergent sequence).

Using the continuity of the product, we know that $rb = \lim_{N \rightarrow \infty} r \sum_{n=0}^N (-a)^n$.

$$\lim_{N \rightarrow \infty} (1 + a) \sum_{n=0}^N (-a)^n = \lim_{N \rightarrow \infty} \sum_{n=0}^N (-a)^n - \sum_{n=0}^N (-a)^{n+1} = \lim_{N \rightarrow \infty} 1 - (-a)^{N+1} = 1$$

For the second statement, notice that if $\pi_1(r)\pi_1(u) = \bar{1}$, then $ru \in 1 + \mathfrak{m}$, and so r is a unit. □

Theorem 1.41 (A version of Hensel's Lemma).

Let R be a ring complete with respect to an ideal $\mathfrak{m}$.

Let $F \in R[x]$ and let $f \in R/\mathfrak{m}[x]$ be the polynomial obtained by reducing the coefficients of F mod $\mathfrak{m}$.

If $f = g_1 g_2$, with $g_1, g_2 \in R/\mathfrak{m}[x]$, g_1 monic and $(g_1, g_2) = (1)$, then there is a unique factorization $F = G_1 G_2$ such that $G_1, G_2 \in R[x]$, G_1 is monic, G_i reduced mod $\mathfrak{m}$ is g_i for both indexes and $\deg(G_1) = \deg(g_1)$.

To prove this result, we first prove the following lemmas.

Lemma 1.42. Let S be a ring, $g_1, g_2 \in S[x]$ such that g_1 is monic and $(g_1, g_2) = (1)$.

1. For any $h \in S[x]$, there exist two unique polynomials $h_1, h_2 \in S[x]$ such that $h = h_1g_1 + h_2g_2$ and $\deg(h_2) < \deg(g_1)$
2. Let $S = R/\mathfrak{m}$ for some ring R and ideal $\mathfrak{m}$, such that $\mathfrak{m} \subseteq \bigcap \text{Max}(R)$ (the Jacobson radical). If $G_1, G_2 \in R[x]$ such that G_1 is monic and they both reduce to g_1 and g_2 respectively, then $(G_1, G_2) = (1)$.

Proof.

1. Since $(g_1, g_2) = (1)$, we can write $h = \alpha g_1 + \beta g_2$ for some $\alpha, \beta \in S[x]$, but since g_1 is monic, it is possible to write $\beta = vg_1 + w$ such that $\deg(w) < \deg(g_1)$. We get what we want by substituting: $h = \alpha g_1 + (vg_1 + w)g_2 = (\alpha + vg_2)g_1 + wg_2$. Now for the uniqueness, assume h_1, h_2 and h'_1, h'_2 to be some desired solutions, so $h_1g_1 + h_2g_2 = h'_1g_1 + h'_2g_2 \implies (h_1 - h'_1)g_1 = (h'_2 - h_2)g_2$. From the hypothesis there exists $a, b \in S[x]$ such that $1 = ag_1 + bg_2$ so

$$(h'_2 - h_2) = a(h'_2 - h_2)g_1 + b(h'_2 - h_2)g_2 = (a(h'_2 - h_2) + b(h'_1 - h_1))g_1$$

and so $g_1 | (h'_2 - h_2)$, but since $\deg(h'_2 - h_2) < \deg(g_1)$ we must have $h'_2 = h_2$ and so $h_1g_1 = h'_1g_1$, but since g_1 is monic it cannot be a zero divisor and so $h_1 = h'_1$.

2. We know that for some $h_1, h_2 \in S[x]$, $1 = h_1g_1 + h_2g_2$, which means that there exists some $H_1, H_2 \in R[x]$ such that $H_1G_1 + H_2G_2 \in 1 + \mathfrak{m}R[x]$. Notice that since G_1 is monic, $R \subseteq R[x]/(G_1)$ is an integral extension of finite type, in particular, by 1.4, $R[x]/(G_1)$ is a finitely generated R -module. Now consider the ideal $\overline{(G_2)} \subseteq R[x]/(G_1)$, it is easy to see that $\overline{(G_2)} = \langle \overline{G_2}, \overline{xG_2}, \dots, \overline{x^{\deg(G_1)-1}G_2} \rangle_R$ and so it is finitely generated as an R -module. We now have the following expression:

$$\overline{H_2G_2} \in \overline{1} + \mathfrak{m}R[x]/(G_1)$$

which implies that $\overline{(G_2)} + \mathfrak{m}R[x]/(G_1) = R[x]/(G_1)$, by Nakayama 5.5 we get $\overline{(G_2)} = R[x]/(G_1)$ and so $(G_1, G_2) = R[x]$

□

Lemma 1.43. Let R be a ring, complete with respect to an ideal $\mathfrak{m} \subseteq R$. Then $\mathfrak{m}$ is contained in the Jacobson ideal, i.e. $\mathfrak{m} \subseteq \bigcap \text{Max}(R)$

Proof. By contradiction, assume that $\mathfrak{n} \in \text{Max}(R)$ such that $\mathfrak{m} \not\subseteq \mathfrak{n}$, then there exists $r \in \mathfrak{m}$ such that $r \notin \mathfrak{n}$.

Since $R/\mathfrak{n}$ is a field, there must exist an element $q \in R$ such that $1 - rq \in \mathfrak{n}$, but since $r \in \mathfrak{m}$, by Remark 1.40, $1 + rq$ must also be a unit; this cannot happen. □

We can now prove the factorization version of Hensel's Lemma.

Proof of Theorem 1.41. Recall the hypothesis:

R is a ring complete with respect to an ideal $\mathfrak{m}$, $F \in R[x]$ and $f \in R/\mathfrak{m}[x]$ is its projection. $f = g_1 g_2$, with $g_1, g_2 \in R/\mathfrak{m}[x]$, g_1 monic and $(g_1, g_2) = (1)$. We want to lift this factorization to $R[x]$, preserving the fact that g_1 is monic.

Let G'_1 and G'_2 be any lifting of g_1 and g_2 such that G'_1 is monic.

Notice that, by the previous lemma, since R is complete with respect to $\mathfrak{m}$, $\mathfrak{m}$ is contained in every maximal ideal of R .

Using the hypothesis and 2 of Lemma 1.42, we know that $(G'_1, G'_2) = (1)$.

We want to find two sequences of polynomials $(G_1^{(k)})_{k \in \mathbb{N}_{>0}}$ and $(G_2^{(k)})_{k \in \mathbb{N}_{>0}}$ such that, for all $k \in \mathbb{N}_{>0}$, the following conditions are satisfied:

1. $F = G_1^{(k)} G_2^{(k)} \pmod{\mathfrak{m}^{2^{k-1}} R[x]}$
2. $G_1^{(k)}$ is monic of degree $\deg(g_1)$
3. for every $l \in \mathbb{N}_{>0}$, $l < k$, $i = 1, 2$, $G_i^{(l)} = G_i^{(k)} \pmod{\mathfrak{m}^{2^{l-1}} R[x]}$
4. $g_i = G_i^{(k)} \pmod{\mathfrak{m} R[x]}$ for $i = 1, 2$
5. $(G_1^{(k)}, G_2^{(k)}) = (1)$

Clearly G'_1, G'_2 satisfy this conditions for $k = 1$.

Let us assume to have found $G_1^{(l)}, G_2^{(l)}$ for all $l \leq k$, we want to construct $G_1^{(k+1)}$ and $G_2^{(k+1)}$.

By 1 of Lemma 1.42 we may write $F - G_1^{(k)} G_2^{(k)} = G_1^{(k)} H_1 + G_2^{(k)} H_2 \in \mathfrak{m}^{2^{k-1}} R[x]$ with $H_i \in R[x]$ and $\deg(H_2) < \deg(G_1^{(k)})$. We first show that $H_1, H_2 \in \mathfrak{m}^{2^{k-1}} R[x]$.

If we denote with h_1, h_2 the projections of H_1, H_2 in $R/\mathfrak{m}^{2^{k-1}}[x]$, we get that $0 = \pi_{2^{k-1}}(G_1^{(k)})h_1 + \pi_{2^{k-1}}(G_2^{(k)})h_2$ with $\deg(h_2) < \deg(g_1)$. We notice that all the hypothesis Lemma 1.42 part 1 are satisfied, thus h_1 and h_2 are unique and they must both be zero (since that is a solution).

Define $G_1^{(k+1)} := G_1^{(k)} + H_2$, $G_2^{(k+1)} := G_2^{(k)} + H_1$, we now show that this choices satisfy the required conditions:

1. $G_1^{(k+1)} G_2^{(k+1)} = G_1^{(k)} G_2^{(k)} + G_1^{(k)} H_1 + G_2^{(k)} H_2 + H_1 H_2 = F + H_1 H_2$.
Thus $G_1^{(k+1)} G_2^{(k+1)} = F \pmod{\mathfrak{m}^{2^k} R[x]}$
2. Since $\deg(H_2) < \deg(G_1^{(k)})$ and $G_1^{(k+1)} = G_1^{(k)} + H_2$, it is clear that $\deg(G_1^{(k+1)}) = \deg(G_1^{(k)}) = \deg(g_1)$ and $G_1^{(k+1)}$ is monic.
3. By induction, it is sufficient to show that $G_i^{(k)} = G_i^{(k+1)} \pmod{\mathfrak{m}^{2^{k-1}} R[x]}$ for $i = 1, 2$; but this is evident.
4. The equation: $g_i = G_i^{(k)} \pmod{\mathfrak{m} R[x]}$ for $i = 1, 2$, follows from point 3 and the induction hypothesis.
5. To prove that $(G_1^{(k+1)}, G_2^{(k+1)}) = (1)$, we make use of part 2 of Lemma 1.42: notice that all the hypothesis are satisfied thanks to point 4.

Clearly condition 3 tells us that all the coefficients of $G_1^{(k)}$ converges, so we get a limit $G_1^{(k)} \rightarrow G_1 \in R[x]$ (in the topology induced by $\mathfrak{m} R[x]$ on $R[x]$).

To get to the same conclusion for $(G_2^{(k)})_{k \in \mathbb{N}_{>0}}$, we need a bound on the degrees. Notice that condition 1 and 3, forces $\deg(G_2^{(k)})$ to be less or equal to $\deg(F)$ for all $k \in \mathbb{N}_{>0}$. In fact, if there was an index $k \in \mathbb{N}_{>0}$ such that $G_2^{(k)}$ has degree greater than $\deg(F)$, then let a be the leading term of $G_2^{(k)}$. Since $a \neq 0$ and R is complete with respect to $\mathfrak{m}$, there exists an index k' such that $a \notin \mathfrak{m}^{2^{k'-1}}$. Now using condition 1 and 2, we get that since $F = G_1^{(k')} G_2^{(k')} \bmod \mathfrak{m}^{2^{k'-1}}$,

$$\deg(F) \geq \deg(g_1) \deg(G_2^{(k)}) > \deg(g_1) \deg(F) \geq \deg(F)$$

which is absurd.

So we get the limits G_1, G_2 , with G_1 monic. Finally, since by 1.38, multiplication is continuous, we also have that $G_1 G_2 = F$.

To prove uniqueness, notice that since G_i reduced mod $\mathfrak{m}$ are g_i , condition 5 still holds for the limits, so we can use part 1 of Lemma 1.42. $\square$

Following exercise 7.21 of Eisenbud [8], this alternative version of Hensel's Lemma implies a subcase of the original version:

Theorem 1.44 (weak Hensel's Lemma).

Let R be a ring that is complete with respect to an ideal $\mathfrak{m}$, and let $f \in R[x]$ be a polynomial. If $a \in R$ such that $f(a) = 0 \bmod \mathfrak{m}$ and $f'(a)$ is a unit, then there exists a unique $b \in R$ such that $f(b) = 0$ and $b = a \bmod \mathfrak{m}$.

Proof. Let us denote with $\bar{f} \in R/\mathfrak{m}[x]$ the projection of f . So we have that $\bar{f}(\bar{a}) = 0$, so $x - \bar{a}$ divides $\bar{f}$.

We get a factorization $\bar{f} = (x - \bar{a})g$. $f'(a)$ is a unit, so $\bar{f}'(\bar{a})$ is also a unit.

$$\bar{f}'(\bar{a}) = \bar{f}'(x)|_{x=\bar{a}} = g(x)|_{x=\bar{a}} + (x - \bar{a})g'(x)|_{x=\bar{a}} = g(\bar{a}).$$

In particular using euclidian division we get $g = (x - \bar{a})h + \bar{f}'(\bar{a})$, and so the unit $\bar{f}'(\bar{a})$ belongs to the ideal $(x - \bar{a}, g)$. So now we can use 1.41 and find a unique lifting of the factorization, which does not change the degrees of the monic polynomial, so $f = (x - b)G$, with $b = a \bmod \mathfrak{m}$. $\square$

Theorem 1.45 (Cohen Structure Theorem).

Let R be a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ and let $K = R/\mathfrak{m}$. If R contains a field, then $R \cong K[[x_1, \dots, x_n]]/I$ for some $n \in \mathbb{N}$ and some ideal $I \subseteq K[[x_1, \dots, x_n]]$.

Remark 1.46. If there exists an inclusion $K \subseteq R$ such that $K \hookrightarrow R \rightarrow R/\mathfrak{m}$ is the identity, then the Cohen's structure Theorem is proved: let $\mathfrak{m} = (a_1, \dots, a_n)$, we can construct (using completeness, see Proposition 1.66) a ring homomorphism:

$$\phi : K[[x_1, \dots, x_n]] \longrightarrow R, \text{ fixing } K, \text{ such that } x_i \mapsto a_i \text{ for all } i = 1, \dots, n$$

We prove that this map is surjective.

Since $\phi((x_1, \dots, x_n)) \subseteq \mathfrak{m}$, by 1.39, ϕ is a continuous function.

If $r \in R$, then, using the fact that $K \hookrightarrow R \rightarrow R/\mathfrak{m}$ is surjective, there exists an element $r' \in K$ such that $\bar{r}' = \bar{r}$ in $R/\mathfrak{m}$, so $r'' := r - r' \in \mathfrak{m}$ and $r = r' + r'' \in \mathfrak{m} + K$. Thus it is sufficient to show that $\mathfrak{m}$ is contained in the image of the homomorphism.

Let $r \in \mathfrak{m}$, then we can write $r = \sum_{i=1}^n b_i a_i$.

As before, for all $i = 1, \dots, n$, there exists $b'_i \in K$ and $b''_i \in \mathfrak{m}$ such that $b_i = b'_i + b''_i$. We define $r_1 := \sum_{i=1}^n b'_i a_i \in \text{Im} \phi$. We have that $r - r_1 \in \mathfrak{m}^2$. Now for all $i = 1, \dots, n$; $b''_i = \sum_{j=1}^n b_{i,j} a_j$ and so $r = \sum_{i=1}^n b_i a_i + \sum_{i=1}^n \sum_{j=1}^n b_{i,j} a_i a_j$. We define $r_2 := \sum_{i=1}^n b_i a_i + \sum_{i=1}^n \sum_{j=1}^n b_{i,j} a_i a_j$ so that $r - r_2 \in \mathfrak{m}^3$. Iterating this process, we find that $K[a_1, \dots, a_n]$ is dense inside R . From the construction of the sequence $(r_k)_{k \in \mathbb{N}_{>0}}$, it is clear that the obvious preimage of the r_n 's converges in $K[[x_1, \dots, x_n]]$, and so, using the fact that ϕ is continuous, we get that $r \in \text{Im} \phi$.

Remark 1.47. The existence of the immersion $R/\mathfrak{m} \subseteq R$ described in the previous remark is not only sufficient but it is also necessary for the Cohen Structure theorem to be hold.

Indeed if $\phi : k[[x_1, \dots, x_n]]/I \simeq R$, for some k , then clearly $\phi((\overline{x_1}, \dots, \overline{x_n})) = \mathfrak{m}$, so it induces an isomorphism $\overline{\phi} : k \simeq R/\mathfrak{m}$.

Clearly $\phi(k) \subseteq R$ is a field isomorphic to k and thus $R/\mathfrak{m}$, following the isomorphisms, we get:

$$\psi : \phi(k) \xrightarrow{\phi^{-1}} k \xrightarrow{\overline{\phi}} R/\mathfrak{m}$$

Since $\overline{\phi} = \pi \circ \phi|_k$, we get $\psi = \pi \circ \phi|_k \circ \phi|_{\phi(k)}^{-1} = \pi|_{\phi(k)} : \phi(k) \longrightarrow R/\mathfrak{m}$.

Finally the inclusion required by Remark 1.46 is

$$\psi^{-1} : R/\mathfrak{m} \rightarrow \phi(k) \subseteq R$$

Since clearly $\pi \circ \psi^{-1} = \text{id}_{R/\mathfrak{m}}$.

Remark 1.48. We need to find the field $k \subseteq R$. Since all homomorphism starting from a field are injective, if we find $k \subseteq R$ such that $\pi(k) = R/\mathfrak{m}$, then it clearly must be isomorphic to $R/\mathfrak{m}$ via the projection and so we are done.

We only prove the theorem in the case when R is a domain of characteristic zero.

We know that R contains a field $l \subseteq R$.

Consider the following non empty partial ordered set:

$$\{k \subseteq R : k \text{ is a field}\}.$$

By Zorn's Lemma, there exists a maximal element k_0 . Notice that $k_0 + \mathfrak{m}$ is a ring. If $k_0 + \mathfrak{m} \neq R$, let $r \in R \setminus \{k_0 + \mathfrak{m}\}$, in particular $r \notin \mathfrak{m}$ and so it is a unit.

r cannot be algebraic on k_0 , otherwise $k_0[r] = k_0(r)$ is a greater field inside R . If $k_0[r] \cap \mathfrak{m} = \{0\}$ then every non zero element in $k_0[r]$ is a unit in R , and so there exists an inclusion $k_0(x) \subseteq R$ where $\frac{f(x)}{g(x)} \mapsto f(r)g(r)^{-1}$. Again, this contradicts the maximality of k_0 , so let $g(r) \in k_0[r] \cap \mathfrak{m} \setminus \{0\}$, such that $g \in k_0[x]$ has the least degree possible (so it is irreducible since $\mathfrak{m}$ is prime).

Note that $g'(r) \notin \mathfrak{m}$ since the degree of $g'(x)$ is less than the degree of g and r is not algebraic in k_0 and $g' \neq 0$. So it is a unit of R .

By weak Hensel's Lemma, there exists a unique $v \in r + \mathfrak{m}$ such that $g(v) = 0$, so v is algebraic on k_0 and by maximality $k_0[v] = k_0$, which means that $r \in \mathfrak{m} + k_0$.

So $k_0 + \mathfrak{m} = R$. Thus for any $\bar{r} \in R/\mathfrak{m}$, $r = r' + r'' \in k_0 + \mathfrak{m}$ and so $\overline{r'} = \bar{r}$, which means that $\pi(k_0) = R/\mathfrak{m}$.

1.4 Commutative algebra

In this section, we recall some basic definitions and theorems of commutative algebra that will later be used in chapter 3.

Definition 1.49. Let $(R, \mathfrak{m})$ be a Noetherian local ring. R is regular if $\mu(\mathfrak{m}) = \text{ht}(\mathfrak{m})$, where μ denotes the minimal number of generators of the ideal.

It is a theorem that "regularity localizes", meaning that if $(R, \mathfrak{m})$ is a regular ring and $\mathfrak{p} \in \text{Spec}(R)$, then $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ is still regular. This property allows us to give the following more general definition.

Definition 1.50. Let R be a Noetherian ring. R is regular if $R_{\mathfrak{m}}$ is regular for all $\mathfrak{m} \in \text{Max}(R)$.

Some classic examples of regular rings are the polynomial rings $k[x_1, \dots, x_n]$ and the formal power series ring $k[[x_1, \dots, x_n]]$; both with coefficients in a field.

Moreover, we know the following statements to be true.

- If R is a standard-graded regular ring, then it must be isomorphic to a polynomial ring $k[x_1, \dots, x_n]$.
- If $(R, \mathfrak{m})$ is complete regular and contains a field, then it is actually isomorphic to a formal power series ring $k[[x_1, \dots, x_n]]$ (see 1.74 and Cohen's structure Theorem).
- Every local regular ring is a Unique Factorization Domain.

What we are saying, is that regular rings are quite rare, we would like to have a bigger class of rings which retains some good properties of regular rings.

Definition 1.51. Let R be a ring and M an R -module, $a \in R$ is an M -regular element if the multiplication $M \xrightarrow{a} M$ is injective.

A sequence of elements $a_1, \dots, a_n \in R$ is said to be M -regular, if for all indexes $i = 1, \dots, n$, a_i is $M/(a_1, \dots, a_{i-1})M$ -regular and $(a_1, \dots, a_n)M \neq M$.

Notice that the last condition, when the ring is local, is equivalent to requiring for all the a_i to belong to the maximal ideal.

When there is no ambiguity we will say that the sequence is regular.

Definition 1.52. Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a finitely generated R -module and $I \subseteq R$ an ideal, we set

$\text{grade}(I, M) := \max\{n \in \mathbb{N} : \exists a_1, \dots, a_n \text{ an } M\text{-regular sequence inside } I\}$, and we let

$\text{depth}(M) := \text{grade}(\mathfrak{m}, M)$.

We say that a finitely generated R -module M is Cohen-Macaulay if $\text{depth}(M) = \dim(M)$.

Recall that $\dim(M) := \dim(R/\text{Ann}_R(M))$, where the latter is the Krull dimension of the ring. A ring is said to be Cohen-Macaulay if it is Cohen-Macaulay as a module.

Similarly to before, the Cohen-Macaulayness property localizes through prime ideals, so we can give a more general definition.

Definition 1.53. Let R be a Noetherian ring. R is Cohen-Macaulay if $R_{\mathfrak{m}}$ is Cohen-Macaulay for all $\mathfrak{m} \in \text{Max}(R)$.

It is a classical result that if $(R, \mathfrak{m})$ is a regular local ring, then any system of parameters is a regular sequence, so all regular rings are Cohen-Macaulay. Let us now lists some useful properties of Cohen-Macaulay local rings and modules.

Theorem 1.54 (Purity). Let $(R, \mathfrak{m})$ be a local ring and M a Cohen-Macaulay R -module, then for any $\mathfrak{p} \in \text{Ass}_R(M)$ we have that $\dim(R/\mathfrak{p}) = \dim(M)$.

Therefore $\text{Ass}_R(M) = \text{Min}(\text{Ann}_R(M))$.

In the case where $M = R/I$ for some ideal $I \subseteq R$, $\text{Ass}_R(R/I) = \text{Min}(I)$.

Theorem 1.55. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring and $a_1, \dots, a_n \in \mathfrak{m}$. Then the following statements are equivalent:

- $a_1, \dots, a_n$ is a regular sequence.
- $ht(a_1, \dots, a_n) = n$.
- $a_1, \dots, a_n$ is part of a system of parameters.

Theorem 1.56. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring and I any ideal of R . Then

- $ht(I) = \text{grade}(I, R)$
- $\dim(R) = \dim(R/I) + ht(I)$

Notice that in general we only know that $ht(I) \geq \text{grade}(I, R)$ and $\dim(R) \geq \dim(R/I) + ht(I)$.

For our purposes, we actually need another version of the equality.

Corollary 1.57. If S is a local Cohen-Macaulay ring and $f : S \twoheadrightarrow R$ is a ring epimorphism onto a local domain R . Then if $\mathfrak{p} \in \text{Spec}(R)$,

$$\dim(R) = \dim(R/\mathfrak{p}) + ht(\mathfrak{p}).$$

Proof. Let us call $P := \ker f \in \text{Spec}(S)$ so that $R \cong S/P$. Using Theorem 1.56:

$$\begin{aligned} \dim(R) &= \dim(S/P) = \dim(S) - ht(P) = \dim(S/f^{-1}(\mathfrak{p})) + ht(f^{-1}(\mathfrak{p})) - ht(P) = \\ &= \dim(R/\mathfrak{p}) + \dim(S_{f^{-1}(\mathfrak{p})}) - ht(PS_{f^{-1}(\mathfrak{p})}) = \dim(R/\mathfrak{p}) + \dim(R_{\mathfrak{p}}) = \dim(R/\mathfrak{p}) + ht(\mathfrak{p}) \end{aligned}$$

Where in the second to last equality, we use that $S_{f^{-1}(\mathfrak{p})}$ is Cohen-Macaulay. $\square$

Remark 1.58. In Corollary 1.57 the hypothesis that R is a domain is necessary.

Consider $S = k[[x, y, z]]$ for some field k , $I = (xy, xz)$ and $R = S/I$.

Then if $\mathfrak{p} = (y, z)/I$, since it is minimal: $\dim(R/\mathfrak{p}) + ht(\mathfrak{p}) = \dim(k[[x]]) + 0 = 1$, however $\dim(R) \geq 2$, since $(x)/I \subseteq (x, y)/I \subseteq (x, y, z)/I$ is a chain of strict inclusion of primes of R . (The dimension of R is indeed 2).

Corollary 1.59. Let $(R, \mathfrak{m})$ be an homomorphic image of a Cohen-Macaulay ring. Then R is catenary. This means that any strictly increasing chains of prime ideals between two primes $\mathfrak{p} = \mathfrak{p}_0 \subseteq \dots \subseteq \mathfrak{p}_n = \mathfrak{q}$, can be completed into a chain of length the maximal possible for such chains.

Proof. Let $\pi : S \twoheadrightarrow R$ be a ring homomorphism such that S is Cohen-Macaulay. If the statement is true for S then it is true for R , since π induces an inclusion preserving bijection between $\text{Spec} R$ and $V(\ker \pi)$. So let us assume that R is Cohen-Macaulay.

Let $\mathfrak{p}, \mathfrak{q} \in \text{Spec}(R)$, $\mathfrak{p} \subseteq \mathfrak{q}$. We prove the statement by induction on n .

If $n = 1$, then there is nothing to prove.

Assume the statement is true for any chains of length less than n , then $\mathfrak{p}_0 \subseteq \dots \subseteq \mathfrak{p}_{n-1}$ can be completed to the maximal possible length l for chains from $\mathfrak{p}_0$ to $\mathfrak{p}_{n-1}$. Notice that $l = \dim(R_{\mathfrak{p}_{n-1}}/\mathfrak{p}_0 R_{\mathfrak{p}_{n-1}})$. Also $\mathfrak{p}_{n-1} \subseteq \mathfrak{p}_n$ can be completed to the maximal possible length l' . Notice that $l' = \dim(R_{\mathfrak{p}_n}/\mathfrak{p}_{n-1} R_{\mathfrak{p}_n})$.

By Theorem 1.56 we know that

$$\begin{aligned} \dim(R_{\mathfrak{p}_n}) &= \dim(R_{\mathfrak{p}_n}/\mathfrak{p}_{n-1} R_{\mathfrak{p}_n}) + ht(\mathfrak{p}_{n-1}) = l' + \dim(R_{\mathfrak{p}_{n-1}}) = \\ &= l' + \dim(R_{\mathfrak{p}_{n-1}}/\mathfrak{p}_0 R_{\mathfrak{p}_{n-1}}) + ht(\mathfrak{p}_0) = l' + l + ht(\mathfrak{p}_0) \end{aligned}$$

Therefore, $l + l' = \dim(R_{\mathfrak{p}_n}) - ht(\mathfrak{p}_0) = \dim(R_{\mathfrak{p}_n}/\mathfrak{p}_0 R_{\mathfrak{p}_n})$, which is exactly the maximal length of a chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$. $\square$

Remark 1.60. If in the previous Corollary R is also a domain, then the maximal length of strictly increasing chains between two primes $\mathfrak{p} \subseteq \mathfrak{q}$ is exactly $ht(\mathfrak{q}) - ht(\mathfrak{p})$ (Remark 1.58 provides a counterexample when R is not a domain).

To prove this, notice that the length is exactly $\dim(R_{\mathfrak{q}}/\mathfrak{p} R_{\mathfrak{q}})$ which, by 1.57, is equal to $\dim(R_{\mathfrak{q}}) - ht(\mathfrak{p} R_{\mathfrak{q}}) = ht(\mathfrak{q}) - ht(\mathfrak{p})$.

Now we shall briefly discuss injective modules.

Definition 1.61. Let R be a ring, an R -module E is said to be injective if for any map $f : N \longrightarrow E$ and $g : N \hookrightarrow M$, there exists $\bar{f} : M \longrightarrow E$ such that $\bar{f} \circ g = f$.

Proposition 1.62. An R -module M is injective if and only if the functor $\text{Hom}_R(-, M)$ is exact.

Proposition 1.63. If M is an R -module, there exists a unique (up to isomorphism) injective R -module E such that $M \hookrightarrow E$ and the inclusion is essential, meaning that any non zero submodule of E intersects M non-trivially. We denote this module with $E_R(M)$.

Definition 1.64. A Noetherian local ring $(R, \mathfrak{m})$ is Gorenstein if it admits a finite injective resolution.

Again, also this property localizes through prime ideals and so we can give the more general definition:

Definition 1.65. A Noetherian ring R is Gorenstein if $R_{\mathfrak{m}}$ is Gorenstein for all $\mathfrak{m} \in \text{Max}(R)$.

It shall be noted that the following implications holds:

$$(R, \mathfrak{m}) \text{ is regular} \implies (R, \mathfrak{m}) \text{ is Gorenstein} \implies (R, \mathfrak{m}) \text{ is Cohen-Macaulay}$$

For our purposes, Gorenstein rings will only make an appearance for the statement of the local duality theorem in 3.13. Although it shall be noted that any time we will make use of Theorem 3.13, it will be applied to a regular or a Cohen-Macaulay ring.

We end this section with some useful results regarding completion.

Proposition 1.66 (Maps from power series). Let R be a S algebra, complete with respect to an ideal $\mathfrak{m}$ and let $a_1, \dots, a_n \in m$.

There exists a unique S -algebras morphism $S[[x_1, \dots, x_n]] \longrightarrow R$ such that $x_i \mapsto a_i$

Proof. We describe two ways to prove this statement, one topological and one categorical. Recall from 1.37 that $S[x_1, \dots, x_n]$ is dense in $S[[x_1, \dots, x_n]]$. Let

$$\phi : S[x_1, \dots, x_n] \longrightarrow R : x_i \mapsto a_i$$

If $\lambda \in S[[x_1, \dots, x_n]]$ there exists a sequence in $S[x_1, \dots, x_n]$ such that $\lambda_n \rightarrow \lambda$, moreover the sequence $(\phi(\lambda_n))_n$ is a Cauchy sequence and thus we can define $\phi(\lambda) := \lim_{n \rightarrow \infty} \phi(\lambda_n)$. Using the fact that ϕ is linear, we deduce that the $\phi(\lambda)$ is independent from the choices of the sequence. At last, we can use continuity of sum and product to prove that the resulting extension is a ring homomorphism.

The more direct way is to use the definition, ϕ induces the following maps:

$$\begin{array}{ccccccc} S[[x_1, \dots, x_n]] & \longrightarrow & \frac{S[x_1, \dots, x_n]}{(x_1, \dots, x_n)^{k+1}} & \xrightarrow{\bar{\phi}^{k+1}} & R/\mathfrak{m}^{k+1} & \longleftarrow & \widehat{R} \\ & \searrow & \downarrow & & \downarrow & & \swarrow \\ & & \frac{S[x_1, \dots, x_n]}{(x_1, \dots, x_n)^k} & \xrightarrow{\bar{\phi}^k} & R/\mathfrak{m}^k & & \end{array}$$

Now, by the universal property of $\widehat{R}$, there exists a morphism $f : S[[x_1, \dots, x_n]] \longrightarrow \widehat{R}$ that commutes with the diagram above. Hence it is an S -algebra morphism and $f(x_i) = a_i$ for all $i = 1, \dots, n$. $\square$

Definition 1.67. Let $(R, \mathfrak{m})$ be a ring and $I \subseteq R$ an ideal.

Recall the characterization of $\widehat{R}$ as a subring of $\prod_{n \in \mathbb{N}} R/\mathfrak{m}^n$.

We define $\bar{I} := \{(r_n)_{n \in \mathbb{N}} \in \widehat{R} : r_n \in \frac{I + \mathfrak{m}^n}{\mathfrak{m}^n}\}$.

It is immediate to see that $\bar{I}$ is an ideal of $\widehat{R}$ containing $I\widehat{R}$, moreover it is exactly the closure of I as a subset of $\widehat{R}$.

When R is also Noetherian, the equality $\bar{I} = I\widehat{R}$ holds and $\bar{I}$ is also isomorphic as an R -module to $\widehat{I}$. (For this last statement see Proposition 10.15 of [9]).

To prove that when R is Noetherian, $I\widehat{R}$ is closed, see Chapter III, §3.3, Proposition 6 of [10].

Theorem 1.68. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $I \subseteq R$ an ideal, then $\widehat{R/I} \cong \widehat{R}/\bar{I}$.

Proof. Recall the characterization given in Remark 1.32. For all $n \in \mathbb{N}$ we have the following maps,

$$b_n : R/\mathfrak{m}^n \longrightarrow R/(\mathfrak{m}^n + I) \cong \frac{R/I}{(\mathfrak{m}^n + I)/I}$$

This maps commutes in the following way:

$$\begin{array}{ccccccc} \widehat{R} & \longrightarrow & R/\mathfrak{m}^n & \xrightarrow{b_n} & \frac{R}{\mathfrak{m}^n + I} & \xlongequal{\sim} & \frac{R/I}{(\mathfrak{m}/I)^n} \\ & \searrow & \downarrow p_n & & \downarrow p'_n & & \\ & & R/\mathfrak{m}^{n-1} & \xrightarrow{b_{n-1}} & \frac{R}{\mathfrak{m}^{n-1} + I} & \xlongequal{\sim} & \frac{R/I}{(\mathfrak{m}/I)^{n-1}} \end{array}$$

So, by the universal property of $\widehat{R/I}$, we get a morphism:

$$\phi : \widehat{R} \longrightarrow \widehat{R/I} : (r_i)_{i \in \mathbb{N}} \mapsto (b_i(r_i))_{i \in \mathbb{N}} .$$

ϕ is surjective, in fact if $(\eta_i)_{i \in \mathbb{N}} \in \widehat{R/I}$, then inductively we can construct a preimage: assume that $b_i(r_i) = \eta_i$ for all $i < n$. When $n = 1$, we can take as r_1 any preimage of η_1 through b_1 , so let n be greater than 1.

We will denote with π_J the projection $R \longrightarrow R/J$ for any ideal $J \subseteq R$.

Let $\tilde{\eta}_n \in R$ be such that $\pi_{\mathfrak{m}^n+I}(\tilde{\eta}_n) = \eta_n$ and

let $\tilde{r}_{n-1} \in R$ be such that $\pi_{\mathfrak{m}^{n-1}}(\tilde{r}_{n-1}) = r_{n-1}$. Then

$$\pi_{\mathfrak{m}^{n-1}+I}(\tilde{r}_{n-1}) = b_{n-1}(r_{n-1}) = \eta_{n-1} = p'_n(\eta_n) = \pi_{\mathfrak{m}^{n-1}+I}(\tilde{\eta}_n)$$

Therefore $\tilde{r}_{n-1} - \tilde{\eta}_n = a + b \in \mathfrak{m}^{n-1} + I$, where $a \in \mathfrak{m}^{n-1}$ and $b \in I$.

$$\tilde{r}_n := \tilde{r}_{n-1} - a = \tilde{\eta}_n + b$$

Let us show that $r_n := \pi_{\mathfrak{m}^n}(\tilde{r}_n)$ is such that $b_n(r_n) = \eta_n$ and $p_n(r_n) = r_{n-1}$.

- $b_n(r_n) = \pi_{\mathfrak{m}^n+I}(\tilde{r}_n) = \pi_{\mathfrak{m}^n+I}(\tilde{\eta}_n + b) = \pi_{\mathfrak{m}^n+I}(\tilde{\eta}_n) = \eta_n$
- $p_n(r_n) = \pi_{\mathfrak{m}^{n-1}}(\tilde{r}_n) = \pi_{\mathfrak{m}^{n-1}}(\tilde{r}_{n-1} - a) = \pi_{\mathfrak{m}^{n-1}}(\tilde{r}_{n-1}) = r_{n-1}$

Hence there exists an element $(r_i)_{i \in \mathbb{N}} \in \widehat{R}$ such that $\phi((r_i)_{i \in \mathbb{N}}) = (\eta_i)_{i \in \mathbb{N}}$.

$$\ker \phi = \{(r_i)_{i \in \mathbb{N}} \in \widehat{R} : r_i \in \frac{I + \mathfrak{m}^i}{\mathfrak{m}^i}\} = \bar{I} = I\widehat{R}.$$

Where last equality is true since R is Noetherian. □

Theorem 1.69. Let $(R, \mathfrak{m})$ be a complete local domain and $R \subseteq S$ be an integral extension into a domain. Then $(S, \sqrt{\mathfrak{m}S})$ is a local domain.

If R is Noetherian and S is a finite R -module, then S is also complete with respect to its maximal ideal.

Proof. Let $s \in S \setminus U(S)$ and let $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in R[x]$ be the minimal monic polynomial that annihilates s . By Lemma 1.8, a_0 cannot be a unit, therefore $a_0 \in \mathfrak{m}$. So taking the modulo of the coefficients, we get a polynomial $xg(x) \in R/\mathfrak{m}[x]$. Since p is minimal, it is actually irreducible, so by the factorization version of Hensel's Lemma 1.41, $g(x)$ cannot contain inside its factorization any irreducible factors different from x , and so $p(x) = x^n \pmod{\mathfrak{m}R[x]}$. This means that all the coefficients a_i belongs to $\mathfrak{m}$ and so $s^n \in \mathfrak{m}S$.

Now we prove that if R is Noetherian and $R \subseteq S$ is finite, then S is complete with respect to $\mathfrak{n} := \sqrt{\mathfrak{m}S}$. The condition $\bigcap_{n \in \mathbb{N}} \mathfrak{n}^n = 0$ follows from Krull's intersection theorem (see Chapter III, §3.2 of [10]). To prove completeness, notice that, by Noetherianity, there is a $k \in \mathbb{N}$ such that $\mathfrak{n}^k \subseteq \mathfrak{m}S$. Clearly $\mathfrak{m}S \subseteq \mathfrak{n}$, therefore S is complete with respect to $\mathfrak{n}$ if and only if it is complete with respect to $\mathfrak{m}S$. Completeness is equivalent to asking for any series $\sum_{n \in \mathbb{N}} \alpha_n$ such that for any $n \in \mathbb{N}$, $\alpha_n \in \mathfrak{m}^n S$, to converge.

Let $s_1, \dots, s_t \in S$ be generators of S . If $\alpha_n \in \mathfrak{m}^n S$ then it is easy to show that we can write $\alpha_n = \sum_{i=1}^t b_{i,n} s_i$ such that $b_{i,n} \in \mathfrak{m}^n$ for all indexes.

So we can define $b_i := \sum_{n \in \mathbb{N}} b_{i,n} \in R$, therefore if $\alpha := \sum_{i=1}^t b_i s_i \in S$, $\sum_{n=0}^N \alpha_n$ converges to α in the topology of $\mathfrak{m}S$. □

Remark 1.70. The first statement of Theorem 1.69 is not true if R is not complete, even when assuming that S is a domain. Consider $R = \mathbb{Q}[t]_{(t)}$,

$$R \subseteq R[x]/(x^2 + x - t)$$

is clearly integral, $x^2 + x - t$ is irreducible since its roots are $\frac{-1 \pm \sqrt{1+5t}}{2}$ which do not belong to R . $\mathbb{Q}[t]_{(t)}$ is an UFD so $(x^2 + x - t)$ is prime.

Both $\bar{x}$ and $-1 - \bar{x}$ satisfy $X^2 + X - t = 0$. Since t is not a unit and R is a UFD, by Lemma 1.8, neither $\bar{x}$ nor $-1 - \bar{x}$ can be units of $R[x]/(x^2 + x - t)$, however their sum is a unit. Thus $R[x]/(x^2 + x - t)$ cannot be local.

Definition 1.71. Let M be an R -module and $I \subseteq R$ an ideal. Consider the following diagram generated by the projections:

$$M/IM \leftarrow M/I^2M \leftarrow M/I^2M \leftarrow \dots$$

We denote with $\widehat{M}^I$ the inverse limit of this diagram. When there is no ambiguity we will simply write $\widehat{M}$. Similarly to rings, there is a topology induced by the ideal I , called I -adic topology, described by taking as a basis for the open sets: $x + I^n M$ where $x \in M$ and $n \in \mathbb{N}$. As for the ring case, this topology is induced by a pseudo-metric that is a metric if and only if $\bigcap_{n \in \mathbb{N}} (I^n M) = 0$. The sum of the module and the scalar multiplication $\cdot : R \times M \rightarrow M$ are both continuous functions, in the respective topologies induced by I . Indeed for any $x, y \in M$, $a \in R$, $n \in \mathbb{N}$,

- $(x + I^n M) + (y + I^n M) \subseteq x + y + I^n M$
- $(a + I^n)(x + I^n M) \subseteq ax + I^n M$

Theorem 1.72. Let $(R, \mathfrak{m})$ be a Noetherian complete local ring of dimension d .

1. $\widehat{R}$ is a faithfully flat R -algebra, meaning that the functor $\otimes_R \widehat{R}$ is exact and for any R -module M , $M \otimes_R \widehat{R} = 0$ if and only if $M = 0$.
2. For any finitely generated R -module M , $\widehat{M} \cong M \otimes_R \widehat{R}$.
3. For any R -modules M of finite length, $M \cong M \otimes_R \widehat{R}$.
4. $\widehat{R}$ is a local Noetherian ring with maximal ideal $\mathfrak{m}\widehat{R}$ and dimension d .
5. R is regular if and only if $\widehat{R}$ is regular.

Proof. For 1 and 2, see Eisenbud [8] Theorem 7.2.

For 3, notice that by 5.15, $\mathfrak{m}^n M = 0$ for some $n \in \mathbb{N}$. Consider the map

$$\varphi : M \longrightarrow M \otimes_R \widehat{R} : m \mapsto m \otimes 1$$

If $r \in \widehat{R}$, then $r = r' + r''$ such that $r' \in R$ and $r'' \in \mathfrak{m}^n \widehat{R}$.

Notice that the R -bilinear map $\phi : M \times \widehat{R} \rightarrow M : (m, r) \mapsto r'm$ is well defined, therefore it induces a morphism $g : M \otimes_R \widehat{R} \rightarrow M$ such that $g(m \otimes r) = r'm$. Clearly $g \circ \varphi = id_M$ and $\varphi \circ g = id_{M \otimes_R \widehat{R}}$.

Statement 5 follows from 4.

To prove 4, for Noetherianity see [8]. Recall Remark 1.34, notice that $\widehat{R}/\widehat{\mathfrak{m}}\widehat{R} = \widehat{R}/\widehat{\mathfrak{m}}_1 = R/\mathfrak{m}$ is a field. Moreover, since $\widehat{R}$ is complete with respect to $\widehat{\mathfrak{m}}_1 = \widehat{\mathfrak{m}}\widehat{R}$, by 1.43, $\widehat{\mathfrak{m}}\widehat{R} \subseteq \bigcap \text{Max}(\widehat{R})$. Hence, $\widehat{\mathfrak{m}}\widehat{R}$ is the only maximal ideal. To see that $\dim(\widehat{R}) = \dim(R)$, notice that, since any system of parameters in R generates up to radical the maximal ideal $\widehat{\mathfrak{m}}\widehat{R}$, hence $\dim(\widehat{R}) \leq \dim(R)$. To prove the other inequality, it is needed some kind of lying over theorem to lift chains of prime ideals. See for example page 10 of [11]. There is another approach. In general given $(R, \mathfrak{m})$, the associated graded ring is $gr_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. One has: $\dim(R) = \dim(gr_{\mathfrak{m}}(R))$. Furthermore, it is clear that $gr_{\mathfrak{m}}(R) = gr_{\widehat{\mathfrak{m}}}(\widehat{R})$ since $\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \widehat{\mathfrak{m}}^i/\widehat{\mathfrak{m}}^{i+1}$. $\square$

The following Lemma is taken from [12].

Lemma 1.73. Let R be a ring complete with respect to an ideal $I \subseteq R$. Let M be an R -module and $w_1, \dots, w_n \in M$ such that

- $\bigcap_{n \in \mathbb{N}} (I^n M) = 0$
- M/IM is generated as an R/I -module by $\overline{w_1}, \dots, \overline{w_n}$.

Then M is generated as an R -module by $w_1, \dots, w_n$.

Proof. Let $w_1, \dots, w_n \in M$ be such that $\overline{w_1}, \dots, \overline{w_n}$ are generators of M/IM . Let us write

$$M = \sum_{i=1}^n R w_i + IM = \sum_{i=1}^n R w_i + I \left(\sum_{i=1}^n R w_i + IM \right) = \sum_{i=1}^n R w_i + I^2 M$$

Continuing the procedure, we deduce that for any $\nu \in \mathbb{N}$, $M = \sum_{i=1}^n R w_i + I^\nu M$. For any $\xi \in M$ we may write $\xi = \sum_{i=1}^n a_{0,i} w_i + \xi_1$ with $a_i \in R$ and $\xi_1 \in IM$. Then we can write $\xi_1 = \sum_{i=1}^n a_{1,i} w_i + \xi_2$ with $a_{1,i} \in I$ and $\xi_2 \in I^2 M$. Inductively we have a sequence of $\xi_\nu = \sum_{i=1}^n a_{\nu,i} w_i + \xi_{\nu+1}$ such that $a_{\nu,i} \in I^\nu$ and $\xi_{\nu+1} \in I^{\nu+1} M$ for all $\nu \in \mathbb{N}$. Since R is complete, $b_i := \sum_{\nu \in \mathbb{N}} a_{\nu,i}$ converges. Let us call $b_i^N := \sum_{\nu=0}^N a_{\nu,i}$:

$$\xi - \sum_{i=1}^n b_i^N w_i = \xi_{N+1} \rightarrow 0$$

Since $\bigcap_{n \in \mathbb{N}} (I^n M) = 0$, the topology is Hausdorff and thus $\xi - \sum_{i=1}^n b_i w_i = 0$ $\square$

Theorem 1.74. Let $(R, \mathfrak{m})$ be a complete local Noetherian ring of dimension d , containing a field, let $K \subseteq R$ be a coefficient field of R found using Cohen Structure Theorem 1.45. Then if $\{a_1, \dots, a_d\}$ is a system of parameters ($\sqrt{(a_1, \dots, a_d)} = \mathfrak{m}$); the map (see 1.66)

$$\phi : K[[x_1, \dots, x_d]] \longrightarrow R : x_i \mapsto a_i$$

is injective and the resulting inclusion is finite.

Proof. We first show that R is a finite $K[[x_1, \dots, x_d]]$ -module. Observe that $R/(a_1, \dots, a_d)$ is a Noetherian zero dimensional local ring, so, since K is a coefficient field, it is a finite K -vector space. Hence $R/(a_1, \dots, a_d) = R/(x_1, \dots, x_d)R$ is a finite $K[[x_1, \dots, x_d]]/(x_1, \dots, x_d)$ -vector space. Since R is complete and $(a_1, \dots, a_d)^n \subseteq \mathfrak{m}^n$ for all $n \in \mathbb{N}$, we are in the

hypothesis of Lemma 1.73 and so R is a finite $K[[x_1, \dots, x_d]]$ -module.

If $J = \ker \phi$, then $K[[x_1, \dots, x_d]]/J \subseteq R$ is an integral extension, thus $\dim(K[[x_1, \dots, x_d]]/J) = \dim(R)$. Since $K[[x_1, \dots, x_d]]$ is a domain of dimension d , any of its proper quotient has dimension less than d , so J must be zero, therefore ϕ is injective. $\square$

Lemma 1.75. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and M a finitely generated R -module, then $\text{Ann}_{\widehat{R}} \widehat{M} = (\text{Ann}_R M) \widehat{R}$

Proof. Let $m \otimes 1 \in M \otimes_R \widehat{R}$,

$$0 \longrightarrow 0 :_R m \longrightarrow R \xrightarrow{1 \mapsto m} M$$

is an exact sequence, so

$$0 \longrightarrow (0 :_R m) \widehat{R} \longrightarrow \widehat{R} \xrightarrow{1 \mapsto m \otimes 1} M \otimes_R \widehat{R}$$

is also exact. Thus $0 :_{\widehat{R}} (m \otimes 1) = (0 :_R m) \widehat{R}$.

M is finitely generated so $M = \langle m_1, \dots, m_n \rangle_R$ and $\text{Ann}_R(M) = \bigcap_{i=1}^n (0 :_R m_i)$.

Therefore

$$\text{Ann}_{\widehat{R}}(\widehat{M}) = \bigcap_{i=1}^n (0 :_{\widehat{R}} (m_i \otimes 1)) = \bigcap_{i=1}^n ((0 :_R m_i) \widehat{R}) = \left(\bigcap_{i=1}^n (0 :_R m_i) \right) \widehat{R} = (\text{Ann}_R M) \widehat{R}$$

where we used flatness to say that intersections commute with $- \otimes_R \widehat{R}$. $\square$

2 Dimension one

In this chapter, we shall study the absolute integral closure of complete local Noetherian domains of dimension 1. Theorem 1.74 tells us that whenever R is a complete local Noetherian domain, it contains a formal power series ring such that the inclusion is integral, so by Remark 1.23, we know that they share the same absolute integral closure. This tells us that, for this class of domains, the only absolute integral closure we need to study are of the form $K[[x_1, \dots, x_d]]^+$. In this chapter we shall give an explicit description of this ring in the case when K is algebraically closed and $d = 1$.

To do this we need to first find the algebraic closure of $K((t))$. Its construction strongly depends on the characteristic of K , so we will study each case separately: for the characteristic zero case, we will follow what it has been done by K. J. Nowak in [13] where the factorization version of Hensel's Lemma plays a crucial role in the proof; instead for the characteristic p case, we will follow the work of K. S. Kedlaya done in [4].

After finding the algebraic closure of $K((t))$, we will be able to describe the absolute integral closure of $K[[t]]$ in both cases.

2.1 Characteristic zero

Proposition 2.1. Let R be a domain, $K = \text{frac}(R)$ and $K \subseteq L$ an algebraic field extension. Then L is algebraically closed if and only if every monic polynomial in $R[x]$ completely factors in L .

Proof. We only need to prove the "if" direction. Let $\bar{L}$ be an algebraic closure of L . Let $\lambda \in \bar{L}$, since $\bar{L}/K$ is algebraic, there exists an irreducible polynomial $p \in K[x]$ such that $p(\lambda) = 0$. Multiplying by the denominators we can assume that $p \in R[x]$, so let us write $p(x) = \sum_{i=0}^n r_i x^i$ with $r_i \in R$ for all $i = 0, \dots, n$ and $r_n \neq 0$.

We want to find a monic polynomial:

$$r_n^{n-1} p(x) = \sum_{i=0}^n r_i r_n^{n-1} x^i = \sum_{i=0}^n r_i r_n^{n-1-i} (r_n x)^i = h(r_n x)$$

Now $h(x) = \sum_{i=0}^n r_i r_n^{n-1-i} x^i \in R[x]$ is a monic polynomial, so it completely factors in $L[x]$, and so since $h(r_n \lambda) = 0$ we have that $\lambda \in L$. □

Theorem 2.2 (Newton-Puiseux field). Let K be an algebraically closed field of characteristic zero. If we denote $K((t)) := \text{frac}(K[[t]])$, an algebraic closure of $K((t))$ is the Newton-Puiseux field: $\bigcup_{n \in \mathbb{N}} K((t^{1/n}))$.

Proof. It is sufficient to show that every monic polynomial of degree greater than one inside $\bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$ factors into monic polynomial of positive degree in $\bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$. In fact $\text{frac}(\bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]) = \bigcup_{n \in \mathbb{N}} K((t^{1/n}))$ and so we can use the previous result. Another (less direct) way to prove this, is to use 1.25 to say that $\bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$ is absolutely integrally closed and then use 1.19 to say that its quotient ring must be algebraically closed.

Let us call $B = \bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$, let $P(z) \in B[z]$ be a monic polynomial of degree $n > 1$, let us write $P(z) = z^n + \sum_{i=0}^{n-1} a_{n-i} z^i$, with $a_i \in B$ for all $i = 1, \dots, n$.

Since B is a filtered infinite union and we only have a finite number of coefficients, there must be a $N \in \mathbb{N}$ such that $a_i \in K[[t^{1/N}]]$ and so $P(z) \in (K[[t^{1/N}]]) [z]$, but clearly $K[[t^{1/N}]] \cong K[[t]]$ and so we can assume without loss of generality that $P(z) \in (K[[t]]) [z]$.

Since $K[[t]]$ is a complete local ring, we can use the factorization version of Hensel's Lemma 1.41 to prove that $P(z)$ factors.

Clearly taking the polynomial mod (t) gives us a polynomial in $K[z]$ which must factor into positive degree monic polynomial because K is algebraically closed and $n > 1$. What is not clear is the existence of a factorization into two relatively prime monic polynomials of positive degree. This is always possible whenever $P(z) \bmod (t)$ is not equal to $(z - \alpha)^n$.

To guarantee this we will use a proper a change of variables.

For starters we want to assume $a_1 = 0$, to do this we can make the change of variables $z \mapsto z - \frac{a_1}{n}$, so that

$$p(z - \frac{a_1}{n}) = (z - \frac{a_1}{n})^n + a_1(z - \frac{a_1}{n})^{n-1} + \dots$$

With an easy calculation we get that the coefficient of z^{n-1} in $p(z - \frac{a_1}{n})$ is zero; if $p(z - \frac{a_1}{n}) = g(z) \cdot w(z)$ with g, h two monic polynomial of positive degree, then clearly $p(z) = g(z + \frac{a_1}{n}) \cdot w(z + \frac{a_1}{n})$ and so we are done.

Now if $P(z) \bmod (t) = (z - \alpha)^n$, then we would have $-n\alpha = \overline{a_1} = 0$, but since we are in characteristic zero this means that $\alpha = 0$, and so $\alpha = 0$ thus $a_i \in (t)$ for all $i = 2, \dots, n$.

Let $a_k = t^{r_k} u_k(t)$ such that $u_k(t)$ is a unit of $K[[t]]$ and $r_k \in \mathbb{N}_{>0}$. For every $r \in \mathbb{Q}$

$$p(t^r y) = t^{rn} y^n + \sum_{i=2}^n a_i t^{r(n-i)} y^{n-i} = t^{rn} y^n + \sum_{i=2}^n u_i t^{r_i + r(n-i)} y^{n-i}$$

We want to factor out t^{rn} , to do this it is sufficient to guarantee $rn \leq r_i + r(n-i)$, i.e., $r \leq \frac{r_i}{i}$ for every $i = 2, \dots, n$, so let us set $r = \min\{\frac{r_i}{i} : i = 2, \dots, n\}$.

Then $p(t^r y) = t^{rn} (y^n + \dots) = t^{rn} h(y)$.

Now we can again see $h(y)$ inside $K[[t^{1/M}]] [y]$ for some $M \gg 0$.

The $(n-1)$ -st coefficient of h is zero, moreover if we consider $i \in \{2, \dots, n\}$ such that $r = \frac{r_i}{i}$, then the i -th coefficient of h is u_i which cannot be zero going the modulo $(t^{1/M})$, so by the above discussion, going modulo $(t^{1/M})$ cannot produce a polynomial of the form $(y - \alpha)^n$. So by Hensel's Lemma: $h(y) = g(y) \cdot w(y)$, with both g and h monic polynomial in $K[[t^{1/M}]] [y]$.

We have $p(z) = p(t^r y) = t^{rn} h(y) = t^{rn} g(y) \cdot h(y) = t^{rn} g(t^{-r} z) \cdot h(t^{-r} z)$.
Set $m = \deg_y(g) > 1$ and $m' = \deg_y(w) > 1$ so that $n = m + m'$,

$$t^{rm} g(t^{-r} z) = t^{rm} ((t^{-r} z)^m + g_1 (t^{-r} z)^{m-1} + \dots + g_m) = z^m + t^r g_1 z^{m-1} + \dots + g_m \in B[z]$$

So $t^{rm} g(t^{-r} z)$ is a monic polynomial in $B[z]$, similarly $t^{rm'} w(y)$ is a monic polynomial in $B[z]$, and so we have found a factorisation of $p(z)$ into two positive degree monic polynomial. $\square$

Corollary 2.3. If K is an algebraically closed field of characteristic zero, then $K[[t]]^+ \cong \bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$

Proof. We want to use the characterization of 1.26. We need for every monic polynomial in $K[[t]]$ to factor completely in $\bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$; the proof of the previous theorem already tells us that every monic polynomial in $\bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$ factors completely in $\bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$ which is more than what we need. The only thing we need to observe is that the domain extension $K[[t]] \subseteq \bigcup_{n \in \mathbb{N}} K[[t^{1/n}]]$ is integral.

But this is clear since $t^{1/n}$ are all integral over $K[[t]]$ and for every $n \in \mathbb{N}$, $K[[t^{1/n}]] = K[[t]][t^{1/n}]$ since every element $f(t^{1/n}) \in K[[t^{1/n}]]$ can be written as:
 $f(t^{1/n}) = \sum_{i=0}^{n-1} t^{\frac{i}{n}} f_i(t)$ with $f_i(t) \in K[[t]]$ for all $i = 0, \dots, n-1$. $\square$

Remark 2.4. In characteristic p the Artin Schreier polynomial $x^p - x + t^{-1}$, does not have a solution in $\bigcup_{n \in \mathbb{N}} K((t^{1/n}))$.

Notice that the Frobenius homomorphism is continuous, so it commutes with infinite sums (limits). If $x = \sum_{i \in \mathbb{Q}} x_i t^i \in \bigcup_{n \in \mathbb{N}} K((t^{1/n}))$, was a solution, then:

$$(x^p - x)_{-1} = x_{-1/p}^p - x_{-1} = -1$$

and for any $k \in \mathbb{N}_{>0}$,

$$(x^p - x)_{-1/p^k} = (x_{-1/p^{k+1}})^p - x_{-1/p^k} = 0$$

so $x_{-1/p} = (x_{-1/p^k})^{p^{k-1}}$ for all $k \in \mathbb{N}_{>0}$.

Since $x \in K((t^{1/n}))$ for some $n \in \mathbb{N}_{>0}$, then eventually $x_{-1/p^k} = 0$, and so $x_{-1/p} = 0$ which implies $x_{-1} = 1$.

Now looking at the $-p^k$ th coefficients we get that for any $k \in \mathbb{N}_{>0}$:

$$(x^p - x)_{-p^k} = (x_{-p^{k-1}})^p - x_{-p^k} = 0$$

Thus $x_{-p^k} = 1$ for all $k \in \mathbb{N}$. This is absurd since there exists a natural number $m \in \mathbb{N}$ such that $t^m x$ has a non negative support.

Remark 2.5. Indeed a solution to this Artin Schreier polynomial is the power series $x = \sum_{i=0}^{\infty} t^{-1/p^i}$ which does not belong to the Newton-Puiseux field.

$$x^p - x = \sum_{i=0}^{\infty} t^{-1/p^{i+1}} - \sum_{i=0}^{\infty} t^{-1/p^i} = -t^{-1}$$

In the next section we shall see how to add this type of series to get an algebraic closure in characteristic p .

2.2 Positive characteristic

Definition 2.6. A totally ordered set $(X, \leq)$ is said to be well ordered if any nonempty subset of X admits a minimum.

Theorem 2.7. Let K be an algebraically closed field of positive characteristic p . The subset of generalized power series L of the form $f = \sum_{i \in S} x_i t^i$, $x_i \in K$ such that:

- $S \subseteq \mathbb{Q}$ is well ordered
- there exists a natural number $m \neq 0$ such that $mS \subseteq \mathbb{Z}[p^{-1}]$

is an algebraically closed field; in particular we have that $K((t)) \subseteq L$, and so L contains an algebraic closure of $K((t))$.

We will not give a proof to this theorem, this result was independently proved by Huang [14], Rayner [15] and Stefanescu [16]. The only important notions to take from this result are the following:

- When the supports of power series are well ordered sets, operations makes sense.
- The algebraic closure we are searching sits inside this bigger field.
- The only denominators that are not well behaved are powers of p .

What follows is a summary of the work done by Kiran Kedlaya in [4], to construct the algebraic closure of $K((t))$ in characteristic p .

From now on K will always be an algebraically closed field of characteristic p .

Lemma 2.8. Let $L/K((t))$ be a normal finite field extension, then there exists $n \in \mathbb{N}$, such that $L/K((t^{1/n}))$ is a tower of Artin-Schreier extensions. (We call Artin-Schreier extension, any extension obtained by adding a solution to an Artin-Schreier polynomial).

Proof. See [4] to get started. □

Definition 2.9. A sequence $(c_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}}$ satisfies the "linearized recurrence relations" (LRR) with respect to $d_0, \dots, d_k \in K$ if for all $n \in \mathbb{N}$

$$\sum_{i=0}^k d_i c_{n+i}^{p^i} = 0 \tag{1}$$

We say that a sequence satisfies LRR, when there exist $k \in \mathbb{N}_{>0}$ and $d_0, \dots, d_k \in K$ such that $d_k \neq 0$ and the sequence satisfies LRR with respect to $d_0, \dots, d_k$.

Remark 2.10. Let us list some easy properties,

1. Any constant sequence $(c)_{n \in \mathbb{N}}$ satisfies LRR with respect to $c^{p-1}, -1$.

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence that satisfies LRR with respect to $d_0, \dots, d_k$.

2. If a sequence $(v_n)_{n \in \mathbb{N}}$ is such that there exists $m \in \mathbb{N}$ such that $v_{m+n} = c_n$ for all $n \in \mathbb{N}$, then it satisfies LRR with respect to $0, 0, \dots, 0, d_0^{p^m}, \dots, d_k^{p^m}$, where the number of zeroes is m .
3. If $\lambda \in K$, then $(\lambda c_n)_{n \in \mathbb{N}}$ satisfies LRR with respect to $d_0 \lambda^{p^k-1}, d_1 \lambda^{p^k-p}, d_2 \lambda^{p^k-p^2}, \dots, d_k$:

$$\sum_{i=0}^k d_i \lambda^{p^k-p^i} (\lambda c_{n+i})^{p^i} = \lambda^{p^k} \sum_{i=0}^k d_i c_{n+i}^{p^i} = 0$$

Putting together 1 and 2, we get that any sequence that is eventually null satisfies LRR.

Lemma 2.11. Let $d_0, \dots, d_k \in K$, $d_0 \neq 0 \neq d_k$

1. Let $P(x) = \sum_{i=0}^k d_i x^{p^i}$, if $Z_P \subseteq K$ is the set of all the roots of P , then it is a k -dimensional $\mathbb{F}_p$ -vector space.
2. Let $z_1, \dots, z_k$ be a basis for Z_P , then

$$\{(c_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}} : (c_n)_{n \in \mathbb{N}} \text{ satisfies LRR wrt } d_0, \dots, d_k\} =$$

$$\{(c_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}} : \exists \lambda_1, \dots, \lambda_k \in K \text{ such that } c_n = \sum_{i=1}^k z_i \lambda_i^{p^{-n}}\}$$

Proof.

1. Z_P is an additive group, since by Frobenius we have that $P(y+z) = P(y) + P(z)$; moreover, since for any $\lambda \in \mathbb{F}_p$ we have that $\lambda^p = \lambda$, then clearly $P(\lambda y) = \lambda P(y)$. This proves that Z_P is an $\mathbb{F}_p$ -vector space.
For the dimension: we notice that $P'(x) = d_0$, and so P doesn't have multiple roots, which means that the cardinality of Z_P is exactly the degree of P , which is p^k . So k must be the dimension.
2. We first prove that both of the above sets are K vector spaces of dimension k . Let $(c_n)_{n \in \mathbb{N}}$ and $(c'_n)_{n \in \mathbb{N}}$ be two sequences satisfying LRR. Then putting $(c_n + c'_n)_{n \in \mathbb{N}}$ inside (1), we get

$$\sum_{i=0}^k d_i (c_{n+i} + c'_{n+i})^{p^i} = \sum_{i=0}^k d_i c_{n+i}^{p^i} + \sum_{i=0}^k d_i c'_{n+i}^{p^i} = 0$$

and so the first set is a group.

We now define the scalar multiplication in a non-conventional way:

$$\lambda \in K, \lambda \cdot (c_n)_{n \in \mathbb{N}} := (\lambda^{p^{-n}} c_n)_{n \in \mathbb{N}}.$$

With this scalar multiplication, the first set is indeed a K -vector space:

$$\sum_{i=0}^k d_i (\lambda^{p^{-n-i}} c_{n+i})^{p^i} = \lambda^{p^{-n}} \sum_{i=0}^k d_i c_{n+i}^{p^i} = 0$$

Let us now prove that the dimension of this vector space is k .
Consider the following map:

$$\{(c_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}} : (c_n)_{n \in \mathbb{N}} \text{ satisfies LRR wrt } d_0, \dots, d_k\} \longrightarrow K^k$$

$$(c_n)_{n \in \mathbb{N}} \mapsto (c_0, c_1^p, \dots, c_{k-1}^{p^{k-1}})$$

It is easy to verify that this map is K -linear, moreover it is bijective, since all the coefficients with indexes greater than $k - 1$ are forced by the LRR conditions (here we use that $d_k \neq 0$).

Now let us show that the second set is a K -vector space, with the same scalar multiplication.

Again, using Frobenius, it is clear that the set is a group and the scalar multiplication is exactly the normal scalar multiplication on the vectors $(\lambda_1, \dots, \lambda_k)$. This proves that the following map from K^k to the second set is K linear and surjective:

$$(\lambda_1, \dots, \lambda_k) \mapsto \left(\sum_{i=1}^k z_i \lambda_i^{p^{-n}} \right)_{n \in \mathbb{N}}$$

The injectivity of this map follows from a result from linear algebra:

If $z_1, \dots, z_k$ are $\mathbb{F}_p$ -linearly independent, then the Moore determinant is non zero (see 5.7), meaning that the following matrix is invertible in $K^{k \times k}$:

$$M = \begin{bmatrix} z_1 & z_2 & \dots & z_k \\ z_1^p & z_2^p & \dots & z_k^p \\ \dots & \dots & \dots & \dots \\ z_1^{p^{k-1}} & z_2^{p^{k-1}} & \dots & z_k^{p^{k-1}} \end{bmatrix}.$$

Let $(\lambda_1, \dots, \lambda_k)$ be in the kernel of the map above, then clearly $\sum_{i=1}^k z_i \lambda_i^{p^{-n}} = 0$ for all n . In particular, $\sum_{i=1}^k z_i^{p^n} \lambda_i = 0$ for $n \in \{0, \dots, k-1\}$.

This means that:

$$M \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_k \end{bmatrix} = 0$$

and so $\lambda_i = 0$ for all $i \in \{1, \dots, k\}$.

Finally let us prove one inclusion (this is enough because they are both K vector spaces with the same finite dimension).

If $\lambda_1, \dots, \lambda_k \in K$, let us prove that $(\sum_{i=1}^k z_i \lambda_i^{p^{-n}})_{n \in \mathbb{N}}$ satisfies LRR with respect to $d_0, \dots, d_k$:

$$\sum_{i=0}^k d_i \left(\sum_{j=1}^k z_j \lambda_j^{p^{-n-i}} \right)^{p^i} = \sum_{i=0}^k \sum_{j=1}^k d_i z_j^{p^i} \lambda_j^{p^{-n}} = \sum_{j=1}^k \lambda_j^{p^{-n}} \sum_{i=0}^k d_i z_j^{p^i} = 0.$$

□

Corollary 2.12. If $(c_n)_{n \in \mathbb{N}}$ and $(c'_n)_{n \in \mathbb{N}}$ satisfies LRR with respect to $d_0, \dots, d_k$ and $d'_0, \dots, d'_l$ respectively, then $(c_n + c'_n)_{n \in \mathbb{N}}$ and $(c_n c'_n)_{n \in \mathbb{N}}$ both satisfy LRR with respect to some coefficients which only depend on d_i and d'_i .

Proof. We want to use the characterization given in 2 of 2.11.

Firstly, notice that we can assume $d_0 \neq 0 \neq d'_0$, since if this is not the case, we can delete all the zeroes and consider the two sequences starting from a sufficiently large $N \in \mathbb{N}$. This is sufficient thanks to point 2 of Remark 2.10.

Let $(c_n)_{n \in \mathbb{N}}$ and $(c'_n)_{n \in \mathbb{N}}$ be as in the hypothesis, let $\{z_i\}, \{z'_i\}$ be basis of the polynomials from 2 of 2.11, then there exists $\lambda_1, \dots, \lambda_k \in K$ and $\lambda'_1, \dots, \lambda'_l \in K$ such that $c_n = \sum_{i=1}^k z_i \lambda_i^{p^{-n}}$ and $c'_n = \sum_{i=1}^l z'_i \lambda'_i^{p^{-n}}$ for all $n \in \mathbb{N}$.

Finally $c_n + c'_n = \sum_{i=1}^k z_i \lambda_i^{p^{-n}} + \sum_{i=1}^l z'_i \lambda'_i^{p^{-n}}$ and so the sum satisfies LRR with respect to the coefficients of the polynomial whose roots generates the $\mathbb{F}_p$ -vector space generated by $\{z_1, \dots, z_k, z'_1, \dots, z'_l\}$.

Notice that the above set needs not to be a basis. To be more precise, if we rewrite the set as $\{\tilde{z}_1, \dots, \tilde{z}_{l+k}\}$ and their coefficients as $\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_{l+k}\}$. We can extract a basis $\{\tilde{z}_1, \dots, \tilde{z}_c\}$ and rewrite the linearly dependent elements in terms of the basis, so $\tilde{z}_j = \sum_{q=1}^c a_{q,j} \tilde{z}_q$ for all $c < j \leq l+k$. Doing this gives another expression for $c_n + c'_n$:

$$\begin{aligned} c_n + c'_n &= \sum_{i=1}^c \tilde{z}_i \tilde{\lambda}_i^{p^{-n}} + \sum_{j=c+1}^{l+k} \tilde{z}_j \tilde{\lambda}_j^{p^{-n}} = \sum_{i=1}^c \tilde{z}_i \tilde{\lambda}_i^{p^{-n}} + \sum_{j=c+1}^{l+k} \sum_{i=1}^c a_{i,j} \tilde{z}_i \tilde{\lambda}_j^{p^{-n}} = \\ &= \sum_{i=1}^c \tilde{z}_i (\tilde{\lambda}_i^{p^{-n}} + \sum_{j=c+1}^{l+k} a_{i,j} \tilde{\lambda}_j^{p^{-n}}) = \sum_{i=1}^c \tilde{z}_i (\tilde{\lambda}_i + \sum_{j=c+1}^{l+k} a_{i,j} \tilde{\lambda}_j)^{p^{-n}} \end{aligned}$$

Where we use that $a^{p^{-1}} = a$ for all $a \in \mathbb{F}_p$.

For the product we write: $c_n c'_n = \sum_{i+j=k} (z_i z'_j) (\lambda_i \lambda'_j)$.

We use the same argument as before to reduce the product of the roots to an $\mathbb{F}_p$ -linearly independent set, and so we are done. $\square$

Definition 2.13. Let $a, b, c \in \mathbb{N}$, $a \neq 0$, we define the following subsets of $\mathbb{Q}$

$$S_{a,b,c} := \frac{1}{a} \left\{ n - \sum_{i=1}^{\infty} b_i p^{-i} : n \in \mathbb{Z}, n \geq -b, b_i \in \{0, \dots, p-1\}, \sum_{i=1}^{\infty} b_i \leq c \right\}.$$

We define $T_c := S_{1,0,c} \cap (-1, 0]$.

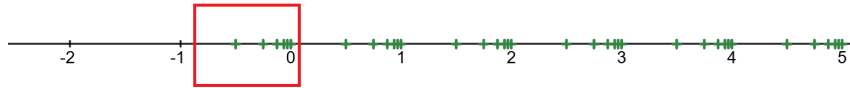


Figure 1: in green $S_{1,0,1}$ and T_1 contained in red.

Remark 2.14. It is easy to see that there is a partition $S_{a,b,c} = \bigsqcup_{n \geq -b} \frac{1}{a}(n + T_c)$. Moreover, all these sets are well ordered:

Let $\emptyset \neq U \subseteq S_{a,b,c}$, to find the minimum it is sufficient to choose the minimum

$n \in \mathbb{Z}_{\geq -b}$ such that some $\frac{1}{a}(n - \sum_{i=1}^{\infty} b_i p^{-i}) \in U$. From this we need to maximize the b_i 's in the first indexes, this is always possible thanks to the constraint on the sum of all the b_i 's.

We also have the following useful facts:

1. $S_{a,b,c} \cup S_{a',b',c'} \subseteq S_{aa',a'(b+1)+a(b'+1),a'c+ac'}$
2. $S_{a,b,c} + S_{a',b',c'} \subseteq S_{aa',a'(b+1)+a(b'+1)+1,a'c+ac'}$

To prove 1, notice that $\frac{1}{a}(n - \sum b_i p^{-i}) = \frac{1}{aa'}(a'n - \sum a'b_i p^{-i})$.

Consider $v = a'n - \sum a'b_i p^{-i}$, we have that, $a'n - a' \leq v \leq a'n$.

Write v in the form $m - \sum f_i p^{-i}$, with $f_i \in \{0, \dots, p-1\}$, so using $-b \leq n$,

$$-a(b'+1) - a'(b+1) \leq -a'(b+1) \leq a'n - a' \leq v \leq m.$$

Furthermore, $\sum f_i \leq \sum a'b_i \leq a'c \leq a'c + ac'$.

We have proved that $S_{a,b,c} \subseteq S_{aa',a'(b+1)+a(b'+1),a'c+ac'}$,

by symmetry we also get $S_{a',b',c'} \subseteq S_{aa',a'(b+1)+a(b'+1),a'c+ac'}$.

For 2, notice that, using the notation as before,

$$\frac{1}{a}(n - \sum b_i p^{-i}) + \frac{1}{a'}(n' - \sum b'_i p^{-i}) = \frac{1}{aa'}(m + m' - \sum (f_i + f'_i) p^{-i})$$

If we now write this in the form $\frac{1}{aa'}(m'' - \sum f''_i p^{-i})$ we get that

$$-a'(b+1) - a(b'+1) - 1 \leq m + m' - 1 \leq m''.$$

Moreover $\sum f''_i \leq \sum (f_i + f'_i) \leq a'c + ac'$.

(Recall from the proof of 1 that $\sum f_i \leq a'c$ and $\sum f'_i \leq ac'$)

Definition 2.15. A function $f : T_c \rightarrow K$ is said to be twist recurrent of order k if there exist $d_0, \dots, d_k \in K$ such that for any "twisted sequence" (defined below) $(a_n)_{n \in \mathbb{N}} \in (T_c)^{\mathbb{N}}$, the sequence $(f(a_n))_{n \in \mathbb{N}}$ satisfies LRR with respect to $d_0, \dots, d_k$.

Twisted sequences are sequences $(a_n)_{n \in \mathbb{N}}$ in T_c such that there exists two elements $\alpha, \beta \in T_c$ such that if $m = \text{ord}_{p^{-1}}(\alpha)$ then $\alpha + p^{-m}\beta \in T_c$ and $a_n := \alpha + p^{-m-n}\beta$ for all $n \in \mathbb{N}$. In base $p > 2$, for $c = 3$, an example is:

$$\text{for } \alpha = -0.01; m = 2; \beta = -0.2;$$

$$a_0 = -0.012; a_1 = 0.0102; a_2 = 0.01002; a_3 = 0.0010002; \dots$$

Definition 2.16. A series $x = \sum_{i \in \mathbb{Q}} x_i t^i$ is twist recurrent if

1. the support of x is contained in a set of the form $S_{a,b,c}$ for some $a, b, c \in \mathbb{N}$, $a \neq 0$
2. for some $S_{a,b,c}$ as in 1, for all $m \geq -b$, the functions $f_m : T_c \rightarrow K$ defined as $f_m(z) = x_{\frac{m+z}{a}}$ are twist recurrent for some order k .
3. the K -vector space generated by all the f_m in 2 is finitely generated.

Remark 2.17. If x is a twist recurrent series supported on $S_{a,b,c}$, meaning that conditions 1, 2 and 3 of the definition are satisfied for this choice of the support, then for any $b' \in \mathbb{N}$ and $c' \in \mathbb{N}$, conditions 2 and 3 are satisfied for $S_{a,b',c'}$.

Condition 2: let $(a_n)_{n \in \mathbb{N}}$ be a twisted sequence in $T_{c'}$, then it is either contained entirely in T_c or it never intersects it (since this only depend on the sums of the digits which never changes for twisted sequences), thus if $m \geq -b'$, $(x_{\frac{m+a_n}{a}})_{n \in \mathbb{N}}$ is either zero or it satisfies LRR. In either case it satisfies LRR.

Condition 3 is also clear, since the resulting f_m are either extension to zero or restriction of the f_m of $S_{a,b,c}$. This operations does not change the dimension of the vector space.

Remark 2.18. If x is a twist recurrent series supported on $S_{a,b,c}$, then for any $a' \in \mathbb{N}_{>0}$, condition 2 is true for $S_{a',b,c}$.

Let us assume p does not divide a' . Let $(d_n)_{n \in \mathbb{N}}$ be a twisted sequence in T_c , by definition $d_n = \alpha + p^{-n-o}\beta$ with $\alpha, \beta \in T_c$ and $o = \text{ord}_{p-1}(\alpha)$. Let $m \geq -b$.

We need to prove that, if we call $(c_n)_{n \in \mathbb{N}} = (\frac{m+\alpha+p^{-n-o}\beta}{a'})_{n \in \mathbb{N}}$, then the sequence $(x_{c_n})_{n \in \mathbb{N}}$ satisfies LRR. We have

$$c_n = \frac{m + \alpha + p^{-n-o}\beta}{a'} = \frac{\frac{a}{a'}(m + \alpha) + \frac{a}{a'}p^{-n-o}\beta}{a} = \frac{v_n}{a}$$

- If $\frac{a}{a'}(m + \alpha) \notin \mathbb{Z}[1/p]$ then it has infinite digits in base p . Notice that $v_n \rightarrow \frac{a}{a'}(m + \alpha)$, thus there exists $N \in \mathbb{N}$ such that for any $n \geq N$ the sum of the digits of v_n is greater than c , so $v_n \notin S_{1,b,c}$ and $\frac{v_n}{a} \notin S_{a,b,c}$, which entails that x_{c_n} is definitely null and so $(x_{c_n})_{n \in \mathbb{N}}$ satisfies LRR.
- If $\frac{a}{a'}(m + \alpha) \in \mathbb{Z}[1/p]$,
 - If $\frac{a}{a'}p^{-n-o}\beta \notin \mathbb{Z}[1/p]$ (this condition doesn't depend on n), then clearly $v_n \notin \mathbb{Z}[1/p]$ for all $n \in \mathbb{N}$, thus $v_n \notin S_{1,b,c}$; $c_n \notin S_{a,b,c}$, and $x_{c_n} = 0$ for all $n \in \mathbb{N}$, and we conclude as above.
 - If $\frac{a}{a'}p^{-n-o}\beta \in \mathbb{Z}[1/p]$,

$$\frac{a}{a'}p^o(m + \alpha) \in \mathbb{Z}[1/a'] \cap \mathbb{Z}[1/p] = \mathbb{Z}$$

This implies that if $o' = \text{ord}_{p-1}(\frac{a}{a'}(m + \alpha))$, then $o = o' + o''$ for some $o'' \in \mathbb{N}$.

Let $k \in \mathbb{N}$ be such that $\frac{a}{a'}p^{-k}\beta < 1$, then

$$v_{n+k} = \frac{a}{a'}(m + \alpha) + \frac{a}{a'}p^{-n-k-o''-o'}\beta = m' + \alpha' + p^{-n-o'}\beta',$$

where $m' = \lceil \frac{a}{a'}(m + \alpha) \rceil$, $\alpha' = \frac{a}{a'}(m + \alpha) - m' \in \mathbb{Z}[1/p] \cap (-1, 0]$, $\beta' = \frac{a}{a'}p^{-k-o''}\beta \in \mathbb{Z}[1/p] \cap (-1, 0]$.

Clearly $o' = \text{ord}_{p-1}(\alpha')$, so the sequence $v_{n+k} - m'$ is twisted in some $T_{c'}$. As already stated before, a twisted sequence is either completely contained in a T_c or it never intersects it.

If $(v_{n+k} - m')_{n \in \mathbb{N}}$ is not in T_c , then $v_{n+k} \notin S_{1,b,c}$ and again the resulting sequence is null.

If $(v_{n+k} - m')_{n \in \mathbb{N}}$ is in T_c , then (depending on m') the resulting sequence is either null or it satisfies LRR using the fact that x is twist recurrent on $S_{a,b,c}$.

If $a' = p^k a''$, then $(c_n)_{n \in \mathbb{N}} = (\frac{m+\alpha+p^{-n-o}\beta}{a'})_{n \in \mathbb{N}} = (\frac{p^{-k}(m+\alpha)+p^{-n-o-k}\beta}{a''})_{n \in \mathbb{N}}$, and so we can choose different α and β and use what we have already proved.

Remark 2.19. If x is a twist recurrent series supported on $S_{a,0,c}$; then for any $a' \in \mathbb{N}_{>0}$, condition 3 is true for $S_{a',0,c}$.

Let us call $f_{m,s} : \mathbb{Q} \cap (-1, 0] \rightarrow K : f_{m,s}(i) = x_{\frac{m+i}{s}}$.

Let us call $V_s := \langle \{f_{m,s} : m \geq 0\} \rangle_K$. We prove that for any $s, s' \in \mathbb{N}_{>0}$, V_s is finitely generated if and only if $V_{ss'}$ is finitely generated, notice that this proves the claim.

Let $m \geq 0$,

$$f_{m,s}(i) = x_{\frac{s'm + s'i}{ss'}} = \sum_{j=0}^{s'-1} f_{s'm-j,ss'}(s'i+j) 1_{(-1,0]}(s'i+j).$$

This decomposition is clearly unique and linear. Therefore the following map exists and it is injective:

$$\phi : V_s \rightarrow V_{ss'}^{s'} : f_{m,s} \mapsto (f_{s'm-j,ss'})_{j=0,\dots,s'-1}$$

It follows that V_s is finitely generated whenever $V_{ss'}$ is, moreover $\dim_K V_s \leq s' \dim_K V_{ss'}$. For the converse, consider the following surjective linear map:

$$V_s^{s'} \rightarrow V_{ss'} : (v_i)_{i=0,\dots,s'-1} \mapsto \sum_{i=0}^{s'-1} \pi_i(\phi(v_i))$$

where $\pi_i : V_s^{s'} \rightarrow V_s$ is the i 'th projection for all $i = 0, \dots, s' - 1$. Thus if $V_{ss'}$ is finitely generated, so it is V_s , moreover $\dim_K V_{ss'} \leq s' \dim_K V_s$.

Corollary 2.20. Putting together the previous remarks, we obtain the following: if x is a twisted series, then conditions 2 and 3 are always valid for any set $S_{a,b,c}$.

Lemma 2.21. Any twist recurrent series supported on $S_{1,b,c}$ is a $K((t))$ -linear combination of twist recurrent series supported on T_c and viceversa.

Proof. Let $x = \sum_{i \in S_{1,b,c}} x_i t^i$ be a twist recurrent series and let $f_m : T_c \rightarrow K$ be as in 2 of definition 2.16. Let $g_1, \dots, g_r$ be a basis of the space generated by all the f_m (we can take all the g_i such that $g_i = f_{m_i}$ for some $m_i \geq -b$).

For any $m \geq -b$, let $k_{m,i} \in K$ such that $f_m = \sum_{j=1}^r k_{m,j} g_j$.

Using the partition $S_{1,b,c} = \bigsqcup_{m \geq -b} (m + T_c)$, we may write:

$$\begin{aligned} x &= \sum_{i \in S_{1,b,c}} x_i t^i = \sum_{m \geq -b} \sum_{i \in T_c} x_{m+i} t^{m+i} = \sum_{m \geq -b} \sum_{i \in T_c} f_m(i) t^{m+i} = \\ &= \sum_{m \geq -b} \sum_{i \in T_c} \sum_{j=1}^r k_{m,j} g_j(i) t^{m+i} = \sum_{j=1}^r \left(\sum_{m \geq -b} k_{m,j} t^m \right) \left(\sum_{i \in T_c} g_j(i) t^i \right). \end{aligned}$$

Since $g_i = f_{m_i}$, the sum $\sum_{i \in T_c} g_j(i) t^i$ is a twist recurrent series supported in T_c (choose $S_{1,0,c}$ and all the conditions are clearly satisfied).

So we have proved the first implication.

To prove the second implication, we shall prove that twist recurrent series supported on T_c times $v \in K((t))$ are twist recurrent supported in $S_{1,b,c}$ for some b . Then we shall prove that sums of twist recurrent series supported on $S_{1,b,c}$ are twist recurrent supported on $S_{1,b,c}$.

- Let $x = \sum_{i \in T_c} x_i t^i$ be a twist recurrent series and $v = \sum_{i \geq -b} v_i t^i$, $b \in \mathbb{N}$.
 v is twist recurrent, since it is supported on $S_{1,b,0}$ and $\bar{T}_0 = \{0\}$; therefore, condition 2 and 3 are tautological.
 vx is supported in $T_c + S_{1,b,0} = T_c + \mathbb{N}_{\geq -b} = S_{1,b,c}$.
Let $m \geq -b$, $i \in T_c$, $f_m(i) = (vx)_{m+i} = v_m x_i$, thus if $(a_n)_{n \in \mathbb{N}}$ is a twisted sequence, $f_m(a_n) = v_m x_{a_n}$ and so it satisfies LRR.
Notice that all the f_m are multiples of the function $i \in T_c \mapsto x_i \in K$, and so 3 is also verified.
- Let $x = \sum_{i \in S_{1,b,c}} x_i t^i$ and $y = \sum_{i \in S_{1,b,c}} y_i t^i$ be two twist recurrent series.
Clearly $x + y$ is supported on $S_{1,b,c}$. Condition 2 is true because if $(a_n)_{n \in \mathbb{N}}$ is a twisted sequence, then $f_m(a_n) = (x + y)_{m+a_n} = x_{m+a_n} + y_{m+a_n}$; using Corollary 2.12, we know that this sequence satisfies LRR.
Condition 3 is trivial since the vector spaces generated by the functions f_m , is contained in the sum of the finite dimensional vector spaces generated by the " f_m s" of x and y , and thus it is finite dimensional.

□

Theorem 2.22. The set of twist recurrent series is an algebraic closure of $K((t))$

Proof. Assume the following statements,

1. Every twist recurrent series is algebraic over $K((t))$.
2. The set of twist recurrent series is a ring.
3. If y is a twist recurrent series then there is a twist recurrent series x such that $x^p - x = y$.

Then the proof is done: we know that the elements of $K((t))$ are all twist recurrent series, so if 2 is true, let us call L' the ring of twist recurrent series. Since $K((t)) \subseteq L'$ is integral because of 1, L' is a field and it is algebraic over $K((t))$, we only need to prove that it is algebraically closed.

If $\lambda \in \bar{L}'$ then let $K((t)) \subseteq l$ be the finite normal extension obtained from adjoining all the roots of the minimal polynomial of λ . Using Lemma 2.8, there is an $n \in \mathbb{N}$ such that $l/K((t^{1/n}))$ is a tower of Artin-Schreier extensions. But $K((t^{1/n})) \subseteq L'$ and using 3 all the Artin-Schreier extensions from L' are inside L' , so $\lambda \in l \subseteq L'$.

Let us now begin to prove the statements.

1. Assume that we have proven that all twist recurrent series supported on any $S_{1,b,c}$ are algebraic over $K((t))$.

Then if $x = \sum_{i \in S_{a,b,c}} x_i t^i = \sum_{i \in S_{1,b,c}} x_{i/a} (t^{1/a})^i$, since $K((t)) \cong K((t^{1/a}))$, we can follow this isomorphism to find a polynomial which annihilates x , so all twist recurrent series are algebraic.

Thanks to Lemma 2.21 we only need to show that twist recurrent series supported on T_c are algebraic over $K((t))$. Let x be a twist recurrent series on T_c .

Let us proceed by induction on $c \in \mathbb{N}$: if $c = 0$ then $x = x_0$ and so it is algebraic.

Let us assume that all the twist recurrent series supported on T_{c-1} are algebraic over $K((t))$. Then by the previous discussions we know that all the twist recurrent series supported on $S_{a,b,c-1}$ are algebraic for any $a \in \mathbb{N}_{>0}$ and $b \in \mathbb{N}$.

Let $d_0, \dots, d_k \in K$ be such that for all twisted sequences $(a_n)_{n \in \mathbb{N}}$, $(x_{a_n})_{n \in \mathbb{N}}$ satisfies LRR with respect to $d_0, \dots, d_k$ and let $y := \sum_{i=0}^k d_i^{p^{-k}} x^{p^{-k+i}}$ (notice that Frobenius commutes with infinite sums).

Clearly y is supported on T_c . We want to show that it is supported on $S_{p^k, p^k, c-1}$. Let $j \in T_c \setminus S_{p^k, p^k, c-1}$, and write $j = -\sum_{i=1}^{\infty} b_i p^{-i}$. Then

$$j = p^{-k} \left(-\sum_{i=1}^k b_i p^{k-i} - \sum_{i=k+1}^{\infty} b_i p^{k-i} \right) \notin S_{p^k, p^k, c-1}$$

Since clearly $-\sum_{i=1}^k b_i p^{k-i} \geq -p^k$, the second condition must fail, so $\sum_{i=k+1}^{\infty} b_i > c-1$. Since $\sum_{i=1}^{\infty} b_i \leq c$ we conclude that $\sum_{i=k+1}^{\infty} b_i = c$ and $b_i = 0$ for all $i \leq k$. So $j = p^{-k} \sum_{i=1}^{\infty} b_{i+k} p^{-i}$ and $p^k j \in T_c$. It follows that

$$y_j = \sum_{i=0}^k d_i^{p^{-k}} (x_{p^{k-i}j})^{p^{-k+i}} \text{ and thus}$$

$$y_j^{p^k} = \sum_{i=0}^k d_i x_{p^{k-i}j}^{p^i}.$$

Now the sequence $(p^{k-n}j)_{n \in \mathbb{N}}$ is in T_c and it is twisted (choose $\alpha = 0, \beta = p^k j$). Thus $y_j^{p^k} = 0$ and so $y_j = 0$ and we have proved that y is supported on $S_{p^k, p^k, c-1}$.

Recall that $y^{p^k} = \sum_{i=0}^k d_i x^{p^i}$, so if y is a twist recurrent series, by induction hypothesis, it must be algebraic, and so x is also algebraic.

We must show that y is a twist recurrent series, we choose the set $S_{1,0,c}$. Since y is supported on T_c , condition 1 and 3 are clear. Let $(a_n)_{n \in \mathbb{N}}$ be a twisted sequence in T_c , then

$$y_{a_n} = \sum_{i=0}^k d_i^{p^{-k}} (x_{p^{k-i}a_n})^{p^{-k+i}}$$

Now, the sequence $(p^{k-i}a_n)_n$ is eventually of the form $(m + a'_n)_n$, where a'_n is a twisted sequence and $m \in \mathbb{N}$, so it satisfies LRR.

Evidently, taking the p 'th roots of a sequence that satisfies LRR for some coefficients results in a sequence satisfying LRR for the p 'th roots of the coefficients; moreover we have already shown that normal scalar multiplications and sums of sequences satisfying certain LRR also satisfy LRR. And so we have proved that y is a twist recurrent series.

2. Again, we want to assume $a = 1$. We already know from the proof of 2.21 that sums of twist recurrent series supported on $S_{1,b,c}$ are twist recurrent sequence supported on $S_{1,b,c}$. (Recall that thanks to Remark 2.17, we can assume to have fixed b and c)

Let $\sum_i x_i t^i$ and $\sum_i y_i t^i$ be twist recurrent series supported (using 2.20 and 2.14) on a common $S_{a,b,c}$. Then, as before, we know that $\sum_i x_i / a t^i$ and $\sum_i y_i / a t^i$ are twist recurrent series supported on some $S_{1,b',c'}$, so their sum is a twist recurrent series supported on $S_{1,b',c'}$ and we conclude that $x+y$ is twist recurrent supported on $S_{a,b,c}$. Now for the multiplication, for the same reasons as before, we only need to prove that multiplication of twist recurrent series supported on T_c are twist recurrent.

Notice that thanks to 2.14, $T_c + T_c \subseteq S_{1,2,2c}$, and thus the support of xy is contained

in $S_{1,2,2c}$. Moreover it is contained in $(-2, 0]$ and so condition 3 is also true.

First of all, the products is well defined, since both the support are well ordered. So $xy = \sum_k (xy)_k t^k$, where $(xy)_k = \sum_{i,j \in T_c, i+j=k} x_i y_j$ is a finite sum.

Let $c_n = \alpha + p^{-n-m}\beta$ be a twisted sequence in T_c , $m = \text{ord}_{p^{-1}}(\alpha)$.

Notice that since $\alpha \in T_c$, the set of pairs (α_1, α_2) in T_c such that $\alpha = \alpha_1 + \alpha_2$ is finite. The same thing is true for β . We argue that for sufficiently large $n \in \mathbb{N}$, the only decompositions $c_n = c_{n,1} + c_{n,2}$ are of the form:

$$c_{n,i} = \alpha_i + p^{-n-m}\beta_i, \text{ for } i = 1, 2$$

$$\alpha = \alpha_1 + \alpha_2$$

$$\beta = \beta_1 + \beta_2$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in T_c$.

If this is the case, since there is only a finite number of pairs, c_n has a uniformly (with respect to n) bounded number of ways to be written as a sum of elements in T_c . Therefore $(xy)_{c_n} = \sum_q x_{c_{n,1,q}} y_{c_{n,2,q}}$. Since all $c_{n,i,q}$ are twisted sequences, by 2.12 $(xy)_{c_n}$ satisfies LRR (recall that it is sufficient for it to eventually satisfy some LRR, to satisfy LRR).

To prove the claim, let $c_n = c_{n,1} + c_{n,2}$; then the sum of the digits in c_n is equal to the sum of the digits in $c_{n,1}$ and $c_{n,2}$ minus $(p-1)$ times the number of carries. Since all the sums are bounded by c , the number of carries is also bounded. Indeed it cannot surpass $2c$.

Hence, if $n > -4c-1$ the $p^{-m-2c-1}$ th digit of $c_{n,1}$ and $c_{n,2}$ must be zero, since in the sum is zero and if it was not zero we would need more then $2c$ carries to annihilate all the digits from p^{-m-1} to $p^{-m-2c-1}$. Hence the sum of the truncation of $c_{n,1}$ and $c_{n,2}$ to the p^{-m-2c} digit gives us α . Then clearly what remains must gives us $p^{-m-n}\beta$.

3. Notice that if $y = \sum_{i \in \mathbb{Q}} y_i t^i$ is a twist recurrent series, we can write $y = y_0 + \sum_{i \in \mathbb{Q} \cap (0, \infty)} y_i t^i + \sum_{i \in \mathbb{Q} \cap (-\infty, 0)} y_i t^i = y_0 + y_+ + y_-$ and y_+, y_- are twist recurrent series.

Since the map $x \mapsto x^p - x$ is additive, it suffice to show that $x^p - x = y$ has solutions whenever $y = y_+$ or $y = y_-$.

Let y be supported in $(-\infty, 0) \cap S_{a,b,c}$,

$$x := \sum_i \sum_{n=1}^{\infty} y_i^{1/p^n} t^{i/p^n} = \sum_i \left(\sum_{n=1}^{\infty} y_i^{1/p^n} \right) t^i$$

Since $S_{a,b,c} \cap (-\infty, 0)$ is limited, all the coefficients of x are finite sums.

x is indeed a solution for $x^p - x = y$,

$$x^p - x = \sum_i \sum_{n=1}^{\infty} y_i^{1/p^{n-1}} t^{i/p^{n-1}} - \sum_i \sum_{n=1}^{\infty} y_i^{1/p^n} t^{i/p^n} = \sum_i y_i t^i$$

We need to show that it is a twisted series:

Firstly x is supported on $S_{a,b,b+c}$. If $j = \frac{1}{a}(m - \sum_i b_i p^{-i}) \notin S_{a,b,b+c}$, then if $m < -b$, clearly $p^n m < -b$ and all the y_{jp^n} are zero; if $\sum_{i=1}^{\infty} b_i > b+c$, then $p^n j =$

$\frac{1}{a}(p^n m - \sum_{i=1}^n b_i p^{n-i} - \sum_{i=n+1}^{\infty} b_i p^{n-i})$, so whenever $\sum_{i=n+1}^{\infty} b_i \leq c$ we have that $-\sum_{i=1}^n b_i p^{n-i} < -b$ so in either case $y_{p^n j} = 0$.

Condition 3 is satisfied since the support of x is contained in $S_{a,b,b+c} \cap (-\infty, 0)$.

Let $-b \leq m \leq 0$ and $a_n = \alpha + p^{-n} \beta$ be a twisted sequence in T_{b+c} .

We need for $c_n = x_{\frac{1}{a}(m+a_n)} = \sum_{k=1}^{\infty} y_{\frac{1}{a}(m+a_n)p^k}^{1/p^k}$ to satisfy LRR.

Whenever m or α are not zero, $\frac{1}{a}(m+a_n)p^k \leq \frac{1}{a}mp^k \leq \frac{1}{a}(-b-1)$ for a fixed $k \in \mathbb{N}$, and so the numbers of addends is uniformly bounded. Moreover $v_n = \frac{1}{a}p^k(m+a_n)$ are sequences for which y_{v_n} satisfy LRR, and so using 2.12 we are done.

If $\alpha = 0 = m$, then we cannot find a bound on the number of addends. Using $x^p - x = y$, we can write:

$$c_{n+1}^p - c_n = y_{\frac{1}{a}p^{-n}\beta}$$

Let $(y_{\frac{1}{a}p^{-n}\beta})_{n \in \mathbb{N}}$ satisfies LRR wrt $d_0, \dots, d_k \in K$, then c_n satisfies LRR wrt $-d_0, d_0 - d_1, d_1 - d_2, \dots, d_{k-1} - d_k, d_k$, indeed:

$$\sum_{i=0}^{k+1} (d_{i-1} - d_i)(c_{n+i})^{p^i} = \sum_{i=-1}^k d_i (c_{n+1+i})^{p^{i+1}} - \sum_{i=0}^{k+1} d_i (c_{n+i})^{p^i} = 0$$

Let y be supported in $(0, +\infty) \cap S_{a,b,c}$,

$$x = - \sum_i \sum_{n=0}^{\infty} y_i^{p^n} t^{ip^n} = - \sum_i \left(\sum_{n=0}^{\infty} y_{i/p^n}^{p^n} \right) t^i$$

Since $i/p^k \xrightarrow{k} 0$ and $(0, +\infty) \cap S_{a,b,c}$ is well ordered, all the coefficients of x are finite sums. x is clearly supported on $S_{a,b,c}$. (since the support is positive, b doesn't play any role and whenever $i/p^n \in S_{a,b,c}$ we have $i \in S_{a,b,c}$)

Let $(a_n)_{n \in \mathbb{N}}$ be a twisted sequence in T_c , $m \geq 0$, $v_n = \frac{1}{a}(m+a_n)$.

$$x_{v_n} = - \sum_{k=0}^{\infty} y_{v_n/p^k}^{p^k}$$

Since $v_n/p^k = \frac{1}{a}(mp^{-k} + p^{-k}a_n) \leq \frac{1}{a}mp^{-k}$ there is a k' such that $y_{v_n/p^k} = 0$ for all $k > k'$ and all $n \in \mathbb{N}$ (whenever $\frac{1}{a}mp^{-k} < \min(S_{a,b,c} \cap (0, \infty))$). Since $v_n/p^k = \frac{1}{a}(mp^{-k} + p^{-k}a_n)$ is induced by a twisted sequence, using 2.12, x_{v_n} satisfies LRR. To conclude the proof, we need to verify condition 3 on x .

Unfortunately, in this setting, the support is not limited, and the problem seems to be overlooked by K. Kedlaya in [4]. We will continue the thesis, assuming that a solution to this problem exists.

□

Theorem 2.23. $K[[t]]^+$ is the set of twist recurrent series supported on $[0, +\infty)$.

Proof. Let $x = \sum_{i \in S_{a,b,c}} x_i t^i$ be an integral twist recurrent series and let $\lambda_0, \dots, \lambda_{n-1} \in K[[t]]$ be such that

$$x^n + \lambda_{n-1}x^{n-1} + \dots + \lambda_0 = 0.$$

Since $\text{supp}(x) \subseteq S_{a,b,c}$ it admits a minimum s . If $s < 0$, then clearly:

$$(x^n + \lambda_{n-1}x^{n-1} + \dots + \lambda_0)_{ns} = x_s^n \neq 0$$

and we have found a contradiction, so $s \geq 0$.

For the other inclusion, we prove, by induction on c , that any twist recurrent series supported on (by an abuse of notation) $S_{a,-1,c}$ is integral. (These are all the series we need, up to sums of elements in K)

Let x be a twisted series supported on $T_c + 1 \subseteq [0, +\infty)$; following proof of 1 of 2.22, we define:

$$y := \sum_{i=0}^k d_i^{p^{-k}} (t^{-1}x)^{p^{-k+i}}$$

We know that y is a twist recurrent series supported on $S_{p^k, p^k, c-1}$ and T_c , and

$$y^{p^k} = \sum_{i=0}^k d_i t^{-p^i} x^{p^i} \text{ gives that } (ty)^{p^k} = \sum_{i=0}^k d_i t^{p^k - p^i} x^{p^i}.$$

Now ty is supported on $S_{p^k, p^k, c-1}$ and $T_c + 1$, thus its support is non negative and so it is actually contained in $S_{p^k, -1, c-1}$. By induction on c it is integral and we notice that the above polynomial is (after multiplying by d_k^{-1}) monic in x with integral coefficients, so x is also integral.

Now we can follow the proof of the first implication of Lemma 2.21, to deduce that all twist recurrent series supported on $S_{1,-1,c}$ are $K[[t]]$ -linear combination of twist recurrent series supported on T_c , such that the coefficients are contained in (t) .

Explicitly, if z is a twist recurrent series supported in $S_{1,-1,c}$,

$$z = \sum_{j=1}^r \left(\sum_{m \geq 1} k_{m,j} t^m \right) g_j = \sum_{j=1}^r \left(\sum_{m \geq 1} k_{m,j} t^{m-1} \right) t g_j$$

therefore any twist recurrent series supported on $S_{1,-1,c}$ is integral.

Finally, since $K[[t]] \subseteq K[[t^{1/a}]]$ is an integral extension, we may again use the isomorphism $t \mapsto t^{1/a}$ to say that all twist recurrent series supported on $S_{a,-1,c}$ are integral over $K[[t^{1/a}]]$ and so over $K[[t]]$. $\square$

3 R^+ is Big Cohen-Macaulay in characteristic p

In this chapter, we shall prove the main result of the thesis. We shall prove that for any characteristic p local domain, which is an image of a Cohen-Macaulay ring, its absolute integral closure is a Big Cohen-Macaulay R -algebra. Notice that when R is a complete local ring, it is always the image of a Cohen-Macaulay ring by Cohen's Structure Theorem.

Definition 3.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and T an R -algebra. We say that T is a Big Cohen-Macaulay R -algebra if any system of parameters of R is a T -regular sequence.

Some references call these algebras "Balanced Big Cohen Macaulay". We shall stick with the name "Big Cohen Macaulay".

Notice that since $R \subseteq R^+$ is an integral extension, R^+ is an R -algebra of dimension equal to the dimension of R . Unfortunately, as observed in Proposition 1.30, R^+ is practically never Noetherian; thus, we will never say that R^+ is Cohen-Macaulay. Note, however, that when $(R, \mathfrak{m})$ is a complete Noetherian domain of characteristic p , $(R^+, \sqrt{\mathfrak{m}}R^+)$ is local and, by the main results, admits an R^+ -regular sequence of length equal to the dimension. What may fail is a result that guarantees that all maximal R^+ -regular sequences have the same length.

To prove the main Theorem, we will first give a brief introduction to some results from the theory of Matlis duality and Local Cohomology. Then we shall give a proper introduction to attached primes and secondary representations. Attached primes will later play an important role in the proof of the main theorem. Finally, following [5], we shall divide the proof of the main theorem in three different results. It will be evident that the real protagonist of our proof will be the local cohomologies of R^+ supported at the maximal ideal of R . Indeed, what we are really trying to show, is that this cohomologies are all null when the indexes of the cohomology are strictly less than the dimension of R (see Corollary 3.38). Those already familiar with local cohomology will recall that this conditions for a finitely generated module of a Noetherian ring is equivalent to the Cohen-Macaulay property.

We end the chapter with a brief discussion on what happens to R^+ when the characteristic of R is not positive.

3.1 Matlis duality and Local Cohomology

Proposition 3.2. Let $(R, \mathfrak{m})$ be a local ring and M an Artinian R -module. Then M is an Artinian $\widehat{R}$ -module (such that the new structure agrees with the natural morphism $R \longrightarrow \widehat{R}$). Furthermore if N is an Artinian R -module and $f : M \longrightarrow N$ is R -linear morphism, then f is also $\widehat{R}$ -linear with the new structures of M and N .

Proof. For any $x \in M$ the following chain of modules stabilizes:

$$\langle x \rangle_R \supseteq \mathfrak{m}\langle x \rangle_R \supseteq \mathfrak{m}^2\langle x \rangle_R \supseteq \dots$$

hence $\mathfrak{m}^{n_x}\langle x \rangle_R = \mathfrak{m}^{n_x+1}\langle x \rangle_R$ for some minimal $n_x \in \mathbb{N}$. By Nakayama, $\mathfrak{m}^{n_x}\langle x \rangle_R = 0$.

Let $x \in M$ and $a \in \widehat{R}$. By completion theory, there is an element $a' \in R$ such that $a - a' \in \mathfrak{m}^{n_x}\widehat{R}$. We define $a \cdot x := a' \cdot x$. This multiplication does

not depend on the choice of a' , indeed if $a'' \in R$ such that $a - a'' \in \mathfrak{m}^{n_x} \widehat{R}$, then $a'' - a' = (a - a') - (a - a'') \in (\mathfrak{m}^{n_x} \widehat{R}) \cap R = \mathfrak{m}^{n_x}$ and so $a'x = a''x$.

All the properties of scalar multiplication are easily verified, furthermore this scalar multiplication clearly agrees with the old one. M is Artinian as an $\widehat{R}$ -module, since any $\widehat{R}$ -submodule of M is clearly an R -module.

Let N be an Artinian R -module and $f : M \rightarrow N$ be an R -linear morphism. If $a \in \widehat{R}$, $x \in M$, $a' \in R$ such that $a - a' \in \mathfrak{m}^{n_x} \widehat{R}$ then $f(a \cdot x) = f(a'x) = a'f(x)$ and notice that $\mathfrak{m}^{n_x} f(x) = 0$ so $n_{f(x)} \leq n_x$ which means that $a - a' \in \mathfrak{m}^{n_x} \widehat{R} \subseteq \mathfrak{m}^{n_{f(x)}} \widehat{R}$. Therefore $a'f(x) = af(x)$. □

Theorem 3.3. Let R be a Cohen-Macaulay local ring and $\mathfrak{p} \in \text{Spec}(R)$. Then $\dim \widehat{R}/\mathfrak{q} = \dim R/\mathfrak{p}$ for all $\mathfrak{q} \in \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$.

Proof. See [17] Theorem 2.1.15. □

The following Corollary is a generalization of Theorem 3.3 and will be used later in the chapter.

Corollary 3.4. Let R be the homomorphic image of a Cohen-Macaulay local ring and $\mathfrak{p} \in \text{Spec}(R)$. Then $\dim \widehat{R}/\mathfrak{q} = \dim R/\mathfrak{p}$ for all $\mathfrak{q} \in \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$.

Proof. We want to use the previous theorem. Recall that when the ring is Noetherian, completion for local rings commutes with taking quotients, meaning that if $R = S/I$ then $\widehat{R} \cong \widehat{S}/\widehat{I}\widehat{S}$ (see Theorem 1.68).

Let $f : S \rightarrow S/I = R$ and $g : \widehat{S} \rightarrow \widehat{S}/\widehat{I}\widehat{S} \cong \widehat{R}$ be the respective projections. Let $\mathfrak{p} \in \text{Spec}(R)$ and $P = f^{-1}(\mathfrak{p})$, we show the following:

$$\{g^{-1}(\mathfrak{q}) : \mathfrak{q} \in \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})\} \subseteq \text{Ass}_{\widehat{S}}(\widehat{S}/P\widehat{S})$$

Notice that $\widehat{S}/P\widehat{S} \cong \widehat{S}/\widehat{P} \cong \widehat{R}/\widehat{\mathfrak{p}} \cong \widehat{R}/\mathfrak{p}\widehat{R}$ as rings.

If $\mathfrak{q} \in \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ then

$$\begin{array}{ccc} \widehat{S}/g^{-1}(\mathfrak{q}) & \hookrightarrow & \widehat{S}/P\widehat{S} \\ \simeq \downarrow g & & \simeq \uparrow g^{-1} \\ \widehat{R}/\mathfrak{q} & \hookrightarrow & \widehat{R}/\mathfrak{p}\widehat{R} \end{array}$$

If we consider the full composition of this maps, we get an injective $\widehat{S}$ -linear morphism. Hence $g^{-1}(\mathfrak{q}) \in \text{Ass}_{\widehat{S}}(\widehat{S}/P\widehat{S})$

Now we can prove the statement.

If $\mathfrak{q} \in \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ then $g^{-1}(\mathfrak{q}) \in \text{Ass}_{\widehat{S}}(\widehat{S}/P\widehat{S})$ so using the previous theorem,

$$\dim \widehat{R}/\mathfrak{q} = \dim \widehat{S}/g^{-1}(\mathfrak{q}) = \dim S/P = \dim R/\mathfrak{p}.$$

□

Definition 3.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring, let $E := E_R(R/\mathfrak{m})$ be the injective envelope of $R/\mathfrak{m}$. Since E is an injective R -module, the functor $\text{Hom}_R(-, E)$ is exact. We denote this functor with $\vee(R)$ so that:

$$M^{\vee(R)} := \text{Hom}_R(M, E) \text{ and if } f : M \longrightarrow N,$$

$$f^{\vee(R)} : N^{\vee(R)} \longrightarrow M^{\vee(R)}$$

When there is no ambiguity, we will just use $\vee$.

Proposition 3.6. Let $(R, \mathfrak{m})$ be a Noetherian local, for any R -module M , $M^\vee = 0$ if and only if $M = 0$.

As a consequence, if $f : M \longrightarrow N$ is an R -linear maps between R -modules, then $f^\vee = 0$ if and only if $f = 0$

Proof. Clearly $M = 0$ always implies that $M^\vee = 0$, so we prove the converse. Consider the natural R -module morphism,

$$\phi_M : M \longrightarrow M^{\vee\vee} \text{ such that } \phi_M(m)(\varphi) = \varphi(m) \text{ for all } m \in M \text{ and } \varphi \in M^\vee$$

We prove that ϕ_M is injective.

Let $m \in \ker \phi_M$, if $m \neq 0$, then $0 :_R m \subseteq \mathfrak{m}$, so the following is an injective map:

$$R/(0 :_R m) \xrightarrow{\bar{1} \mapsto m} M.$$

Hence the map $R/(0 :_R m) \twoheadrightarrow R/\mathfrak{m} \hookrightarrow E_R(R/\mathfrak{m})$, extend to a map

$$\varphi : M \longrightarrow E_R(R/\mathfrak{m})$$

such that $\varphi(m) \neq 0$. This contradicts the fact that $m \in \ker \phi_M$.

So now if $M^\vee$ is zero then clearly $M^{\vee\vee}$ is zero and $\phi_M : M \hookrightarrow M^{\vee\vee}$ implies that $M = 0$.

For the second statement, notice that any morphism $f : M \longrightarrow N$ can be factored through the image: $f : M \twoheadrightarrow \text{Im} f \hookrightarrow N$.

Clearly $f = 0$ if and only if $\text{Im} f = 0$. Similarly, using exactness, we get

$$f^\vee : N^\vee \twoheadrightarrow (\text{Im} f)^\vee \hookrightarrow M^\vee$$

and so $f^\vee = 0$ if and only if $(\text{Im} f)^\vee = 0$. But we know that $\text{Im} f = 0$ if and only if $(\text{Im} f)^\vee = 0$, therefore $f^\vee = 0$ if and only if $f = 0$. $\square$

Theorem 3.7 (Matlis duality). Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, M an R -module, then

- M is Noetherian if and only if $M^\vee$ is Artinian.
- M is Artinian if and only if $M^\vee$ is Noetherian.
- If M is either Artinian or Noetherian, then $\text{Ann}_R(M) = \text{Ann}_R(M^\vee)$ and

$$\phi_M : M \longrightarrow M^{\vee\vee} : m \in M \longmapsto \left(\varphi \in M^\vee \mapsto \varphi(m) \in E_R(R/\mathfrak{m}) \right)$$

is an isomorphism.

Proof. See Theorem 10.2.12 of [18]. $\square$

Definition 3.8 (Local Cohomology 1). Let R be a Noetherian ring and $I \subseteq R$ an ideal. We define the functor of R -modules $\Gamma_I : M \mapsto \bigcup_{n \in \mathbb{N}} 0 :_M I^n$, where the latter is an R -module, since it is the union of a nested chain of R -modules.

It is easy to verify that R -linear maps $f : M \rightarrow N$ restricts to $\Gamma_I(M) \rightarrow \Gamma_I(N)$ so we define $\Gamma_I(f)$ as this restriction.

Γ_I is a left exact functor from and to the category of R -modules. For any $i \in \mathbb{N}$, the i th local cohomology supported on I is the i th right derived functor of Γ_I and we denote it with H_I^i .

Explicitly, if M is an R -module, to compute $H_I^i(M)$, we need to take an injective resolution of M , say $0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \dots$, apply the functor Γ_I :

$$0 \rightarrow \Gamma_I(M) \rightarrow \Gamma_I(E_0) \rightarrow \Gamma_I(E_1) \rightarrow \Gamma_I(E_2) \rightarrow \dots$$

and finally $H_I^i(M)$ is the i th cohomology of the complex $\Gamma_I(E_0) \rightarrow \Gamma_I(E_1) \rightarrow \dots$. Since maps from R -modules $M \rightarrow N$, induce complexes morphism on the respective injective resolutions, they also induce maps on the local cohomology. It is a theorem to prove that the cohomologies and these maps do not depend on the chosen resolutions.

Another definition for local cohomology, which is easier to compute, is the following:

Definition 3.9 (Local Cohomology 2). Let R be a Noetherian ring, $I = (a_1, \dots, a_n)$ an ideal of R and M an R -module. If $\underline{a}$ is the list of all a_i , we define the associated Čech complex $C^*(\underline{a}; M)$ as: $C^0(M) = M$, $C^k(M) := \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} M_{a_{i_1} \dots a_{i_k}}$

$$0 \rightarrow M \rightarrow \bigoplus_{i=1}^n M_{a_i} \rightarrow \dots \rightarrow \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} M_{a_{i_1} \dots a_{i_k}} \rightarrow \dots \rightarrow M_{a_1 \dots a_n} \rightarrow 0$$

Let $k \in \mathbb{N}$ and $1 \leq i_1 < \dots < i_k \leq n$. Denote with loc_j the localization from $M_{a_{i_1} \dots \widehat{a_{i_j}} \dots a_{i_{k+1}}}$ to $M_{a_1 \dots a_{i_k}}$, then the map from $C^k(M)$ to $C^{k+1}(M)$ sends

$$(x_{i_1, \dots, i_k})_{i_1 < \dots < i_k} \mapsto (y_{i_1, \dots, i_{k+1}})_{i_1 < \dots < i_{k+1}}$$

where $y_{i_1, \dots, i_{k+1}} := \sum_{j=1}^{k+1} (-1)^j loc_j(x_{i_1, \dots, \widehat{i_j}, \dots, i_{k+1}})$.

We denote with $H^i(\underline{a}^\infty; M)$ the i th Cohomology of the Čech complex. It is a theorem that $H^i(\underline{a}^\infty; M) \cong H_I^i(M)$, moreover since R -linear maps induce morphism on the Čech complexes, they induce morphism on $H^i(\underline{a}^\infty; M)$. This morphisms are naturally isomorphic to those induced by the functor H_I^i . See [18] Theorem 5.1.20.

Proposition 3.10 (Base Change). Let $\phi : R \rightarrow S$ be a ring homomorphism, $I \subseteq R$ an ideal, N a S -module, then $H_I^i(N) \cong H_{IS}^i(N)$ as R -modules.

If S is a flat R -algebra and M is a R -module, then $H_I^i(M) \otimes_R S \cong H_{IS}^i(M \otimes_R S)$.

Proof. For the first statement, simply notice that if $a \in R$, $N_a = N_{\phi(a)}$, then by the second definition $H_I^i(N) = H^i(\underline{a}^\infty; N) = H^i(\phi(\underline{a})^\infty; N) = H_{IS}^i(N)$.

For the second statement, notice that cohomology commutes with exact functors, so $H_I^i(M) \otimes_R S \cong H_I^i(M \otimes_R S) \cong H_{IS}^i(M \otimes_R S)$. $\square$

Proposition 3.11. If $M = \lim_{\rightarrow \alpha} M_\alpha$ is a direct limit of R -modules, then $H_I^i(M) = \lim_{\rightarrow \alpha} H_I^i(M_\alpha)$, i.e., the local cohomology functor preserves direct limits.

Proof. See Theorem 3.4.10 from [18]. $\square$

Remark 3.12. If R is a ring of positive characteristic, then the Frobenius homomorphism $f : R \rightarrow R$ induces a natural morphism in local cohomology: for any $i \in \mathbb{N}$, $I \subseteq R$ ideal,

$$f_*^i : H_I^i(R) \rightarrow H_I^i(R)$$

Notice that the Frobenius homomorphism is not generally R -linear, so to construct f_*^i we shall use the second definition of local cohomology: since localizations commutes with the Frobenius homomorphism, there is an induced map between the Čech complexes and thus between the cohomologies.

Theorem 3.13 (Local Duality). Let $(R, \mathfrak{m})$ be an image of a local Gorenstein ring $(S, \mathfrak{n})$ of dimension n , then for any finitely generated R -module M ,

$$H_{\mathfrak{m}}^i(M) \cong \text{Ext}_S^{n-i}(M, S)^{\vee(R)} \text{ for all } i \in \mathbb{Z}$$

Moreover, if $(R, \mathfrak{m})$ is complete, then

$$H_{\mathfrak{m}}^i(M)^{\vee(R)} \cong \text{Ext}_S^{n-i}(M, S) \text{ for all } i \in \mathbb{Z}$$

Proof. See Theorem 11.2.6 from [18] and Matlis Duality. $\square$

Theorem 3.14. Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R -module, then for all $i \in \mathbb{N}$, $H_{\mathfrak{m}}^i(M)$ is an Artinian R -module.

Proof. Notice that $H_{\mathfrak{m}}^i(M)$ is an Artinian R -module if $H_{\mathfrak{m}}^i(M \otimes_R \widehat{R})$ is an Artinian $\widehat{R}$ -module. (by exactness of $\otimes_R \widehat{R}$). Therefore we can assume R to be complete. By Cohen's Theorem, we know that $R \cong S/I$ for some regular local ring $(S, \mathfrak{n})$. So in conclusion, using base change and local duality:

$$H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{n}}^i(M) \cong \text{Ext}_S^{d-i}(M, S)^{\vee}$$

and thus, by Matlis duality, it is Artinian since $\text{Ext}_S^{d-i}(M, S)$ is finitely generated. $\square$

Corollary 3.15. Let $(R, \mathfrak{m})$ be a Noetherian local ring, if M is a finitely generated R -module, then $H_{\mathfrak{m}}^i(M) \cong H_{\widehat{\mathfrak{m}}}^i(\widehat{M})$.

3.2 Attached primes and secondary representations

We shall develop secondary representation theory in order to introduce attached primes. Let R be a ring.

Definition 3.16. An R -module M is said to be secondary if, for any $a \in R$, either $aM = M$ or there exists some $n \in \mathbb{N}$ such that $a^n M = 0$.

Remark 3.17. If M is a non zero secondary R -module, then $\mathfrak{p} = \sqrt{\text{Ann}_R(M)}$ is a prime ideal and we say that M is $\mathfrak{p}$ -secondary.

Indeed, since for any $a \in R$, either $aM = M$ or $a \in \sqrt{\text{Ann}_R(M)}$, then the set $R \setminus \sqrt{\text{Ann}_R(M)}$ is a multiplicative set, and so $\sqrt{\text{Ann}_R(M)}$ is prime.

Remark 3.18. Let M be an R -module.

If $N, T \subseteq M$ are both $\mathfrak{p}$ -secondary, then $N + T$ is $\mathfrak{p}$ -secondary.

In fact, if $a \in \mathfrak{p}$ then there exist $n_1, n_2 \in \mathbb{N}$ such that $a^{n_1}N = 0$ and $a^{n_2}T = 0$, so $a^{n_1+n_2}(N + T) = 0$. Furthermore, if $a \notin \mathfrak{p}$ then $a(N + T) = aN + aT = N + T$.

Definition 3.19. We say that an R -module admits a secondary representation (or that it is secondary representable), if there exist some secondary R -submodules of M , $N_1, \dots, N_n \subseteq M$, such that $M = \sum_{i=1}^n N_i$.

If we call $\mathfrak{p}_i := \sqrt{\text{Ann}_R(N_i)}$ for all $i = 1, \dots, n$, then we say that the secondary representation is minimal if $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all $i \neq j$ and $N_t \not\subseteq \sum_{i \neq t} N_i$ for all $t = 1, \dots, n$.

Corollary 3.20. If an R -module M is secondary representable, then it admits a minimal secondary representation.

Proof. Let us write a secondary representation: $M = \sum_{i=1}^n N_i$.

From Remark 3.18, we can assume that $\mathfrak{p}_i \neq \mathfrak{p}_j$ for any i, j such that $i \neq j$. Now if for some $t = 1, \dots, n$ we have that $N_t \subseteq \sum_{i \neq t} N_i$, then we can just remove N_t from the list. Since there are a finite number of modules, we can keep removing redundant modules until we get a minimal secondary representation. $\square$

Remark 3.21. Let N be a $\mathfrak{p}$ -secondary R -module and N' be a $\mathfrak{q}$ -secondary R -module. If $N' \subseteq N$ then $\mathfrak{p} \subseteq \mathfrak{q}$. In fact, if $a \in \mathfrak{p}$, then the map $a^n N = 0$ for some n , and so $a^n N' = 0$, which means that $a \in \mathfrak{q}$.

Theorem 3.22. Let M be a non zero R -module and $M = \sum_{i=1}^n N_i = \sum_{j=1}^m N'_j$ be two minimal secondary representations.

Call $\mathfrak{p}_i := \sqrt{\text{Ann}_R(N_i)}$ and $\mathfrak{q}_j := \sqrt{\text{Ann}_R(N'_j)}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Then $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$.

Proof. Let us prove the statement by induction on $n + m$, $n, m \in \mathbb{N}_{>0}$.

Since the case $n + m = 2$ implies both n and m to be one, it is clear.

Assume the statement to be true for pairs of representation such that the sum of the number of modules in each representation is less than $n + m$.

Consider the multiplicative set $S = \{a \in R : aM = M\}$.

It is easy to see that $S = R \setminus \bigcup_{l=1}^s \tilde{\mathfrak{p}}_l$ for any primes $\{\tilde{\mathfrak{p}}_1, \dots, \tilde{\mathfrak{p}}_s\}$ from a minimal secondary representation $M = \sum_{l=1}^s \tilde{N}_l$: indeed, if $a \notin \bigcup_{l=1}^s \tilde{\mathfrak{p}}_l$ then $a\tilde{N}_l = \tilde{N}_l$ for all l , hence $aM = M$. While if $a \in S$ it cannot belong to any $\tilde{\mathfrak{p}}_{l_0}$ since if that was the case we would have $M = a^k M = \sum_{l=1}^s a^k \tilde{N}_l \subseteq \sum_{l \neq l_0} \tilde{N}_l$ for some $k \in \mathbb{N}$. This contradicts the non redundancy of the representation.

So we have proved that $R \setminus \bigcup_{i=1}^n \mathfrak{p}_i = R \setminus \bigcup_{j=1}^m \mathfrak{q}_j$ and so $\bigcup_{i=1}^n \mathfrak{p}_i = \bigcup_{j=1}^m \mathfrak{q}_j$.

Let $\mathfrak{p}_{i_0}$ be a maximal element in $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, then by prime avoidance, there exists j_0 such that $\mathfrak{p}_{i_0} \subseteq \mathfrak{q}_{j_0}$, and again by prime avoidance, $\mathfrak{p}_{i_0} \subseteq \mathfrak{q}_{j_0} \subseteq \mathfrak{p}_{i_1}$, but since $\mathfrak{p}_{i_0}$ was a maximal element, we obtain that $\mathfrak{p}_{i_0} = \mathfrak{q}_{j_0}$.

We deduce that $\text{Max}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \text{Max}\{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$.

Let us reorder the primes such that $\mathfrak{p}_i$ is maximal for all $i \geq k$ and $\mathfrak{q}_j$ is maximal for all $j \geq k'$. By prime avoidance, there exists $a \in R$ such that it belongs to all the maximal elements but not to any other primes $\mathfrak{p}_i$ or $\mathfrak{q}_j$. Then, for some sufficiently large $n \in \mathbb{N}$, we have $a^n M = \sum_{i=1}^{k-1} N_i = \sum_{i=1}^{k'-1} N'_i$ which are minimal secondary representations such that $k - 1 + k' - 1 < k + k' \leq n + m$.

By induction we conclude that $\{\mathfrak{p}_1, \dots, \mathfrak{p}_k\} = \{\mathfrak{q}_1, \dots, \mathfrak{q}_{k'}\}$, therefore $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$. $\square$

Definition 3.23. Let M be a secondary representable R -module and $M = \sum_{i=1}^n N_i$ a minimal secondary representation. Using Theorem 3.22, we define

$$\text{Att}_R(M) := \{\sqrt{\text{Ann}_R(N_i)} : i = 1, \dots, n\}.$$

Notice that $\text{Att}_R(0) = \emptyset$.

Corollary 3.24. Let M be an R -module and $M = \sum_{i=1}^n N_i = \sum_{i=1}^n N'_i$ be minimal secondary representations such that $\mathfrak{p}_i = \sqrt{\text{Ann}_R(N_i)} = \sqrt{\text{Ann}_R(N'_i)}$. Then if $\mathfrak{p}_{i_0}$ is minimal in $\text{Att}_R(M)$, then $N_i = N'_i$.

Proof. By prime avoidance, let $a \in \bigcup_{i \neq i_0} \mathfrak{p}_i \setminus \mathfrak{p}_{i_0}$, then for some sufficiently large $n \in \mathbb{N}$ we have that $a^n M = N_{i_0} = N'_{i_0}$. $\square$

Remark 3.25. If M is a secondary representable R -module and $I \subseteq M$ is a submodule, then M/I is secondary representable.

Let $M = \sum_{i=1}^n N_i$ be a minimal secondary representation such that M_i is $\mathfrak{p}_i$ -secondary for any i . We show that if $N_i \not\subseteq I$ then $\frac{N_i+I}{I}$ is $\mathfrak{p}_i$ -secondary. Indeed if $a \in \mathfrak{p}_i$ then for some $n \in \mathbb{N}$, $a^n N_i = 0$ so $a^n(\frac{N_i+I}{I}) = 0$ therefore $a \in \sqrt{\text{Ann}(\frac{N_i+I}{I})}$. If $a \notin \mathfrak{p}_i$, then $aN_i = N_i$ so $a(\frac{N_i+I}{I}) = \frac{N_i+I}{I}$, therefore it is not in $\sqrt{\text{Ann}(\frac{N_i+I}{I})}$ and we have proved what that $\frac{N_i+I}{I}$ is $\mathfrak{p}_i$ -secondary. This also proves that $\text{Att}_R(M/I) \subseteq \text{Att}_R(M)$.

We would like to have a more direct way to establish if a prime belongs to $\text{Att}_R(M)$ whenever M is secondary representable. The next two Propositions give us exactly that.

Proposition 3.26. Let M be a secondary representable R -module, then $\mathfrak{p} \in \text{Att}_R(M)$ if and only if there exists a surjective R -module homomorphism $M \twoheadrightarrow L$ such that L is $\mathfrak{p}$ -secondary.

Proof. Let us fix a minimal secondary representation $M = \sum_{i=1}^n N_i$. If $\mathfrak{p} \in \text{Att}_R(M)$ and $\mathfrak{p} = \sqrt{\text{Ann}_R(N_{i_0})}$, then let us call $N := \sum_{i \neq i_0} N_i$ so that $M = N_{i_0} + N$. We show that M/N is $\mathfrak{p}$ -secondary; in fact, if $a \in \mathfrak{p}$, then $a^n N_{i_0} = 0$ for some $n \in \mathbb{N}$, thus $a^n M \subseteq a^n N_{i_0} + N = N$, and so $a^n(M/N) = 0$. If $a \notin \mathfrak{p}$, then $aN_{i_0} = N_{i_0}$, and so $a(M/N) = \frac{N_{i_0} + aN + N}{N} = M/N$. Note that $M/N \neq 0$ by non redundancy, and so $\mathfrak{p} = \sqrt{\text{Ann}(M/N)}$.

Let $M \twoheadrightarrow M/N$ be an R -module epimorphism such that M/N is $\mathfrak{p}$ -secondary (therefore not zero). Let us reorder the indexes such that $\mathfrak{p}_j \subseteq \mathfrak{p}$ if and only if $j \leq s$. If $a \notin \mathfrak{p}$, then $a(M/N) = M/N$, and so $aM + N = M$. We can choose $a \in R$ such that $a \in \mathfrak{p}_j \setminus \mathfrak{p}$ for all $j > s$, so for some $k \in \mathbb{N}$,

$$a^k M + N = (N_1 + \dots + N_s) + N = M.$$

If $b \in \mathfrak{p}$ then, for some $n \in \mathbb{N}$, $b^n(M/N) = 0$, and so $b^n M \subseteq N$. Now if $\mathfrak{p}_j \subsetneq \mathfrak{p}$ for all $j \leq s$, then we can choose $b \in \mathfrak{p} \setminus \bigcup_{j \leq s} \mathfrak{p}_j$, so that $b^n M \subseteq N$ implies that $b^n(N_1 + \dots + N_s + N) = N_1 + \dots + N_s + b^n N \subseteq N$. So we have that $M = N_1 + \dots + N_s + N \subseteq N$ and so $M = N$ which is absurd. Thus for some $j \leq s$, we must have the equality $\mathfrak{p}_j = \mathfrak{p}$. $\square$

Corollary 3.27. Let R be a Noetherian and M a secondary representable R -module, $\mathfrak{p} \in \text{Spec}(R)$, then the following statements are equivalent:

- $\mathfrak{p} \in \text{Att}_R(M)$
- there exists an epimorphism $M \twoheadrightarrow L$, such that $\text{Ann}_R(L) = \mathfrak{p}$

Proof. Let $\mathfrak{p} = (a_1, \dots, a_n)$.

If $\mathfrak{p} \in \text{Att}_R(M)$, then by Proposition 3.26, $M \twoheadrightarrow L$ for some $\mathfrak{p}$ -secondary R -module L . Consider:

$$M \twoheadrightarrow L \twoheadrightarrow L/\mathfrak{p}L.$$

We want to show that $\text{Ann}(L/\mathfrak{p}L) = \mathfrak{p}$, clearly $\mathfrak{p} \subseteq \text{Ann}(L/\mathfrak{p}L)$.

If $a \in \text{Ann}(L/\mathfrak{p}L) \setminus \mathfrak{p}$, then, since L is $\mathfrak{p}$ -secondary, $aL = L$ and $aL \subseteq \mathfrak{p}L$ so $L = \mathfrak{p}L$. For any $i = 1, \dots, n$, there exists a natural number m_i such that $a_i^{m_i}L = 0$. Let m be the sum of all the m_i , then $L = \mathfrak{p}^m L = 0$, which is absurd.

If $M \twoheadrightarrow L$, such that $\text{Ann}_R(L) = \mathfrak{p}$. Recall that L is secondary representable and $\text{Att}_R(L) \subseteq \text{Att}_R(M)$. Let $\text{Att}_R(L) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, we show that $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{p}_i$. If $a \in \mathfrak{p} \setminus \bigcap \text{Att}_R(L)$, then $a \notin \mathfrak{p}_i$ for some $\mathfrak{p}_i \in \text{Att}_R(L)$; thus $0 = aL \supseteq L_i \neq 0$, where L_i is a $\mathfrak{p}_i$ -secondary component of L . which is absurd. If $a \in \bigcap \text{Att}_R(L)$, then $a^n \in \mathfrak{p}$ for some sufficiently large n , and since $\mathfrak{p}$ is prime, $a \in \mathfrak{p}$. Finally, we have $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{p}_i$. Since they are all prime ideals, one of the $\mathfrak{p}_i$ must be equal to $\mathfrak{p}$ (for example, we can deduce this from primary decomposition or prime avoidance). $\square$

Lemma 3.28. Let M be an Artinian R -module, if M is not secondary, then there exist some proper submodules $N_1, N_2 \subsetneq M$ such that $M = N_1 + N_2$.

Proof. Since M is not secondary, there exists some $r \in R$ such that $rM \subsetneq M$ and $r^k M \neq 0$ for all $k \in \mathbb{N}$. Consider the following descending inclusions

$$M \supseteq rM \supseteq r^2M \supseteq r^3M \supseteq \dots,$$

since M is Artinian, there is a $t \in \mathbb{N}$ such that $r^t M = r^{t+1} M$.

Recall that $r^t M \neq 0$, so $0 :_M r^t = \{x \in M : r^t x = 0\}$ is strictly contained in M .

We want to show that $M = rM + (0 :_M r^t)$. If $m \in M$, then there exists $a \in R$ such that $r^t m = ar^{t+1} m$, so $r^t(m - arm) = 0$.

This means that $m = ram + (m - ram) \in rM + (0 :_M r^t)$. $\square$

Theorem 3.29. All Artinian R -modules admit a secondary representation.

Proof. Let M be an Artinian R -module. By contradiction assume that M is not secondary representable. Let us consider the non empty set

$$\{N \subseteq M : N \text{ is not secondary representable}\}.$$

By Artinianity, this set contains a minimal element N_0 which is Artinian and not secondary, thus by the previous Lemma it admits a proper decomposition $N_0 = N_{0,0} + N_{0,1}$. By minimality we know that $N_{0,0}$ and $N_{0,1}$ both admits a secondary representation, and so N_0 is secondary representable which is absurd. Thus M must be secondary representable. $\square$

Lemma 3.30. Let $f : R \rightarrow S$ be a ring homomorphism and M a secondary representable S -module, then it is secondary representable as an R -module and $\text{Att}_R M = \{f^{-1}(P) : P \in \text{Att}_S M\}$.

Proof. Let $M = N_1 + \dots + N_n$ be a minimal secondary representation of M as an S -module and let $P_i := \sqrt{\text{Ann}_S N_i}$.

For all $i = 1, \dots, n$, N_i is an R -module and

$$\mathfrak{p}_i := f^{-1}(P_i) = f^{-1}(\sqrt{\text{Ann}_S N_i}) = \sqrt{f^{-1}(\text{Ann}_S N_i)} = \sqrt{\text{Ann}_R N_i} \in \text{Spec}(R).$$

Moreover, if $a \in R \setminus \mathfrak{p}_i$ then $f(a) \in S \setminus P_i$, and so $a \cdot N_i = f(a)N_i = N_i$; therefore N_i is $\mathfrak{p}_i$ -secondary. None of the N_i are redundant, hence $\text{Att}_R M = \{\mathfrak{p}_i : i = 1, \dots, n\}$. $\square$

From now on let $(R, \mathfrak{m})$ be a local Noetherian ring.

Lemma 3.31. Let R be a complete local Noetherian ring and let A be an Artinian R -module, then $\text{Att}_R(A) = \text{Ass}_R(A^\vee)$

Proof. Note that $\mathfrak{p} \in \text{Ass}_R(A^\vee)$ if and only if one has an inclusion $R/\mathfrak{p} \hookrightarrow A^\vee$, if and only if there is a surjection $A \twoheadrightarrow (R/\mathfrak{p})^\vee$. By Matlis duality, $\text{Ann}((R/\mathfrak{p})^\vee) = \text{Ann}(R/\mathfrak{p}) = \mathfrak{p}$. This implies, by Corollary 3.27, that $\mathfrak{p} \in \text{Att}_R(A)$.

For the converse, if $\mathfrak{p} \in \text{Att}_R(A)$, again using Corollary 3.27, $A \twoheadrightarrow L$ for some L such that $\text{Ann}(L) = \mathfrak{p}$, so after dualizing $L^\vee \hookrightarrow A^\vee$.

By Matlis Duality $\text{Ann}(L^\vee) = \mathfrak{p} \in \text{Spec}(R)$. Since $A^\vee$ is Noetherian, $L^\vee$ is also Noetherian and $\text{Min}(\text{Ann}(L^\vee)) \subseteq \text{Ass}_R(L^\vee) \subseteq \text{Ass}_R(A^\vee)$, so $\mathfrak{p} \in \text{Ass}_R(A^\vee)$. $\square$

Corollary 3.32. Let A be an Artinian R -module, $\text{Att}_R A = \{P \cap R : P \in \text{Att}_{\widehat{R}} A\}$

Proof. By Proposition 3.2, A is an Artinian $\widehat{R}$ -module, so by Lemma 3.30 we are done. $\square$

Lemma 3.33. Let A be an Artinian R -module,

1. $A \neq 0 \iff \text{Att}_R A \neq \emptyset$
2. $A \neq 0$ has finite length $\iff \text{Att}_R A = \{\mathfrak{m}\}$
3. If $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is an exact sequence of Artinian R -modules then

$$\text{Att}_R A'' \subseteq \text{Att}_R A \subseteq \text{Att}_R A' \cup \text{Att}_R A''$$

4. $\min \text{Att}_R A = \min V(\text{Ann}_R A)$. In particular,

$$\dim(R/\text{Ann}_R A) = \max\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in \text{Att}_R A\}$$

Proof.

1. If $A = 0$ then clearly $\text{Att}_R A = \emptyset$. If $\text{Att}_R(A) = \emptyset$, then, since for non zero Artinian modules a minimal representation exists, A must be zero.
2. If A has finite length, then $\mathfrak{m}^n A = 0$ for some $n \in \mathbb{N}$ (see appendix 5.15). So $\mathfrak{m} = \sqrt{\text{Ann}_R(A)} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \text{Att}_R(M)$, hence $\text{Att}_R(A) \subseteq \{\mathfrak{m}\}$. If $\text{Att}_R(A) = \{\mathfrak{m}\}$, then A is $\mathfrak{m}$ -secondary, therefore (since $\mathfrak{m}$ is finitely generated) there is some $n \in \mathbb{N}$ such that $\mathfrak{m}^n A = 0$. Hence, A has finite length (again, see appendix 5.15).

3. Since all the modules are Artinian, we can think of them as $\widehat{R}$ -modules by 3.2. From the exact sequence we deduce:

$$0 \longrightarrow A''^\vee \longrightarrow A^\vee \longrightarrow A'^\vee \longrightarrow 0.$$

So we have the following inclusions:

$$\text{Ass}_{\widehat{R}}(A''^\vee) \subseteq \text{Ass}_{\widehat{R}}(A^\vee) \subseteq \text{Ass}_{\widehat{R}}(A'^\vee) \cup \text{Ass}_{\widehat{R}}(A''^\vee)$$

Using 3.33 we get

$$\text{Att}_{\widehat{R}}(A'') \subseteq \text{Att}_{\widehat{R}}(A) \subseteq \text{Att}_{\widehat{R}}(A') \cup \text{Att}_{\widehat{R}}(A'')$$

If we call $f : R \longrightarrow \widehat{R}$, the completion map, by an abuse of notation we have, using 3.32, $\text{Att}_R(A) = f^{-1}(\text{Att}_{\widehat{R}}(A))$ so

$$f^{-1}(\text{Att}_{\widehat{R}}(A'')) \subseteq f^{-1}(\text{Att}_{\widehat{R}}(A)) \subseteq f^{-1}(\text{Att}_{\widehat{R}}(A')) \cup f^{-1}(\text{Att}_{\widehat{R}}(A'')),$$

and thus $\text{Att}_R A'' \subseteq \text{Att}_R A \subseteq \text{Att}_R A' \cup \text{Att}_R A''$.

4. We need only to show that $\min \text{Att}_R A = \min(\text{Ann}_R A)$. It is sufficient to show that $\bigcap \text{Att}_R(A) = \sqrt{\text{Ann}_R(A)}$, since removing the redundant primes gives the primary decomposition. If $a \in \bigcap \text{Att}_R(A)$, then if $A = \sum_{i=1}^n N_i$ is a minimal secondary representation, we know that there is a natural number $k \in \mathbb{N}$ such that $a^k N_i = 0$ for all indexes, so $a \in \sqrt{\text{Ann}_R(A)}$. If $a \notin \bigcap \text{Att}_R(A)$, then it cannot be a nilpotent of A since there is a component of the minimal secondary representation that never annihilate.

□

Theorem 3.34. The following statements are equivalent:

1. R is an image of a Cohen-Macaulay local ring
2. For every finitely generated R -module M , $i \in \mathbb{N}$ and $\mathfrak{p} \in \text{Spec}(R)$,

$$\text{Att}_{R_{\mathfrak{p}}}(H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-\dim(R/\mathfrak{p})}(M_{\mathfrak{p}})) = \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \text{Att}_R(H_m^i(M)), \mathfrak{q} \subseteq \mathfrak{p}\}$$

3. For every finitely generated R -module M and $i \in \mathbb{N}$,

$$\text{Att}_{\widehat{R}}(H_m^i(M)) = \bigcup_{\mathfrak{p} \in \text{Att}_R(H_m^i(M))} \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$$

And if $P \in \text{Att}_{\widehat{R}}(H_m^i(M))$, $P \in \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ whenever $\mathfrak{p} = R \cap P$

Proof. See [19].

□

3.3 The main theorem

Lemma 3.35. Let R be a commutative Noetherian domain containing a field of characteristic $p > 0$, let K be the algebraic closure of $\text{frac}(R)$. Let $I \subseteq R$ be an ideal, M a finitely generated R -submodule of $H_I^i(R)$ and $\alpha \in H_I^i(R)$ such that for all $n \in \mathbb{N}$, $(f_*)^n(\alpha) \in M$ (where f_* is the Frobenius induced map in local cohomology, see 3.12). Then there exists an R -algebra R' inside K such that it is finite as an R -module and the natural map $H_I^i(R) \rightarrow H_I^i(R')$ induced by the inclusion sends α to zero.

Proof. We will use $m^p := f_*(m)$ for the Frobenius induced morphism f_* .

For all $t \in \mathbb{N}$ let $A_t := \langle \alpha, \alpha^p, \dots, \alpha^{p^t} \rangle_R \subseteq H_I^i(R)$. Clearly $A_1 \subseteq \dots \subseteq A_t \subseteq \dots \subseteq M$; since M is a Noetherian module, the ascending chains stabilize, hence there is an index $s \in \mathbb{N}$ such that $A_s = A_{s-1}$. Therefore there exist some coefficients $r_i \in R$ such that $\alpha^{p^s} = r_1 \alpha^{p^{s-1}} + r_2 \alpha^{p^{s-2}} + \dots + r_{s-1} \alpha$.

Let $g \in R[T]$, be defined as $g(T) = T^{p^s} - r_1 T^{p^{s-1}} - \dots - r_{s-1} T$; clearly $g(\alpha) = 0$. Let $I = (x_1, \dots, x_n)$, recall that if M is an R -module, the local cohomology can be computed via the cohomology of the Čech complex $C^*(x_1, \dots, x_n; M)$ of M .

In our case $M = R$ and $C^i(R) = \bigoplus_{1 \leq j_1 < \dots < j_i \leq n} R_{x_{j_1} \dots x_{j_i}}$

$$0 \longrightarrow C^0(R) \xrightarrow{d_0} \dots \xrightarrow{d_{i-2}} C^{i-1}(R) \xrightarrow{d_{i-1}} C^i(R) \xrightarrow{d_i} C^{i+1}(R) \xrightarrow{d_{i+1}} \dots \xrightarrow{d_{n-1}} C^d(R) \longrightarrow 0$$

So $\alpha \in Z^i(C^*)/B^i(C^*)$, let $\tilde{\alpha} \in Z^i(C^*) \subseteq C^i(R)$ be a cycle that represents α .

Now the equation $g(\alpha) = 0$ means that there exists an element $\beta \in C^{i-1}(R)$ such that $g(\tilde{\alpha}) = d_{i-1}(\beta)$.

Let

$$\beta = \left(\frac{r_{j_1, \dots, j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} \right)_{1 \leq j_1 < \dots < j_{i-1} \leq n} \in C^{i-1}(R)$$

Where $e \in \mathbb{N}$ does not depend on any indexes.

Consider, for any $1 \leq j_1 < \dots < j_{i-1} \leq n$, the following equations:

$$g\left(\frac{Z_{j_1, \dots, j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e}\right) - \frac{r_{j_1, \dots, j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} = 0$$

Where $Z_{j_1, \dots, j_{i-1}}$ are free variables. Multiplying them by $(x_{j_1} \dots x_{j_{i-1}})^{ep^s}$, we get monic polynomials in $R[Z_{j_1, \dots, j_{i-1}}]$. Let us call with $z_{j_1, \dots, j_{i-1}} \in K$ a solution related to the respective polynomial.

We define the R -module $R'' := R[z_{j_1, \dots, j_{i-1}} : 1 \leq j_1 < \dots < j_{i-1} \leq n]$. Let

$$\tilde{\alpha} := \left(\frac{z_{j_1, \dots, j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} \right)_{1 \leq j_1 < \dots < j_{i-1} \leq n} \in C^{i-1}(R'')$$

The natural inclusion $R \rightarrow R''$, clearly localizes for each $R_{x_{j_1} \dots x_{j_{i-1}}} \rightarrow R''_{x_{j_1} \dots x_{j_{i-1}}}$.

This maps induce a natural inclusion of complexes $C^*(R) \hookrightarrow C^*(R'')$; so we can identify $\tilde{\alpha}$ with its image inside $C^i(R'')$ and β with its image in $C^{i-1}(R'')$.

Let us define $\bar{\alpha} := \tilde{\alpha} - d_{i-1}(\tilde{\alpha})$, clearly in cohomology, $\bar{\alpha}$ is equivalent to $\tilde{\alpha}$ which is equivalent to the image of α via the morphism $H_I^i(R) \rightarrow H_I^i(R'')$. Then we have:

$$g(\bar{\alpha}) = g(\tilde{\alpha}) - g(d_{i-1}(\tilde{\alpha})) = d_{i-1}(\beta) - d_{i-1}(g(\tilde{\alpha})) = 0$$

Let $\bar{\alpha} = (\rho_{j_1, \dots, j_i})_{1 \leq j_1 \leq \dots \leq j_i \leq n} \in C^i(R'')$. We have just showed that $g(\rho_{j_1, \dots, j_i}) = 0$ for all $1 \leq j_1 \leq \dots \leq j_i \leq n$, so let us call R' the ring obtained by adjoining all the $\rho_{j_1, \dots, j_i}$ to R'' .

Now the image of α in $H_I^i(R')$ is represented by the cycle $\bar{\alpha} = (\rho_{j_1, \dots, j_i}) \in C^i(R')$ which has all its components in R' .

For any $j_1, \dots, j_i$; we clearly have $R' \subseteq R'_{x_{j_1} \dots x_{j_i}}$.

This inclusion can be extended for all i to $R'^{\oplus k_i} \subseteq C^i(R')$ (where k_i is the number of summand of $C^i(R')$). Recall that the maps d_i are all sums and differences of localization maps, thus d_i can be restricted to $R'^{\oplus k_i} \rightarrow R'^{\oplus k_{i+1}}$.

Taking this restrictions, we get a subcomplex of $C^*(R')$, which is just the Čech complex where all the localizations are replaced with the identity. Thus the cohomology of this subcomplex are exactly the local cohomology of R' , with respect to the unit ideal $(1, \dots, 1)$; hence they are null.

What we have proved is that this subcomplex is exact, and since $\bar{\alpha}$ belongs to this subcomplex and it is a cycle, it is in the image of d_{i-1} and so α in $H_I^i(R')$ must be zero. $\square$

Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring that is an image of a Cohen-Macaulay local ring.

Proposition 3.36. Let M and N be finitely generated R -modules and $\varphi : M \rightarrow N$ a morphism. For each $i \in \mathbb{N}$, φ induces the morphism $\varphi^i : H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{m}}^i(N)$. Fix $i \in \mathbb{N}$, suppose that for all $\mathfrak{p} \in \text{Att}_R(H_{\mathfrak{m}}^i(M))$ and $\mathfrak{p} \neq \mathfrak{m}$, the map $\varphi_{\mathfrak{p}}$ induces the zero map

$$\varphi_{\mathfrak{p}}^{i-t_{\mathfrak{p}}} : H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(M_{\mathfrak{p}}) \rightarrow H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(N_{\mathfrak{p}})$$

where $t_{\mathfrak{p}} = \dim(R/\mathfrak{p})$. Then $\text{Im}(\varphi^i)$ has finite length.

Proof. Let us assume by contradiction that $\text{Im}(\varphi^i)$ does not have finite length. Notice that by Theorem 3.14, $H_{\mathfrak{m}}^i(N)$ is an Artinian R -module, so by 2 of Lemma 3.33, $\text{Att}_R(\text{Im}(\varphi^i)) \neq \{\mathfrak{m}\}$. Let $\mathfrak{p} \in \text{Att}_R(\text{Im}(\varphi^i))$, $\mathfrak{p} \neq \mathfrak{m}$. Using 3 of Lemma 3.33, we know that $\text{Att}_R(\text{Im}(\varphi^i)) \subseteq \text{Att}_R(H_{\mathfrak{m}}^i(M))$, so $\mathfrak{p} \in \text{Att}_R(H_{\mathfrak{m}}^i(M))$.

From Proposition 3.2, we know that $\varphi^i : H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{m}}^i(N)$ is actually an $\widehat{R}$ -modules morphism. Moreover $H_{\mathfrak{m}}^i(N)$ is still Artinian as an $\widehat{R}$ -module and so $\text{Im}(\varphi^i)$ is an Artinian $\widehat{R}$ -module.

From Corollary 3.32, we know that $\text{Att}_R \text{Im}(\varphi^i) = \{P \cap R : P \in \text{Att}_{\widehat{R}} \text{Im}(\varphi^i)\}$. So there exists some $P \in \text{Att}_{\widehat{R}}(\text{Im} \varphi^i) \subseteq \text{Att}_{\widehat{R}}(H_{\mathfrak{m}}^i(M))$ such that $P \cap R = \mathfrak{p}$.

From 3 of theorem 3.34, we know that $P \in \text{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$. Therefore, using Corollary 3.4, $\dim(\widehat{R}/P) = \dim(R/\mathfrak{p}) = t_{\mathfrak{p}}$.

Now by Cohen Structure Theorem, $\widehat{R}$ is an image of a Gorenstein ring $S = K[[x_1, \dots, x_n]]$ of dimension n , so let $f : S \twoheadrightarrow \widehat{R}$. Applying Local duality (Theorem 3.13), and Corollary 3.15, we get:

$$\text{Ext}_S^{n-i}(\widehat{M}, S)^{\vee(\widehat{R})} \cong H_{\mathfrak{m}}^i(\widehat{M}) \cong H_{\mathfrak{m}}^i(M)$$

Since $\widehat{R}$ is complete, we also have:

$$\text{Ext}_S^{n-i}(\widehat{M}, S) \cong H_{\mathfrak{m}}^i(M)^{\vee(\widehat{R})}$$

Let $\mathcal{I} = \text{Im}(\varphi^i)$, then:

$$\begin{aligned}\varphi^i : H_{\mathfrak{m}}^i(M) &\rightarrow \mathcal{I} \hookrightarrow H_{\mathfrak{m}}^i(N) \\ (\varphi^i)^{\vee(\widehat{R})} : H_{\mathfrak{m}}^i(N)^{\vee(\widehat{R})} &\rightarrow \mathcal{I}^{\vee(\widehat{R})} \hookrightarrow H_{\mathfrak{m}}^i(M)^{\vee(\widehat{R})}\end{aligned}$$

by local duality:

$$(\varphi^i)^{\vee(\widehat{R})} : \text{Ext}_S^{n-i}(\widehat{N}, S) \rightarrow \mathcal{I}^{\vee(\widehat{R})} \hookrightarrow \text{Ext}_S^{n-i}(\widehat{M}, S)$$

Since $\mathcal{I}$ is an Artinian $\widehat{R}$ -module, $\mathcal{I}^{\vee(\widehat{R})}$ is Noetherian by Matlis Duality (Theorem 3.7), thus it is finitely generated.

Thanks to Lemma 3.31, $P \in \text{Att}_{\widehat{R}}(\mathcal{I}) = \text{Ass}_{\widehat{R}}(\mathcal{I}^{\vee(\widehat{R})})$. Let $P' = f^{-1}(P)$, taking the localization by P' :

$$((\varphi^i)^{\vee(\widehat{R})})_{P'} : \text{Ext}_{S_{P'}}^{n-i}(\widehat{N}_P, S_{P'}) \rightarrow (\mathcal{I}^{\vee(\widehat{R})})_P \hookrightarrow \text{Ext}_{S_{P'}}^{n-i}(\widehat{M}_P, S_{P'}).$$

Since $P \in \text{Ass}_{\widehat{R}}(\mathcal{I}^{\vee(\widehat{R})}) \subseteq \text{supp}(\mathcal{I}^{\vee(\widehat{R})})$, we know that $(\mathcal{I}^{\vee(\widehat{R})})_P \neq 0$ and so $((\varphi^i)^{\vee(\widehat{R})})_P \neq 0$. Since $S/P' \cong \widehat{R}/P$, we have that $t_{\mathfrak{p}} = \dim(\widehat{R}/P) = \dim(S/P')$ so

$$d = \dim S_{P'} = \dim(S) - \dim(S/P') = n - t_{\mathfrak{p}} \text{ (since } S \text{ is Cohen-Macaulay, see 1.56).}$$

$$\text{Hence } n - i = (n - t_{\mathfrak{p}}) - (i - t_{\mathfrak{p}}) = \dim(S_{P'}) - (i - t_{\mathfrak{p}}).$$

By applying Matlis duality with respect to $S_{P'}$, we get $L_i := (((\varphi^i)^{\vee(\widehat{R})})_{P'})^{\vee(S_{P'})}$ a map as follows:

$$L_i : \text{Ext}_{S_{P'}}^{d-(i-t_{\mathfrak{p}})}(\widehat{M}_P, S_{P'})^{\vee(S_{P'})} \rightarrow \text{Ext}_{S_{P'}}^{d-(i-t_{\mathfrak{p}})}(\widehat{N}_P, S_{P'})^{\vee(S_{P'})}$$

By Local duality we have

$$L_i : H_{P'S_{P'}}^{i-t_{\mathfrak{p}}}(\widehat{M}_P) \rightarrow H_{P'S_{P'}}^{i-t_{\mathfrak{p}}}(\widehat{N}_P)$$

and finally with a base change, we get:

$$L_i : H_{P\widehat{R}_P}^{i-t_{\mathfrak{p}}}(\widehat{M}_P) \rightarrow H_{P\widehat{R}_P}^{i-t_{\mathfrak{p}}}(\widehat{N}_P)$$

Since $(\mathcal{I}^{\vee(\widehat{R})})_P \neq 0$ is the image of $((\widehat{\varphi}^i)^{\vee(\widehat{R})})_{P'}$, by Proposition 3.6, $L_i \neq 0$.

Since all the isomorphism that we have used to get to this formula are natural, (i.e. isomorphism of functors) we have that

$$L_i = H_{P\widehat{R}_P}^{i-t_{\mathfrak{p}}}(\varphi \otimes_R \widehat{R}_P) = H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(\varphi_{\mathfrak{p}}) \otimes_R \widehat{R}_P = \varphi_{\mathfrak{p}}^{i-t_{\mathfrak{p}}} \otimes_R \widehat{R}_P ,$$

where in the second equality we have used base change since the composition of flat homomorphism is flat, and that $\sqrt{\mathfrak{p}\widehat{R}_P} = P\widehat{R}_P$.

Since for our hypothesis $\varphi_{\mathfrak{p}}^{i-t_{\mathfrak{p}}} = 0$, we have found a contradiction. $\square$

Theorem 3.37. Let $(R, \mathfrak{m})$ be a Noetherian local domain of positive characteristic p , that is an image of a Cohen-Macaulay local ring. Let K be an algebraic closure of $\text{frac}(R)$. Let R' be an R -algebra inside K that is a finite R -module. Then there is an R' -algebra R'' inside K that is finite as an R -module such that the natural map $H_{\mathfrak{m}}^i(R') \rightarrow H_{\mathfrak{m}}^i(R'')$ is zero for all $i < \dim R$.

Proof. We proceed by induction on the dimension $d = \dim(R)$.

If $d = 0$, then there are no indexes less than d , and we are done.

Assume that the theorem holds whenever the dimension of the ring is less than d .

Let R' be an R -algebra inside K that is a finite R -module. For any $i < d$, if $\mathfrak{p} \in \text{Att}_R(H_{\mathfrak{m}}^i(R')) \setminus \{\mathfrak{m}\}$, then the pair of rings $(R_{\mathfrak{p}}, R'_{\mathfrak{p}})$ satisfies all the hypotheses of the theorem for a dimension strictly less than d :

Indeed, if S is Cohen-Macaulay and $\pi : S \twoheadrightarrow R$, then $S_{\pi^{-1}(\mathfrak{p})}$ is Cohen-Macaulay and $S_{\pi^{-1}(\mathfrak{p})} \twoheadrightarrow R_{\mathfrak{p}}$. Furthermore, $R_{\mathfrak{p}}$ is a local Noetherian domain of characteristic p , of dimension $ht(\mathfrak{p}) < ht(\mathfrak{m}) = d$.

Notice that R and $R_{\mathfrak{p}}$ share the same quotient field, and $R_{\mathfrak{p}} \subseteq R'_{\mathfrak{p}}$ is still finite. So, by induction hypothesis, there exists an $R'_{\mathfrak{p}}$ -algebra $\tilde{R}^{i,\mathfrak{p}} \subseteq K$, that is finite as an $R'_{\mathfrak{p}}$ -module, such that

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^j(R'_{\mathfrak{p}}) \longrightarrow H_{\mathfrak{p}R_{\mathfrak{p}}}^j(\tilde{R}^{i,\mathfrak{p}})$$

is zero for all $j < \dim(R_{\mathfrak{p}}) = ht(\mathfrak{p})$.

Notice that if $t_{\mathfrak{p}} := \dim(R/\mathfrak{p})$, then $i - t_{\mathfrak{p}} < d - t_{\mathfrak{p}} = ht(\mathfrak{p})$, where the last equality follows from Corollary 1.57. Therefore, for any $i < d$, we know that

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(R'_{\mathfrak{p}}) \longrightarrow H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(\tilde{R}^{i,\mathfrak{p}})$$

is zero.

Let $\tilde{R}^{i,\mathfrak{p}} = R'_{\mathfrak{p}}[z_1, \dots, z_k]$, where all the z_i are integral over $R'_{\mathfrak{p}}$. Multiplying, if necessary, by elements of $R \setminus \mathfrak{p}$, we can assume that all the z_i are integral over R (see 1.17). Hence, we can define the finitely generated R -module $R^{i,\mathfrak{p}} = R'[z_1, \dots, z_k]$. Finally, we define

$$R^* = R[R^{i,\mathfrak{p}} : i < d, \mathfrak{p} \in \text{Att}_R(H_{\mathfrak{m}}^i(R)) \setminus \{\mathfrak{m}\}]$$

This is also a finitely generated R -module inside K , but more importantly, it is an $R^{i,\mathfrak{p}}$ algebra for all $i < d$ and $\mathfrak{p} \in \text{Att}_R(H_{\mathfrak{m}}^i(R)) \setminus \{\mathfrak{m}\}$.

The inclusions $R' \longrightarrow R^{i,\mathfrak{p}} \longrightarrow R^*$ induce the natural maps:

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(R'_{\mathfrak{p}}) \longrightarrow H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(\tilde{R}^{i,\mathfrak{p}}) \longrightarrow H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-t_{\mathfrak{p}}}(R_{\mathfrak{p}}^*)$$

Since the first map is zero, the composition is also zero. Thus, by Proposition 3.36, the natural map

$$\varphi^i : H_{\mathfrak{m}}^i(R') \longrightarrow H_{\mathfrak{m}}^i(R^*)$$

induced by the inclusion, has an image of finite length for all $i < d$.

Since the Frobenius homomorphism commutes with inclusions, we get in local cohomology the following diagram:

$$\begin{array}{ccc} H_{\mathfrak{m}}^i(R') & \xrightarrow{\varphi^i} & H_{\mathfrak{m}}^i(R^*) \\ \downarrow f_*^i & & \downarrow f_*^i \\ H_{\mathfrak{m}}^i(R') & \xrightarrow{\varphi^i} & H_{\mathfrak{m}}^i(R^*) \end{array}$$

therefore $f_*^i(\text{Im}(\varphi^i)) \subseteq \text{Im}(\varphi^i)$. Since $\text{Im}(\varphi^i)$ has finite length it is finitely generated and so we can use Lemma 3.35 (with respect to R^*) to deduce that there exists an R^* algebra R'' inside K , such that it is finite as an R -module and the natural map

$H_{\mathfrak{m}}^i(R^*) \longrightarrow H_{\mathfrak{m}}^i(R'')$ is zero (the lemma tells us how to annihilate an element, but since $Im(\varphi^i)$ is finitely generated, we can construct R'' by adjoining all the rings that in cohomology annihilate the generators).

Therefore, the natural map $H_{\mathfrak{m}}^i(R) \longrightarrow H_{\mathfrak{m}}^i(R'')$ is also zero, and we are done. $\square$

Corollary 3.38. Let $(R, \mathfrak{m})$ be a Noetherian local domain of positive characteristic p , that is an image of a Cohen-Macaulay local ring. Then the following hold:

1. $H_{\mathfrak{m}}^i(R^+) = 0$ for all $i < \dim(R)$
2. R^+ is a big Cohen-Macaulay R -algebra

Proof. For 1, recall from Remark 1.24, that R^+ is the direct limit of all the finite R -algebras extensions of R inside $K = \overline{frac(R)}$. Hence, by Proposition 3.11, $H_{\mathfrak{m}}^i(R^+) = \lim_{R \subseteq R'} H_{\mathfrak{m}}^i(R')$. Theorem 3.37, tells us that whenever $i < \dim(R)$, the maps we are taking the limits of are eventually zero. So the direct limit is zero and $H_{\mathfrak{m}}^i(R^+) = 0$.

Now to prove 2. Let $x_1, \dots, x_d$ be a system of parameters of R , $d = \dim(R)$. We prove by induction on $j \leq d$ that $x_1, \dots, x_j$ is a regular sequence on R^+ . Since R^+ is a domain, the case $j = 1$ is easily verified.

Assume that $j > 1$ and $x_1, \dots, x_{j-1}$ is a regular sequence in R^+ .

Let $I_t = (x_1, \dots, x_t) \subseteq R$, then for $t \leq j - 1$ we have the short exact sequences:

$$0 \longrightarrow R^+/I_{t-1}R^+ \xrightarrow{\cdot x_t} R^+/I_{t-1}R^+ \longrightarrow R^+/I_tR^+ \longrightarrow 0$$

which induce the long exact sequences:

$$\begin{aligned} 0 \longrightarrow H_{\mathfrak{m}}^0(R^+/I_{t-1}R^+) \longrightarrow H_{\mathfrak{m}}^0(R^+/I_{t-1}R^+) \longrightarrow H_{\mathfrak{m}}^0(R^+/I_tR^+) \longrightarrow \\ \longrightarrow H_{\mathfrak{m}}^1(R^+/I_{t-1}R^+) \longrightarrow H_{\mathfrak{m}}^1(R^+/I_{t-1}R^+) \longrightarrow H_{\mathfrak{m}}^1(R^+/I_tR^+) \longrightarrow \dots \end{aligned}$$

We know from 1 that $H_{\mathfrak{m}}^i(R^+/I_0) = H_{\mathfrak{m}}^i(R^+) = 0$ for $i = 0, 1, \dots, d - 1$; we prove by induction that $H_{\mathfrak{m}}^q(R^+/I_t) = 0$ for all $t \leq j - 1$ and $q < d - t$.

The case $t = 0$ is already done, since $d > 1$. Let us assume to have the thesis for $t - 1$ and that $t \leq j - 1$, then consider long exact sequence generated by $\cdot x_t$:

$$\dots \longrightarrow H_{\mathfrak{m}}^q(R^+/I_{t-1}) \longrightarrow H_{\mathfrak{m}}^q(R^+/I_t) \longrightarrow H_{\mathfrak{m}}^{q+1}(R^+/I_{t-1}) \longrightarrow \dots$$

If $q < d - (t - 1) - 1$, then by induction hypothesis, the left and right modules are both zero, hence also $H_{\mathfrak{m}}^q(R^+/I_t)$ is zero. This proves the induction.

In particular, we proved that $H_{\mathfrak{m}}^0(R^+/(x_1, \dots, x_{j-1})R^+) = 0$.

This implies, by definition of $\Gamma_{\mathfrak{m}} = H_{\mathfrak{m}}^0$, that $0 :_{R^+/(x_1, \dots, x_{j-1})R^+} \mathfrak{m} = 0$, therefore $\mathfrak{m} \notin Ass_R(R^+/(x_1, \dots, x_{j-1})R^+)$.

Now we want to show that $Ass_R(R^+/(x_1, \dots, x_{j-1})R^+) \subseteq Min((x_1, \dots, x_{j-1}))$. If $P \in Ass_R(R^+/(x_1, \dots, x_{j-1})R^+) \setminus Min((x_1, \dots, x_{j-1}))$, then $ht(P) > j - 1$ (In fact, since $x_1, \dots, x_{j-1}$ is part of a system of parameters, all the primes in $Min((x_1, \dots, x_{j-1}))$ have height $j - 1$ and $(x_1, \dots, x_{j-1}) \subseteq P$). Moreover

$$PR_P \in Ass_{R_P}(R^+/(x_1, \dots, x_{j-1})R^+)_P$$

Notice that, by Proposition 1.29, $(R^+/(x_1, \dots, x_{j-1})R^+)_P = (R_P)^+/(x_1, \dots, x_{j-1})(R_P)^+$. Furthermore PR_P is maximal and $x_1, \dots, x_{j-1}$ is still a regular sequence in $(R_P)^+$. So $H_{PR_P}^0((R_P)^+/(x_1, \dots, x_{j-1})(R_P)^+) = 0$ hence

$$PR_P \notin \text{Ass}_{R_P}((R_P)^+/(x_1, \dots, x_{j-1})(R_P)^+)$$

This is a contradiction. Hence every element of $\mathfrak{m}$ not in any minimal prime of $(x_1, \dots, x_{j-1})$ (such as x_j) are not in $\bigcup \text{Ass}_R(R^+/(x_1, \dots, x_{j-1})R^+)$, hence they are $R^+/(x_1, \dots, x_{j-1})R^+$ -regular. $\square$

Remark 3.39. Whenever $(R, \mathfrak{m})$ is a Noetherian local domain, if R^+ is a big Cohen-Macaulay algebra then $H_{\mathfrak{m}}^i(R^+) = 0$ for all $i < d$.

Let $x_1, \dots, x_d$ be a system of parameter of R , and therefore an R^+ -regular sequence. What we use is the fact that the Čech complex $C^*(x_1, \dots, x_d; R^+)$ can be constructed by taking the direct limit of a diagram whose rows are the Koszul complex $K(x_1^i, \dots, x_d^i; R^+)$, which are exact in all position different from d . Direct limits preserves exact sequences, therefore the cohomologies of $C^*(x_1, \dots, x_d; R^+)$ are zero in position different from d , hence $H_{\mathfrak{m}}^i(R^+) = 0$ for all $i < d$.

Remark 3.40 (Equal Characteristic zero). In characteristic zero, the cohomologies of R^+ supported on the maximal ideal are in general not zero.

Suppose that k has characteristic zero and let $(R, \mathfrak{m})$ be complete Noetherian local integrally closed domain such that $R/\mathfrak{m} = k$. Let $K = \text{frac}(R)$ and L be a finite field extension of K . It is a classical result that $R' := \overline{R}^L$ is a free R -module of dimension $n = [L : K]$.

Therefore it is well defined the trace map $tr : R' \rightarrow R$. If we consider $g = \frac{1}{n}tr$, then clearly $R \hookrightarrow R' \xrightarrow{g} R$ is the identity. Therefore $R' = R \oplus N$, which entails that $H_{\mathfrak{m}}^i(R') = H_{\mathfrak{m}}^i(R) \oplus H_{\mathfrak{m}}^i(N)$.

The situation just described is "general enough", meaning that if $(R, \mathfrak{m})$ is such a ring, any finite extension R'' of R can be completed to $R' := \overline{R}^{\text{frac}(R'')}$ and we get the case described before. Thus this inclusion map in cohomology are zero only when $H_{\mathfrak{m}}^i(R)$ is already zero (for example if R is Cohen-Macaulay).

Question. Does the previous remark implies that in equal characteristic zero, R^+ cannot ever be a Big Cohen-Macaulay algebra in dimension greater than 2?

(We are asking if R^+ can ever be a direct limit of Cohen-Macaulay finite R -modules, in dimension greater than 2.)

4 Applications

In this last chapter, we show some applications of the main result proven in Chapter 3. The idea behind these applications is the following: "If a result is true when the ring is Cohen-Macaulay, it may be possible to generalize it by making use of the absolute integral closure". Of course, all these applications will inevitably deal only with positive characteristic rings.

It is important to notice that when working with Noetherian local positive characteristic rings, even though the main result only works when the ring is a domain and a homomorphic image of a Cohen-Macaulay ring, it is often possible to obtain such a ring by simply completing and then taking the quotient by a prime ideal. Indeed, any complete local ring is the image of a Cohen-Macaulay ring by Cohen's structure theorem; taking the quotient doesn't change that. In many of these examples, we will first reduce the problem to these other hypotheses and then exploit the properties of completeness to deduce the general case.

The following results are taken directly from [6].

Theorem 4.1 (Monomial conjecture). Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d and positive characteristic p . Let $x_1, \dots, x_d$ be a system of parameters. Then for all $t \in \mathbb{N}_{>0}$, $(x_1 \dots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})$

Proof. Let us prove this by induction on d .

When $d = 1$, if $x_1^t \in (x_1^{t+1})$ then $x_1(x_1^t) = (x_1^{t+1})$, by Nakayama $(x_1^{t+1}) = 0$ which is absurd. Suppose that $d > 1$ and $(x_1 \dots x_d)^t \in (x_1^{t+1}, \dots, x_d^{t+1})$. Recall that, by Theorem 1.72, $\dim(\widehat{R}) = d$. Let P be a minimal prime of $\widehat{R}$ such that the dimension of $\widehat{R}/P$ is still d . Consider the following maps:

$$f : R \longrightarrow \widehat{R} \longrightarrow \widehat{R}/P$$

Following the ring homomorphism, the relation $(f(x_1) \dots f(x_d))^t \in (f(x_1)^{t+1}, \dots, f(x_d)^{t+1})$ is preserved; moreover $f(x_1), \dots, f(x_d)$ is still a system of parameters, since

$$\sqrt{(f(x_1), \dots, f(x_d))} = \frac{\sqrt{(x_1, \dots, x_d)\widehat{R}}}{P} = \frac{\mathfrak{m}\widehat{R}}{P}.$$

Hence, we can assume that R is a local complete domain of dimension d .

By Corollary 3.38, we know that $x_1, \dots, x_d$ is an R^+ -regular sequence. Write

$$(x_1 \dots x_d)^t = \sum_{i=1}^d s_i x_i^{t+1},$$

where $s_i \in R$. Then $x_d^t((x_1 \dots x_{d-1})^t - s_d x_d) \in (x_1^{t+1}, \dots, x_{d-1}^{t+1})$. Since the powers of regular sequences are still regular sequences, it follows that $(x_1 \dots x_{d-1})^t - s_d x_d \in (x_1^{t+1}, \dots, x_{d-1}^{t+1})R^+$. So $(x_1 \dots x_{d-1})^t - s_d x_d = \sum_{i=1}^{d-1} w_i x_i^{t+1}$ for some $w_i \in R^+$. Adding all the w_i to R we obtain, by Theorem 1.69, a Noetherian complete local domain S which is a finite R -module, such that $(x_1 \dots x_{d-1})^t \in (x_1^{t+1}, \dots, x_{d-1}^{t+1}, x_d)S$. So now $(\overline{x_1} \dots \overline{x_{d-1}})^t \in (\overline{x_1}^{t+1}, \dots, \overline{x_{d-1}}^{t+1})$ in the ring $S/x_d S$ which has dimension $d - 1$ (since S is a domain and $x_d \neq 0$). This contradicts the inductive hypothesis, in fact by Theorem 1.69, the maximal ideal of S is $\sqrt{\mathfrak{m}S}$, thus $x_1, \dots, x_d$ is a system of parameters in S and $\overline{x_1}, \dots, \overline{x_{d-1}}$ is a system of parameters of $S/x_d S$. $\square$

Lemma 4.2. Let $(R, \mathfrak{m})$ be a local ring, M, N two finitely generated non zero R -modules. Then

$$M \otimes_R N \text{ has finite length if and only if } \sqrt{\text{Ann}_R(M) + \text{Ann}_R(N)} = \mathfrak{m}$$

Proof. We know that $M \otimes_R N$ is finitely generated and has finite length, so, by 5.17, $\text{Supp}(M \otimes_R N) = \{\mathfrak{m}\}$. It is a classical result that $\text{Supp}(M \otimes_R N) = \text{Supp}(M) \cap \text{Supp}(N)$. So $\{\mathfrak{m}\} = \text{Supp}(M) \cap \text{Supp}(N) = V(\text{Ann}_R(M)) \cap V(\text{Ann}_R(N)) = V(\text{Ann}_R(M) + \text{Ann}_R(N))$, therefore we get $\sqrt{\text{Ann}_R(M) + \text{Ann}_R(N)} = \mathfrak{m}$. The converse follows using 5.17, indeed:

$$\mathfrak{m} \supseteq \sqrt{\text{Ann}_R(M \otimes_R N)} \supseteq \sqrt{\text{Ann}_R(M) + \text{Ann}_R(N)} = \mathfrak{m},$$

so $\sqrt{\text{Ann}_R(M \otimes_R N)} = \mathfrak{m}$ and thus $M \otimes_R N$ has finite length. $\square$

Theorem 4.3. Let $(R, \mathfrak{m})$ be a local Noetherian ring of positive characteristic p , and let M, N be two finitely generated non zero R -modules such that $M \otimes_R N$ has finite length. Then $\dim(N) \leq \text{projdim}_R(M)$.

Proof. Step zero: Let us assume to have proved the statement when R is complete. Let M and N be R -modules as in the hypothesis, then $\widehat{M} \cong M \otimes_R \widehat{R}$ and $\widehat{N} \cong N \otimes_R \widehat{R}$ are non zero finitely generated $\widehat{R}$ -modules since $- \otimes_R \widehat{R}$ is a faithful functor. By exactness and faithfulness we also deduce that

- $\text{projdim}_R(M) = \text{projdim}_{\widehat{R}}(M \otimes_R \widehat{R})$,
- $\dim(N) = \dim(N \otimes_R \widehat{R})$,
- $\widehat{M} \otimes_{\widehat{R}} \widehat{N}$ has finite length.

The first statement follows from the exactness of the functor and the fact that minimal projective resolution are free resolution for local rings, so $R^{\oplus n} \otimes_R \widehat{R} \cong \widehat{R}^{\oplus n}$. To prove the second statement, notice that, by 1.75, $\text{Ann}(N \otimes_R \widehat{R}) = \text{Ann}(N)\widehat{R}$. Hence, by 1.68, $\widehat{R}/\text{Ann}(\widehat{N}) = (R/\text{Ann}(N))^{\wedge}$, so, by Theorem 1.72, $(R/\text{Ann}(N))^{\wedge}$ has the same dimension of $R/\text{Ann}(N)$, therefore $\dim(N) = \dim(N \otimes_R \widehat{R})$.

For the third statement, notice that tensoring by $\widehat{R}$ preserves maximal decomposition sequences. So the equation follows from the complete case:

$$\dim(N) = \dim(\widehat{N}) \leq \text{projdim}_{\widehat{R}}(\widehat{M}) = \text{projdim}_R(M).$$

Let R be a complete local Noetherian ring, M and N as in the hypothesis, $I = \text{Ann}_R(N)$ and $J = \text{Ann}_R(M)$.

First step: Let P be a prime containing I such that $\dim(R/P) = \dim(N)$, then $\sqrt{I + J} = \sqrt{P + J} = \mathfrak{m}$, so, by Lemma 4.2, $R/P \otimes_R M$ has finite length. Hence it is sufficient to prove the statement for $N = R/P$.

Second step: Consider $S = (R/P)^+$. Notice that, by Cohen's structure theorem, R/P satisfies the condition of Corollary 3.38, so S is Big Cohen-Macaulay. Since $\sqrt{P + J} = \mathfrak{m}$, we can choose $x_1, \dots, x_d \in J$ such that their image on R/P forms a system of parameters. (we are just using that $\sqrt{\frac{J+P}{P}} = \frac{\mathfrak{m}}{P}$)

So they are a regular sequence in S .

Now if, by contradiction, $d = \dim(R/P) > \text{projdim}_R(M)$ then we would have that $\text{Tor}_d^R(S/(x_1, \dots, x_d)S, M) = 0$.

We prove by induction that $\text{Tor}_i^R(S/(x_1, \dots, x_i)S, M) \neq 0$ for all $i = 0, \dots, d$.

If $i = 0$, then $\text{Tor}_0^R(S, M) = S \otimes_R M$, which cannot be zero since neither of the two modules is zero and we are working with a local ring.

Let $0 \leq i < d$ and assume that $\text{Tor}_i^R(S/(x_1, \dots, x_i)S, M) \neq 0$. Set $S_j := S/(x_1, \dots, x_j)S$ for all $j = 1, \dots, d$. Consider the short exact sequence:

$$0 \longrightarrow S_i \xrightarrow{\cdot x_{i+1}} S_i \longrightarrow S_{i+1} \longrightarrow 0.$$

This sequence induces a long exact sequence:

$$\dots \longrightarrow \text{Tor}_{i+1}^R(S_{i+1}, M) \longrightarrow \text{Tor}_i^R(S_i, M) \xrightarrow{\cdot x_{i+1}} \text{Tor}_i^R(S_i, M) \longrightarrow \dots$$

We know that $x_{i+1} \in J$, so the multiplication map is actually the zero map, therefore:

$$\text{Tor}_{i+1}^R(S_{i+1}, M) \twoheadrightarrow \text{Tor}_i^R(S_i, M) \neq 0.$$

Hence $\text{Tor}_{i+1}^R(S_{i+1}, M)$ cannot be zero. $\square$

Theorem 4.4. Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p , and let M be a non zero finitely generated R -module with finite projective dimension. If $x \in R$ is not a zero divisor of M , then it not a zero divisor of R .

Proof. Since $Z_R(M) = \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} \mathfrak{p}$, we need to show that $R \setminus Z_R(M) \subseteq R \setminus Z_R(R)$, which is equivalent to $\bigcup_{\mathfrak{p} \in \text{Ass}_R(R)} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} \mathfrak{p}$.

Let us prove this by induction on the dimension of M . If $\dim(M) = 0$, then $\sqrt{\text{Ann}(M)} = \mathfrak{m}$, so $\text{Ass}_R(M) = \{\mathfrak{m}\}$, and since R is local, we are done.

Assume $\dim(M) = d > 0$. Let $P \in \text{Ass}_R(R)$.

- If there is a prime $Q \in \text{Supp}(M)$ such that $Q \neq \mathfrak{m}$ and $P \subseteq Q$, then M_Q is a non zero R_Q -module of finite projective dimension over R_Q . $P_Q \in \text{Ass}_{R_Q}(R_Q)$ and $\dim(M_Q) = \dim(R_Q/\text{Ann}_{R_Q} M_Q) = \dim((R/\text{Ann}_R M)_Q) < d$, so by induction $P_Q \subseteq \mathfrak{q}R_Q \in \text{Ass}_{R_Q}(M_Q)$, where $\mathfrak{q} \in \text{Ass}_R(M)$. Hence $P \subseteq \mathfrak{q}$ and we are done.
- Recall that $\text{Supp}(M) = V(\text{Ann}(M))$, so since $d > 0$ there must be a prime $Q \in \text{Supp}(M)$ such that $Q \neq \mathfrak{m}$. If $P \not\subseteq Q$ for all primes of $\text{Supp}(M)$ different from $\mathfrak{m}$ then for any $Q \in \text{Supp}(M) \setminus \{\mathfrak{m}\}$, the only prime that can stay in $\text{Min}(P + Q)$ is $\mathfrak{m}$, so, by Lemma 4.2, $R/P \otimes_R M$ has finite length. So by the previous theorem, $\dim(R/P) \leq \text{projdim}_R(M) = \text{depth}(R) - \text{depth}(M)$. $P \in \text{Ass}_R(R)$, hence we know that $\text{depth}(R) \leq \dim(R/P)$. It follows that $\text{depth}(M) = 0$. Therefore $\mathfrak{m} \in \text{Ass}(M)$ and $\mathfrak{p} \subseteq \mathfrak{m}$.

$\square$

Theorem 4.5. Let $(R, \mathfrak{m})$ be a complete local Noetherian domain of characteristic p and dimension d , and let $x_1, \dots, x_k \in \mathfrak{m}$, where $k \leq d - 2$. Then the punctured spectrum of $R/(x_1, \dots, x_k)$ is connected.

Proof. Let us first assume that $x_1, \dots, x_k$ are parameters. The punctured spectrum of a local ring $(R, \mathfrak{m})$ is defined as $\text{Spec}(R) \setminus \{\mathfrak{m}\}$, with the Krull topology.

By contradiction, let us assume that $\text{Spec}(R/(x_1, \dots, x_k)) \setminus \{\mathfrak{m}\}$ is not connected, thus there exist some ideals I and J of R containing $(x_1, \dots, x_k)$ such that, $V(I) \cup V(J) = V(IJ) = \text{Spec}((x_1, \dots, x_k))$, $V(I) \cap V(J) = V(I+J) = \{\mathfrak{m}\}$ and $V(I) \neq \{\mathfrak{m}\} \neq V(J)$.

The first equation tells us that $\sqrt{IJ} = \sqrt{(x_1, \dots, x_k)}$, hence for some $n \in \mathbb{N}$ we have $I^n J^n \subseteq (x_1, \dots, x_k)$. Therefore we can assume that $IJ \subseteq (x_1, \dots, x_k)$.

The second equation tells us that $\sqrt{I+J} = \mathfrak{m}$.

Since $k \leq d - 2$, we can choose $y + z \in I + J$ and $u + v \in I + J$ such that $x_1, \dots, x_k, y + z, u + v$ is part of a system of parameters.

Consider the following equation:

$$y(u + v) - u(y + z) = yv - uz \in IJ,$$

since $IJ \subseteq (x_1, \dots, x_k)$, $y(u + v) - u(y + z)$ is zero modulo $(x_1, \dots, x_k)$.

Using the fact that in R^+ the chosen parameters form a regular sequence, we get $y \in (y + z, x_1, \dots, x_k)R^+$. We deduce that $z \in (y + z, x_1, \dots, x_k)R^+$. Now write:

$$y = c(y + z) \text{ modulo } (x_1, \dots, x_k)R^+, \text{ for some } c \in R^+ \text{ and}$$

$$z = d(y + z) \text{ modulo } (x_1, \dots, x_k)R^+, \text{ for some } d \in R^+$$

We observe that $(1 - c - d)(y + z) = 0$ modulo $(x_1, \dots, x_k)$, so one between c and d must be a unit, say c . Hence, $y = c(y + z)$ implies that $x_1, \dots, x_k, y$ is a regular sequence. For any $j \in J$, $yj \in IJ \subseteq (x_1, \dots, x_k)$, therefore $j \in (x_1, \dots, x_k)R^+$, so $J \subseteq (x_1, \dots, x_k)R^+$, then $IR^+ + JR^+ = IR^+$ must be $\mathfrak{m}R^+$ -primary, which is a contradiction because of lying over. (we know that $V(I) \neq \{\mathfrak{m}\}$, so by lying over $V(IR^+) \neq \{\mathfrak{m}R^+\}$).

We need to show that we can always assume $x_1, \dots, x_k$ to be parameters. Let us first prove that for any ideal U of height at least k , containing $x_1, \dots, x_k$, there are some parameters $y_1, \dots, y_k \in U$ such that $(x_1, \dots, x_k) \subseteq \sqrt{(y_1, \dots, y_k)}$.

We prove the case $k = 1$ for any Noetherian local ring:

If (x_1) has height 1, it is a parameter and so we are done.

If x_1 is nilpotent, then we can just choose any $y \in U$ that it is a parameter, because the condition $(x_1) \subseteq \sqrt{(y)}$ is always true.

Assume that x_1 is neither a parameter nor a nilpotent of R , let $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$, so, by reordering the indexes, we can say that there is some $k \in \{2, \dots, s\}$ such that $x_1 \in \mathfrak{p}_j$ if and only if $j < k$.

By prime avoidance, we can construct an element $z' \in \bigcap_{j \geq k} \mathfrak{p}_j \setminus \bigcup_{j < k} \mathfrak{p}_j$. Multiplying z' by a parameter in U we obtain an element $z \in U \cap \bigcap_{j \geq k} \mathfrak{p}_j \setminus \bigcup_{j < k} \mathfrak{p}_j$. Then $x_1 z$ belongs to all the minimal primes of R , and thus it is nilpotent. So $x_1 z \in \sqrt{(x_1^2 + x_1 z)}$ hence $x_1^2 \in \sqrt{(x_1^2 + x_1 z)}$ so $x_1 \in \sqrt{(x(x + z))} \subseteq \sqrt{(x + z)}$. Finally $y_1 := x_1 + z \in U$ is a parameter since it cannot belong to any minimal primes of R , and $(x_1) \subseteq \sqrt{(y_1)}$.

Now suppose that we have already found some parameters $y_1, \dots, y_{k-1} \in U$ such that $(x_1, \dots, x_{k-1}) \subseteq \sqrt{(y_1, \dots, y_{k-1})}$. We can replace R with $R/(y_1, \dots, y_{k-1})$, so we consider the projections of x_k and U . Now $U/(y_1, \dots, y_{k-1})$ is an ideal of height at least 1. In fact, using 1.60, $ht(U/(y_1, \dots, y_{k-1})) = ht(\mathfrak{q}) - ht(\mathfrak{p})$ for some $\mathfrak{q} \in Min(U)$ and $\mathfrak{p} \in Min((y_1, \dots, y_{k-1}))$. By Krull, we know that $ht(\mathfrak{p}) \leq k-1$ and we already knew that $ht(\mathfrak{q}) \geq k$ therefore $ht(\mathfrak{q}) - ht(\mathfrak{p}) \geq k - (k-1) = 1$.

Hence we are in the hypothesis of the case $k = 1$, so we find some $\overline{y_k} \in U/(y_1, \dots, y_{k-1})$ such that $(\overline{x_k}) \subseteq \sqrt{(\overline{y_k})}$ and $\overline{y_k} \notin \bigcup Min(U/(y_1, \dots, y_{k-1}))$. Lifting all this properties, we get that $y_1, \dots, y_k \in U$ is a part of a system of parameters and $(x_1, \dots, x_k) \subseteq \sqrt{(y_1, \dots, y_k)}$.

To conclude the proof, assume that $V(I)$ and $V(J)$ disconnect the punctured spectrum of $R/(x_1, \dots, x_k)$ as in the first part of the proof. So

- $IJ \subseteq \sqrt{(x_1, \dots, x_k)}$
- $V(I) \neq \{\mathfrak{m}\} \neq V(J)$
- $\sqrt{I+J} = \mathfrak{m}$

We choose two ideal $I' \in V(I)$ and $J' \in V(J)$ such that $k \leq ht(I') < d$ and $k \leq ht(J') < d$. So $U = I' \cap J'$ contains $(x_1, \dots, x_k)$ and $ht(U) \geq k$.

Hence, we can choose parameters $y_1, \dots, y_k \in U$ such that $(x_1, \dots, x_k) \subseteq \sqrt{(y_1, \dots, y_k)}$. Now let $K = (y_1, \dots, y_k)$, $I + K$ and $J + K$ disconnects the punctured spectrum of $R/(y_1, \dots, y_k)$, in fact:

- $(I + K)(J + K) \subseteq \sqrt{(y_1, \dots, y_k)}$
- $\sqrt{I + K} = \sqrt{K} \subseteq U \neq \mathfrak{m}$
- $\sqrt{J + K} = \sqrt{K} \subseteq U \neq \mathfrak{m}$
- $\sqrt{I + J + K} = \mathfrak{m}$

Since $y_1, \dots, y_k$ are parameters, we have already proved that this cannot happen, so we have found a contradiction. $\square$

5 Appendix

5.1 Miscellaneous

Remark 5.1 (On the converse of transitivity). Let $R \subseteq S \subseteq T$ be rings inclusions.

1. It is easy to see that if $R \subseteq S$ is finite and $S \subseteq T$ is finite then so is $R \subseteq T$. (i.e. finite rings extensions forms a category)
If $R \subseteq T$ is finite, then clearly $S \subseteq T$ must also be finite, but is it true that also $R \subseteq S$ must be finite?

- Surprisingly, in general the answer is no, here is a counterexample:

Let k be a field, $R := k[x_i : i \in \mathbb{N}]$,

$T := R[t]/(t^2)$ and $S := R[x_i \bar{t} : i \in \mathbb{N}] \subseteq T$.

$$R \subseteq R[x_i \bar{t} : i \in \mathbb{N}] \subseteq R[t]/(t^2)$$

We have that $R \subseteq T$ is finite, since $\{1, \bar{t}\}$ is a basis of T as an R -module, but clearly $R \subseteq S$ cannot be finite.

To show this assume that it is finite, then let $\{s_1, \dots, s_n\}$ be a finite set of generators of S as an R -module;

using that $S \subseteq T$, for all indexes we can write unequivocally $s_i = f_i + g_i \bar{t}$, where $f_i, g_i \in R$; moreover we know that neither of the g_i has a zero degree term, since all $s_i \in S$.

Now f_i, g_i are all polynomials, so there must exist an index $k \in \mathbb{N}$ such that x_k is not in the support of any f_i or g_i .

Since $x_k \bar{t} \in S$, there exists a linear combination such that

$$x_k \bar{t} = \sum_{i=1}^n b_i s_i$$

Where $b_i \in R$ for all indexes, and so

$$x_k \bar{t} = \sum_{i=1}^n b_i f_i + b_i g_i \bar{t} \implies x_k = \sum_{i=1}^n b_i g_i$$

This cannot happen, because none of the g_i 's has a zero degree term or has x_k in the support, and so x_k cannot be in the support of any of the products $b_i g_i$.

- When R is Noetherian, the extension $R \subseteq S$ is finite when $R \subseteq T$ is.
To show this we can just use the fact that when the ring is Noetherian, a module is finitely generated if and only if it is Noetherian (as a module).
If R is Noetherian, then T is a Noetherian R -module, and so since S is a submodule of T , it must be finitely generated.

2. It is easy to see that if $R \subseteq S$ is of finite type and $S \subseteq T$ is of finite type then so is $R \subseteq T$. (i.e. finite type rings extensions forms a category)
If $R \subseteq T$ is of finite type, then clearly $S \subseteq T$ must also be of finite type, but is it true that also $R \subseteq S$ must be of finite type?

- The answer is no, even when R is assumed to be Noetherian.
Consider the following example:

$$k[x] \subseteq k[xy^i : i \in \mathbb{N}] \subseteq k[x, y]$$

Clearly the most outer extension is of finite type, but the first one is not.
To show that, we can argue in a similar way of the previous example.

Remark 5.2 (Algebraically closed quotient field). A domain whose quotient field is algebraically closed, need not to be normal. Hence this property does not characterize absolutely integrally closed domains (see poin 2 of Remark 1.19).

Let K be an algebraically closed field of characteristic zero. Consider the ring

$$R' := \left\{ \sum_{i=0}^{\infty} \lambda_i t^i \in K[[t]] : \lambda_1 = 0 \right\} \text{ and define } R := R' [t^{1+\frac{1}{n}} : n \in \mathbb{N}_{\geq 1}].$$

Firstly, we show that $\text{frac}(R)$ is equal to $\bigcup_{n=1}^{\infty} K((t^{1/n}))$, which we know is algebraically closed from Theorem 2.2. Since $\{1 + \frac{1}{n} : n \in \mathbb{N}_{\geq 1}\}$ generates $\mathbb{Q}$ as a $\mathbb{Z}$ -module and $\{t^{1+\frac{1}{n}} : n \in \mathbb{N}_{\geq 1}\} \subseteq R$, $t^q \in \text{frac}(R)$ for any $q \in \mathbb{Q}$.

Therefore $\text{frac}(R) \supseteq R'[t^q : q \in \mathbb{Q}] = K[[t]][t^q : q \in \mathbb{Q}] = \bigcup_{n=1}^{\infty} K((t^{1/n}))$.

Now to prove that R is not integrally closed, notice that $t \in \text{frac}(R)$ and $t^2 \in R$. Since $t \notin R'[t^{1+\frac{1}{1}}, \dots, t^{1+\frac{1}{n}}]$ for any $n \geq 1$, $t \notin R$. Hence R is not integrally closed.

Theorem 5.3 (Cayley-Hamilton). Let M be a finitely generated R -module and $I \subseteq R$ an ideal of R . Let $f \in \text{End}(M)$ be an R -linear endomorphism such that $f(M) \subseteq IM$, then there exists a monic polynomial $p \in R[x]$ such that: $p(f) = 0$ in $\text{End}(M)$ and if $p(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$, then $a_i \in I^i \forall i = 1, \dots, n$.

Lemma 5.4 (Nakayama). Let R be a ring, $I \subseteq \bigcap \text{Max}(R)$

If M is a finitely generated R -module such that $M = IM$, then $M = 0$.

Proof. Let $f = \text{id} \in \text{End}(M)$. Then, by the previous Theorem, since $f(M) \subseteq IM$, there is a monic polynomial $p(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$ such that $a_i \in I^i \forall i = 1, \dots, n$ and $p(f) = 0$. Since f is the identity, for any $m \in M$,

$$0 = m + \sum_{i=1}^n a_i m = (1 + \sum_{i=1}^n a_i) m.$$

Since $\sum_{i=1}^n a_i \in I \subseteq \bigcap \text{Max}(R)$, $1 + \sum_{i=1}^n a_i$ cannot belong to any maximal ideal. Hence, it is a unit and $m = 0$. $\square$

Corollary 5.5 (Nakayama). Let R be a ring, $I \subseteq \bigcap \text{Max}(R)$

If $N \subseteq M$ are finitely generated R -modules such that $M = N + IM$, then $N = M$.

Proof. Apply the previous Lemma to M/N . $\square$

Proposition 5.6 (Moore determinant). Consider the polynomial ring $R = \mathbb{F}_p[Z_1, \dots, Z_n]$ of positive characteristic p . We define the Moore Matrix $M \in R^{n \times n}$ as following:

$$M = \begin{bmatrix} Z_1 & Z_2 & \dots & Z_n \\ Z_1^p & Z_2^p & \dots & Z_n^p \\ \dots & \dots & \dots & \dots \\ Z_1^{p^{n-1}} & Z_2^{p^{n-1}} & \dots & Z_n^{p^{n-1}} \end{bmatrix}.$$

Let $V_n \subseteq (\mathbb{F}_p)^n$ be the set of non zero vectors such that the last non zero coordinate is 1. Then the determinant of M is $\Pi_{c \in V_n} (c_1 Z_1 + \dots + c_n Z_n) \in R$.

Proof. Let us call $d \in \mathbb{F}_p[Z_1, \dots, Z_n]$ the determinant of M .

Let $c = (c_0, \dots, c_{i-1}, 1, 0, \dots, 0) \in V_n$ and set $v_c := c_1 Z_1 + \dots + Z_i$. Consider the matrix M after projecting the coefficient into $R/(v_c)$. Clearly $\overline{Z_i} = c_1 \overline{Z_1} + \dots + c_{i-1} \overline{Z_{i-1}}$. Since for any $\eta \in \mathbb{F}_p$, $\eta^p = \eta$, we actually know that $\overline{Z_i}^{p^j} = c_1 \overline{Z_1}^{p^j} + \dots + c_{i-1} \overline{Z_{i-1}}^{p^j}$ for all j . Thus, the i th column of the projection of M is an $\mathbb{F}_p$ -linear combination of the first $i - 1$ columns. Hence $\overline{d} = 0$, and v_c divides d .

We have proved that $q := \Pi_{c \in V_n} (c_1 Z_1 + \dots + c_n Z_n)$ divides d . It is easy to prove that d is, by construction, a homogeneous polynomial of degree $p^n - 1$. Since the cardinality of V_n is also $p^n - 1$, the two polynomials have the same degree, so they are equal up to a unit of $\mathbb{F}_p$.

The coefficient of the monomial $Z_1 Z_2^p Z_3^{p^2} \dots Z_n^{p^{n-1}}$ of q is 1 (We are multiplying all the last 1s of the vectors in V_n , and no other combination can produce this monomial). Similarly, from the definition of the determinant, we see that the only way to obtain this monomial is by multiplying the elements on the main diagonal of M . Hence, the coefficient of $Z_1 Z_2^p \dots Z_n^{p^{n-1}}$ in d is also 1, so $q = d$. $\square$

Corollary 5.7. Let K be a field of positive characteristic and let $z_1, \dots, z_n \in K$. If M is the Moore matrix evaluated in $z_1, \dots, z_n$, then $\det(M) = 0$ if and only if $z_1, \dots, z_n$ are $\mathbb{F}_p$ -linearly independent.

Proof. By the previous Theorem, $\det(M) = \Pi_{c \in V_n} (c_1 z_1 + \dots + c_n z_n)$. So $\det(M)$ is zero if and only if one of the $c_1 z_1 + \dots + c_n z_n = 0$ for some $c \in V_n$. If $z_1, \dots, z_n$ are linearly dependent, then $\sum_{i=1}^k \tilde{c}_i z_i = 0$ for some $\tilde{c}_i \in \mathbb{F}_p$, such that $\tilde{c}_k \neq 0$. Multiplying the coefficients by $\tilde{c}_k^{-1}$, we get a vector $c \in V_n$ for which $c_1 z_1 + \dots + c_n z_n = 0$. The converse is tautological. $\square$

5.2 Length of a Module

Definition 5.8. Let R be a ring. A non zero R -module M is simple if it does not admit any proper non zero submodule.

Remark 5.9. An R -module M is simple if and only if there is a maximal ideal $\mathfrak{m} \subseteq R$ such that $M \cong R/\mathfrak{m}$.

Indeed $R/\mathfrak{m}$ is a field, therefore its only proper R -submodule is zero. Conversely, if M is simple and $m \in M$, $m \neq 0$, then $\langle m \rangle_R = M$, so $R/(0 :_R m) \cong M$. If $0 :_R m$ was not maximal, there would be a strictly bigger ideal $0 :_R m \subsetneq I \subsetneq R$, and so $0 \neq I/(0 :_R m) \subsetneq R/(0 :_R m)$, which is a contradiction.

Definition 5.10. Let R be a ring, M an R -module, a simple decomposition series of M is a chain of strictly increasing submodule of this form:

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M,$$

such that for all $i = 1, \dots, n$, M_i/M_{i-1} is simple.

The length of a module M , denoted $l(M)$, is the length of any such series when it exists and infinity when it does not exists. It is a theorem that, when they exists, all simple decomposition series have the same length.

We present some easy properties of length.

Proposition 5.11. If $0 \rightarrow N \rightarrow M \rightarrow T \rightarrow 0$ is a short exact sequence of R -modules, then $l(M) = l(N) + l(T)$.

Proposition 5.12. If $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_r = M$ is a nested chains of R -modules, then

$$l(M) = \sum_{i=1}^n l(M_i/M_{i-1}).$$

Proposition 5.13. An R -module M has finite length if and only if it is both Noetherian and Artinian.

Remark 5.14. If R is a field, any R -module M is a vector space, hence it admits a basis. So M has finite length if and only if it is finitely generated, since simple modules are simply one dimensional vector spaces.

Indeed, length is a type of generalization of vector space dimension.

Theorem 5.15. Let $(R, \mathfrak{m})$ be a local ring, M an R -module, if $\mathfrak{m}M = 0$ then the following properties are equivalent:

- M has finite length.
- M is Noetherian.
- M is Artinian.

Proof. Notice that M is an $R/\mathfrak{m}$ -vector space. In this setting, the three conditions are all equivalent to having finite dimension. $\square$

Corollary 5.16. Let $(R, \mathfrak{m})$ be a local ring, M an R -module, then the following are equivalent:

1. M has finite length
2. M is Noetherian and $\mathfrak{m}^n M = 0$ for some $n \in \mathbb{N}$
3. M is Artinian and $\mathfrak{m}^n M = 0$ for some $n \in \mathbb{N}$

Proof. Clearly 1 implies both 2 and 3, because having finite length is equivalent to being both Artinian and Noetherian and to prove that $\mathfrak{m}^n M = 0$ for some $n \in \mathbb{N}$, notice that by Artinianity $\mathfrak{m}^n M = \mathfrak{m}^{n+1} M$ for some $n \in \mathbb{N}$, so by Nakayama, $\mathfrak{m}^n M = 0$. To prove that 2 or 3 implies 1, notice that $\mathfrak{m}^n M \subseteq \mathfrak{m}^{n-1} M \subseteq \dots \subseteq M$ so

$$l(M) = \sum_{i=1}^n l(\mathfrak{m}^{n-i} M / \mathfrak{m}^{n-i+1} M)$$

Since $\mathfrak{m}(\mathfrak{m}^{n-i} M / \mathfrak{m}^{n-i+1} M) = 0$ for all i , by the previous theorem $\mathfrak{m}^{n-i} M / \mathfrak{m}^{n-i+1} M$ has finite length for all i , therefore M has finite length. $\square$

Corollary 5.17. Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a non zero, finitely generated R -module, then the following are equivalent:

1. M has finite length
2. $\sqrt{\text{Ann}_R(M)} = \mathfrak{m}$
3. $\text{Supp}(M) = V(\text{Ann}_R(M)) = \{\mathfrak{m}\}$

Proof. By the previous Corollary and the Noetherianity of R , 1 is equivalent to 2. 2 is clearly equivalent to 3. $\square$

5.3 Direct and inverse limits

Definition 5.18 (Diagrams). Let R be a ring and $(\Gamma, \leq)$ a partially ordered set. A diagram of R -modules is a functor $F : (\Gamma, \leq) \rightarrow R\text{-mod}$.

Explicitly, for any $i \in \Gamma$, $F(i)$ is an R -module, for any pair $i \leq j$, $F_{i \leq j} := F(i \leq j) : F(i) \rightarrow F(j)$ is an R -linear morphism and for any $i \leq j \leq k$, $F_{j \leq k} \circ F_{i \leq j} = F_{i \leq k}$.

Definition 5.19 (Direct Limits). Let R be a ring and $F : (\Gamma, \leq) \rightarrow R\text{-mod}$ a diagram of R -modules. A direct cone of F is a pair $(M, \{f_i\}_{i \in \Gamma})$ where M is an R -module and $f_i : F(i) \rightarrow M$ are all R -linear morphisms such that whenever $i \leq j$, $f_j \circ F_{i \leq j} = f_i$.

$$\begin{array}{ccc} F(i) & \xrightarrow{F_{i \leq j}} & F(j) \\ \downarrow f_i & \searrow f_j & \\ M & & \end{array}$$

A direct limit of F is a direct cone of F $(\lim_{\rightarrow} F, \{\lambda_i\}_{i \in \Gamma})$, such that for any direct cone $(M, \{f_i\}_{i \in \Gamma})$ of F there is a unique morphism $h : \lim_{\rightarrow} F \rightarrow M$, such that $h \circ \lambda_i = f_i$ for all $i \in \Gamma$.

$$\begin{array}{ccc} \lim_{\rightarrow} F & \xrightarrow{h} & M \\ \lambda_i \uparrow & \nearrow f_i & \\ F(i) & & \end{array}$$

Definition 5.20 (Inverse Limits). Let R be a ring and $F : (\Gamma, \leq) \rightarrow R\text{-mod}$ a diagram of R -modules. An inverse cone of F is a pair $(M, \{f_i\}_{i \in \Gamma})$ where M is an R -module and $f_i : M \rightarrow F(i)$ are all R -linear morphisms such that whenever $i \leq j$, $F_{i \leq j} \circ f_i = f_j$.

$$\begin{array}{ccc} F(i) & \xrightarrow{F_{i \leq j}} & F(j) \\ f_i \uparrow & \nearrow f_j & \\ M & & \end{array}$$

An inverse limit of F is an inverse cone of F $(\lim_{\leftarrow} F, \{\lambda_i\}_{i \in \Gamma})$, such that for any inverse cone $(M, \{f_i\}_{i \in \Gamma})$ of F there is a unique morphism $h : M \longrightarrow \lim_{\leftarrow} F$, such that $\lambda_i \circ h = f_i$ for all $i \in \Gamma$.

$$\begin{array}{ccc} \lim_{\leftarrow} F & \xleftarrow{h} & M \\ \downarrow \lambda_i & \swarrow f_i & \\ F(i) & & \end{array}$$

Remark 5.21. The characterizing property of direct (inverse) limit is called "Universal property". By the universal property it is easy to verify that if a limit exist it is unique up to isomorphism.

In our settings, inverse and direct limits always exist. Indeed, if $F : (\Gamma, \leq) \longrightarrow R\text{-mod}$ is a diagram, then

- $\lim_{\rightarrow} F = \frac{\bigoplus_{i \in \Gamma} F(i)}{N}$, where
 $N = \langle v_i - v_j : v_i \in F(i), v_j \in F(j), \exists k \geq i, j \text{ such that } F_{i \leq k}(v_i) = F_{j \leq k}(v_j) \rangle_R$
 Moreover for any $j \in \Gamma$, $\lambda_j : \lim_{\rightarrow} F \longrightarrow F(j)$ is the projection of the inclusion $F(j) \subseteq \bigoplus_{i \in \Gamma} F(i)$.
- $\lim_{\leftarrow} F = \{(v_i)_{i \in \Gamma} \in \prod_{i \in \Gamma} F(i) : \text{for any } i \leq j, F_{i \leq j}(v_i) = v_j\}$
 Moreover for any $j \in \Gamma$, $\lambda_j : \lim_{\leftarrow} F \longrightarrow F(j)$ is the restriction of the projection $\prod_{i \in \Gamma} F(i) \longrightarrow F(j)$.

Remark 5.22. Let us study some useful set of cases:

- If $F : (\Gamma, \leq) \longrightarrow R\text{-mod}$ is a diagram such there is an R -module M for which $F(i) \subseteq M$ for all i and $F_{i \leq j} : F(i) \longrightarrow F(j)$ is exactly the inclusion $F(i) \subseteq F(j)$ inside M ; then if $M = \bigcup_{i \in \Gamma} F(i)$, M is the direct limit and $\bigcap_{i \in \Gamma} F(i)$ is the inverse limit.

By the previous remark, $\lim_{\rightarrow} F \cong \frac{\bigoplus_{i \in \Gamma} F(i)}{N}$. Since all the $F_{i \leq j}$ are inclusion, if $v \in F(i)$ and $w \in F(j)$; $v \sim w \text{ mod } N$ if and only if $v = w$ inside M . Therefore $\lim_{\rightarrow} F = M$.

- If $F : (\Gamma, \leq) \longrightarrow R\text{-mod}$ is a diagram such for any $i \in \Gamma$, there is an element $j \in \Gamma$, $i \leq j$ such that $F_{i \leq j} = 0$. Then $\lim_{\rightarrow} F = 0$.

By the previous remark, $\lim_{\rightarrow} F \cong \frac{\bigoplus_{i \in \Gamma} F(i)}{N}$. Our hypothesis tells us that for any $v \in F(i)$ $v - 0 \in N$. Hence $M = 0$.

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