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Derived categories from model categories

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Chapter 1

Introduction

A classical tool of commutative algebra and algebraic geometry is $\mathcal{D}(\mathcal{C})$, the derived category of an abelian category $\mathcal{C}$, the localization of the category $\mathrm{Ch}(\mathcal{C})$ of chain complexes of $\mathcal{C}$ with respect to the class of quasi-isomorphisms, and the most natural setting for derived functors and cohomological constructions. The usual construction begins by defining the so-called homotopy category $\mathcal{K}(\mathcal{C})$ of $\mathcal{C}$ (a name that we will reserve for the homotopy category associated to a model category), which is the category whose objects are chain complexes and whose morphisms are equivalence classes of morphisms of complexes up to homotopy, and then localizing this category with respect to the class of quasi-isomorphisms by introducing a calculus of fractions. This approach presents some issues: firstly, it is necessary to be very careful to avoid size issues, given that we would like to work with locally small categories; furthermore mapping cones are only defined up to homotopy and therefore the triangulated structure on $\mathcal{D}(\mathcal{C})$ can only be defined indirectly.

We will instead move along another route: after summarizing some preliminary results in chapter 2, we will proceed to introduce the concept of model categories in chapter 3, which provide another method of localizing certain categories. The idea of a model category has been introduced by Daniel Quillen in “Homotopical Algebra” ([Qui67]); it consists of a complete and cocomplete category $\mathcal{C}$ where three classes of morphisms have been selected: fibrations, cofibrations and weak equivalences; these classes satisfy certain axioms and we will make use of the first two to define a category, the homotopy category $\mathrm{Ho}(\mathcal{C})$, where the third is invertible.

In chapter 4, we begin with a basic exposition of cotorsion pairs: a cotorsion pair is a pair $(\mathcal{D}, \mathcal{E})$ of classes of objects of $\mathcal{C}$ where the elements D of $\mathcal{D}$ are those objects of $\mathcal{C}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(D, E) = 0$ for all $E \in \mathcal{E}$ and where the elements E of $\mathcal{E}$ are those objects of $\mathcal{C}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(D, E) = 0$ for all $D \in \mathcal{D}$. We pay particular attention to the case where $\mathcal{C}$ is a Grothendieck category; we will also define what it means for a cotorsion pair to be complete, imitating the definition of “enough injectives” and “enough projectives” in abelian categories.

We will then present the work of Mark Hovey on abelian model categories, where model structures on an abelian category (with suitable compatibility conditions) are identified by Hovey's correspondence theorem with objects called Hovey triples: a Hovey triple is a triple (C,W,F) of classes of objects of $\mathcal{C}$ where W is a thick class and $(C \cap W, F)$ and $(C, F \cap W)$ are complete cotorsion pairs.
The final chapter 5 is devoted to specializing our study to cotorsion pairs on categories of chain complexes and examining, as shown in the work of James Gillespie (see for example [Gil04]), that a cotorsion pair on an abelian category induces a pair of cotorsion pairs on the associated category of chain complexes. We will continue by discussing the work of Xiaoyan Yang and Nanqing Ding whose results tell us when these induced cotorsion pairs give rise to a Hovey triple in the category of chain complexes; combining these results, we obtain a model structure on $\mathrm{Ch}(\mathcal{C})$ whose homotopy category is the derived category $\mathcal{D}(\mathcal{C})$.

Chapter 2

Abelian categories

This chapter will serve as an introduction to the basics of Abelian categories, first introduced by Alexander Grothendieck in his seminal "Tohoku Paper" [Gro57]; for an accessible introduction see [Mur06a].

2.1 Basic definitions

Let $\mathcal{C}$ be a category. Let $u\colon A \longrightarrow B$ and $u'\colon A' \longrightarrow B$ be monomorphisms in $\operatorname{Ar}\mathcal{C}$. We say that u majorizes u', written $u \leq u'$, if there exists a morphism $v : A \longrightarrow A'$ such that $u = u' \circ v$.
This relation defines a preorder on the class of arrows into B, from which we can extract an equivalence relation: $u \cong v$ if and only if $u \leq v$ and $v \leq u$.

Definition 2.1.1 (Subobject). *Let $B \in \operatorname{ob}\mathcal{C}$, we define a **subobject** of B to be an equivalence class of monomorphisms into B.*

Remark 2.1.2. *While taking a quotient, we could run into size issues when $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is a proper class; we will generally avoid discussing set-theoretic issues.*

Definition 2.1.3 (Well-powered categories). *We say that a category $\mathcal{C}$ is well-powered if any object of $\mathcal{C}$ has at most a set of subobjects.*

Remark 2.1.4. *Let A and B be representatives of the same subobject of X, then we have a diagram*

$$\begin{tikzcd} & X \\ A \arrow[ur, hook, "a"] \arrow[r, shift left, "u"] & B. \arrow[u, hook, "b"'] \arrow[l, shift left, "v"] \end{tikzcd}$$

By applying the definition of monomorphism, it is easy to see that $vu = \mathrm{id}_A$ and $uv = \mathrm{id}_B$ (also: u and v are unique), therefore A and B are isomorphic. This *implies that being the same subobject is equivalent to being isomorphic in the slice category $\mathcal{C}/X$.*

We can now discuss the operation we can take between subobjects; none of the constructions described depends on the choice of representative of the equivalence class of monomorphisms.

Definition 2.1.5 (Intersection and union)**.** *Let $A \longrightarrow X$ and $B \longrightarrow X$ be subobjects, we define the **intersection** $A \cap B$ as the following pullback (when it exists):*

$$\begin{tikzcd}
 & A \cap B \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dd, dashed] & \\
A \arrow[dr] & & B \arrow[dl] \\
 & X. &
\end{tikzcd}$$

*We can also define the **sum** $A + B$ of two subobjects as the following pushout (when it exists):*

$$\begin{tikzcd}
 & A \cap B \arrow[dl] \arrow[dr] & \\
A \arrow[dr, dashed] \arrow[ddr] & & B \arrow[dl, dashed] \arrow[ddl] \\
 & A + B \arrow[d, dashed] & \\
 & X. &
\end{tikzcd}$$

Remark 2.1.6. *The preorder relation on monomorphisms induces a partial order on the subobjects of a specific object; the intersection is the meet of this poset, while the sum is the join.*

We can dualize all the previous constructions, to obtain the following definition.

Definition 2.1.7 (Quotient)**.** *Let $B \in \operatorname{ob} \mathcal{C}$, we define a **quotient** of B to be an equivalence class of epimorphisms from B.*

All the dual results obviously hold.

2.2 Additive categories

We start this section introducing additive categories and functors between them.

Definition 2.2.1 (Additive category)**.** *An **additive category** is a (locally small) category $\mathcal{C}$ such that:*

- *for every couple of objects $A, B \in \operatorname{ob}\mathcal{C}$ the set of arrows $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is an abelian group such that composition distributes over its group operation;*
- *$\mathcal{C}$ has binary products.*
- *$\mathcal{C}$ has a zero object, that is an object that is both initial and terminal.*

We will refer to the identity element of $\operatorname{Hom}_{\mathcal{C}}(A, B)$ as the 0-morphism.

Definition 2.2.2 (Additive functor)**.** *An **additive functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between additive categories such that for every pair of objects $A, B \in \operatorname{ob}\mathcal{C}$ the induced morphism*

$$F_{A,B} : \operatorname{Hom}_{\mathcal{C}}(A, B) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(FA, FB)$$

is a group homomorphism.

We will now let $\mathcal{C}$ be an additive category.

Remark 2.2.3. *From the distributivity of composition we deduce that composition must be bilinear; therefore for every triplet of objects A, B, C we get a morphism of abelian groups*

$$\operatorname{Hom}_{\mathcal{C}}(A, B) \otimes \operatorname{Hom}_{\mathcal{C}}(B, C) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, C) \tag{2.1}$$

$$f \otimes g \mapsto gf. \tag{2.2}$$

An immediate consequence of this fact is that the composition of 0-morphisms is the 0-morphism.

Proposition 2.2.4. *Let f be a morphism in $\mathcal{C}$, the following are equivalent:*

- *f is monic;*
- *$fg = 0$ implies $g = 0$ for every composable g.*

Proof. Recall that the monomorphisms between two objects A, B are exactly the morphisms f such that for every morphism $h : C \longrightarrow B$ we have at most one morphism $g : C \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "f"] & B \\ C. \arrow[u, "g"] \arrow[ur, "h"'] & \end{tikzcd}$$

Let g_1, g_2 be two such morphisms, then the commutativity of the diagram above is equivalent to requiring that $fg_1 = h = fg_2$, which implies $f(g_1 - g_2) = 0$. Given that h can vary among all morphisms that factor through f, we can set $g = g_1 - g_2$ and get the thesis.

□

The dual of the above proposition is also true.

Proposition 2.2.5. *Let f be a morphism in $\mathcal{C}$, the following are equivalent:*

- *f is epic;*
- *$gf = 0$ implies $g = 0$ for every composable g.*

Suppose that A, B are two objects in $\mathcal{C}$, we must have a (unique) terminal morphism $A \longrightarrow 0$ and a (unique) initial morphism $0 \longrightarrow B$; by uniqueness they must both be 0-morphisms, so their composition is the zero morphism. We have shown that the 0-morphism is the unique morphism that factors through the zero object.

We can easily check that every binary product $A \times B$ is also a binary coproduct, where the canonical morphisms in_1, in_2 are given by $(\mathrm{id}_A, 0)$ and $(0, \mathrm{id}_B)$.

$$\begin{tikzcd}
& A \arrow[d, dashed, "\mathrm{in}_1"] \arrow[ddl, "\mathrm{id}_A"'] \arrow[ddr, "0"] & \\
& A \times B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}$$

We will denote the object that is both the product and the coproduct with the direct sum symbol: we have $A \times B \cong A \amalg B \cong A \oplus B$, along all the standard relationships between the canonical morphisms.

$$\begin{tikzcd}
A \arrow[dr, "\mathrm{in}_1"] \arrow[dd, "\mathrm{id}_A"] & & B \arrow[dl, "\mathrm{in}_2"'] \arrow[dd, "\mathrm{id}_B"] \\
& A \oplus B \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "\mathrm{in}_1"] \arrow[dd, "0"] & & B \arrow[dl, "\mathrm{in}_2"'] \arrow[dd, "0"] \\
& A \oplus B \arrow[dl, "\pi_2"] \arrow[dr, "\pi_1"'] & \\
B & & A
\end{tikzcd}$$

We also denote the morphisms produced from f, g (resp. f', g') by the universal property of the (co)product as (f, g) (resp. $f + g$), taking care to note that $f + g$ has not (yet) any relation with the sum in $\mathrm{Hom}(A \oplus B, B)$.

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, dashed, "{(f,g)}"] \arrow[dr, "g"] & \\
A & A \oplus B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{in}_1"] \arrow[dr, "f'"'] & A \oplus B \arrow[d, dashed, "f+g"] & B \arrow[l, "\mathrm{in}_2"'] \arrow[dl, "g'"] \\
& Y &
\end{tikzcd}$$

We will also denote the diagonal morphism $(\mathrm{id}_A, \mathrm{id}_A)$ as $\Delta_A \colon A \longrightarrow A \oplus A$ and the codiagonal morphism $\mathrm{id}_B + \mathrm{id}_B$ as $\nabla_B \colon B \oplus B \longrightarrow B$.
With an analogous construction we get direct sum of morphisms $f \oplus g$, with all the relevant properties.

$$\begin{tikzcd}
 & A \arrow[r, "f"] & B \arrow[dr] & \\
A \oplus A' \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rrr, "f \oplus g"] & & & B \oplus B' \\
 & A' \arrow[r, "g"'] & B' \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[dr, "\mathrm{in}_1"'] \arrow[rrr, "f"] & & & B \\
 & A \oplus A' \arrow[r, "f \oplus g"] & B \oplus B' \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A' \arrow[ur, "\mathrm{in}_2"] \arrow[rrr, "g"] & & & B'
\end{tikzcd}$$

Remark 2.2.6. *Let us consider the morphism $\pi_1 + \pi_2 \colon B \oplus B \longrightarrow B$ given by the additive structure on $\mathrm{Hom}(B \oplus B, B)$; we have that $(\pi_1 + \pi_2)i_1 = \pi_1 \mathrm{in}_1 + \pi_2 \mathrm{in}_1 = \mathrm{id}_B + 0 = \mathrm{id}_B$ and that $(\pi_1 + \pi_2)i_2 = \mathrm{id}_B$, therefore we can write $\nabla_B = \pi_1 + \pi_2$. A consequence of this result is that, given two morphisms $f, g \colon A \longrightarrow B$, we have*

$$f + g = \nabla_B (f \oplus g) \Delta_A.$$

Definition 2.2.7 (Kernel and cokernel)**.** *Let $f : A \longrightarrow B$ be an arrow of $\mathcal{C}$, the* ***kernel*** *of f is the equalizer*

$$\ker(f) \longrightarrow A \underset{0}{\overset{f}{\rightrightarrows}} B,$$

while the ***cokernel*** *is the coequalizer of the diagram*

$$A \underset{0}{\overset{f}{\rightrightarrows}} B \longrightarrow \mathrm{coker}(f).$$

We will often refer to both the object and the arrow of the equalizer as the kernel. We also define the image of f, denoted $\mathrm{im}\, f$, as $\ker(\mathrm{coker}(f))$ and the coimage of f, denoted $\mathrm{coim}(f)$, as $\mathrm{coker}(\ker(f))$.

Example 2.2.8. *let $0 \colon A \longrightarrow B$ be the 0 morphism, then $\ker(0) = A$ and $\mathrm{coker}(0) = B$ (with the identity morphism).*

Remark 2.2.9. *Due to the fact that every equalizer is monic (and every coequalizer is epic), it is immediate to see that the kernel of a morphism is a subobject, while the cokernel is a quotient. This also implies that the image is a subobject while the coimage is a quotient.*

Applying the relevant universal properties it is easy to see that there exists a canonical morphism $\mathrm{coim}(f) \longrightarrow \mathrm{im}(f)$ such that

$$\begin{tikzcd}
\ker(f) \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] & \mathrm{coker}(f) \\
& \mathrm{coim}(f) \arrow[r] & \mathrm{im}(f) \arrow[u] &
\end{tikzcd}$$

commutes.

2.3 Abelian Categories

Definition 2.3.1 (Abelian category). *An* ***abelian category*** *is an additive category $\mathcal{C}$ that satisfies the following axioms:*

(AB1) *any morphism admits a kernel and a cokernel;*

(AB2) *for every morphism $f : A \longrightarrow B$, the canonical morphism* $\mathrm{coim}(f) \longrightarrow \mathrm{im}(f)$ *is an isomorphism.*

The existence of the kernel and the cokernel is remarkably powerful: the equalizer of f and g is exactly the kernel of $f - g$, while their coequalizer is its cokernel; therefore in any abelian category equalizers and coequalizers must exist. It is a well known result that it is possible to construct a finite limit using equalizers and finite products, thus every abelian category has finite limits and (dually) finite colimits. One consequence of this result is the fact that we can always take the intersection and the sum of subobjects, therefore the subobjects of a given object form a lattice.

Remark 2.3.2. *Let f be a monomorphism, because of proposition 2.2.4 every morphisms that equalizes f and the* 0*-morphism must be the* 0*-morphism itself; given that we also know that the* 0*-morphism factors through the* 0*-object, we conclude that the kernel of f must be* 0*, and that this is a sufficient condition for f to be monic.*
The same can be said for the cokernel of an epimorphism.

From the previous remark follows that the coimage of the monomorphism $f : A \longrightarrow B$ is exactly A (or more precisely the identity morphism $A \longrightarrow A$), therefore by applying (AB2) we deduce that every monomorphism is its image and therefore that it is a kernel (as a subobject).

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r] & \mathrm{coker}(f) \\ & A \arrow[r, "\cong"] & \mathrm{im}(f) \arrow[u] & \end{tikzcd}$$

Dually every epimorphism is its coimage and therefore it is a cokernel.

We will now state a standard result that we will often need.

Proposition 2.3.3. *Epimorphisms are stable under pullback, while monomorphisms are stable under pushout.*

Proof. See [Sta25, Tag 08N4]. □

We continue by introducing one of the main objects used in the study of abelian categories: exact sequences.

Definition 2.3.4 (Exact sequence)**.** *Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be morphisms in $\mathcal{C}$, we say that the sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

*is **exact** if $\mathrm{im}(f)$ and $\ker(g)$ are the same subobject of B. More generally if we have a succession of morphisms*

$$\cdots \longrightarrow A_{k-1} \xrightarrow{f_{k-1}} A_k \xrightarrow{f_k} A_{k+1} \longrightarrow \cdots$$

*we say that it is exact at A_k if $A_{k-1} \longrightarrow A_k \longrightarrow A_{k+1}$ is exact; we say that it is exact if it is exact at A_k for every k. A **short exact sequence** is a sequence of the form*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

that is exact at A, B and C; we will often shorten the name short exact sequence as SES.

Let us now consider a SES

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0;$$

exactness in A implies that $\ker(i) = \mathrm{im}(0) = 0$, therefore i is monic; similarly p is epic.
If we take a monomorphism $i\colon A \longrightarrow B$, we can obtain a SES

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow \mathrm{coker}(i) \longrightarrow 0.$$

Similarly for an epimorphism p we get

$$0 \longrightarrow \ker(p) \longrightarrow B \xrightarrow{p} C \longrightarrow 0.$$

The study of exact sequences is a very large and complex field that cannot be summarized in a couple of short pages, a fantastic reference is [Wei94].

2.4 Extensions

In this section we will essentially follow [Wei94] and [Mac12]. Let $\mathcal{C}$ be an abelian category, let A, B be objects of $\mathcal{C}$.

Definition 2.4.1 (Extension)**.** *An **extension** ξ of A and B is a short exact sequence of the form*

$$\xi\colon \qquad 0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0.$$

*More generally, for $n \geq 1$, an n-**extension** ξ of A and B is an exact sequence*

$$\xi\colon \qquad 0 \to B \to X_{n-1} \to X_{n-2} \to \cdots \to X_0 \to A \to 0$$

We can now define an equivalence relation on the class of n-extensions as follows: we say that the extension ξ and ξ' are equivalent if there is a commutative diagram

$$\begin{array}{llllllllllllllll}
\xi\colon & 0 & \to & B & \to & X_{n-1} & \to & X_{n-2} & \to & \cdots & \to & X_0 & \to & A & \to & 0 \\
& & & \uparrow{\scriptstyle \mathrm{id}_B} & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}_A} & & \\
& 0 & \to & B & \to & X''_{n-1} & \to & X''_{n-2} & \to & \cdots & \to & X''_0 & \to & A & \to & 0 \\
& & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} & & \\
\xi'\colon & 0 & \to & B & \to & X'_{n-1} & \to & X'_{n-2} & \to & \cdots & \to & X'_0 & \to & A & \to & 0.
\end{array}$$

Remark 2.4.2. *If $n = 1$, the relation is much simpler: in the following diagram both middle morphisms must necessarily be isomorphism, therefore we need only to look for a morphism $X \to X'$ that makes sure that the following diagram commutes.*

$$\begin{array}{llllllllll}
\xi\colon & 0 & \to & B & \to & X & \to & A & \to & 0 \\
& & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} & & \\
\xi'\colon & 0 & \to & B & \to & X' & \to & A & \to & 0
\end{array}$$

Definition 2.4.3 (Split extensions)**.** *We say that an extension of A and B is **split** if it is equivalent to*

$$0 \longrightarrow B \xrightarrow{i_A} B \oplus A \xrightarrow{\pi_B} A \longrightarrow 0.$$

It is easy to see that an extension $B \longrightarrow X \longrightarrow A$ is split if and only if there exists an isomorphism $f\colon X \longrightarrow B \oplus A$ such that

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "i"] \arrow[dr, "\mathrm{in}_1"'] & X \arrow[r] \arrow[d, "f", "\cong"'] & A \arrow[r, "p"] & 0 \\
& & B \oplus A \arrow[ur, "\pi_2"'] & &
\end{tikzcd}$$

commutes. From this formulation follows that an extension is split if and only if there exists a retraction of i (i.e. a morphism $q\colon X \longrightarrow B$ such that $qi = \mathrm{id}_B$) or a section of p (i.e. a morphism $j\colon A \longrightarrow X$ such that $pj = \mathrm{id}_A$).

Definition 2.4.4 (Ext). *We define $\mathrm{Ext}^n_{\mathcal{C}}(A, B)$ to be the quotient of the family of all n-extensions by the relation of equivalence. We will also define $\mathrm{Ext}^0_{\mathcal{C}}(A, B)$ as $\mathrm{Hom}_{\mathcal{C}}(A, B)$. In the cases where it is obvious from the context, we will refrain from writing the subscript.*

Remark 2.4.5. *$\mathrm{Ext}^n_{\mathcal{C}}(A, B)$ is not necessarily small (i.e. a set), see [Sta25, Section 07JS], but it will be in the relevant cases (Grothendieck categories). From now on, in this section, we will assume it is a set.*

Remark 2.4.6. *It is clear that $\mathrm{Ext}^0_{\mathcal{C}}(A, B)$ is an abelian group, given that $\mathcal{C}$ is an additive category. Our objective will be to show that this is always the case.*

We will show that Ext is actually a functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\underline{\mathrm{Set}}$, in the case $n = 1$ (for the general case see [Mac12]). If we are given an extension

$$0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$$

and $h\colon B \longrightarrow B'$ we can take the pushout $B' \times_B X$; then, applying the relevant universal property to the morphisms $0\colon B' \longrightarrow A$ and $g\colon X \longrightarrow A$, we get a a commuting diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "f"] \arrow[d, "h"] & X \arrow[r, "g"] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & B' \arrow[r, "f'"] & B' \amalg_B X \arrow[r, "g'"] & A \arrow[r] & 0.
\end{tikzcd}$$

Therefore, by passing to the quotient, we get a (natural) map

$$h_*\colon \mathrm{Ext}^1_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(A, B').$$

Similarly, given a morphism $k : A' \longrightarrow A$, we get a (natural) map

$$k^*\colon \mathrm{Ext}^1_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(A', B).$$

We will now introduce an operation, called the *Baer sum*, on $\mathrm{Ext}^n_{\mathcal{C}}(A, B)$: given two elements in $\mathrm{Ext}^n_{\mathcal{C}}(A, B)$ with representatives

$$\xi\colon \qquad 0 \to B \to X_{n-1} \to X_{n-2} \to \cdots \to X_0 \to A \to 0$$

$$\xi'\colon \qquad 0 \to B \to X'_{n-1} \to X'_{n-2} \to \cdots \to X'_0 \to A \to 0$$

and define their direct sum as

$$\xi \oplus \xi' \colon \qquad 0 \to B \oplus B \to X_{n-1} \oplus X'_{n-1} \to \cdots \to X_0 \oplus X'_0 \to A \oplus A \to 0;$$

then the sum $[\xi] + [\xi']$ is given by $[(\nabla_B)_* (\Delta_A)^* (\xi \oplus \xi')]$.

Theorem 2.4.7. *Baer sum makes the sets* $\mathrm{Ext}^n_{\mathcal{C}}$ *(for* $n \geq 1$*) into abelian groups, where the identity is given by the class of a split extension for* $n = 1$ *and otherwise by the complex*

$$0 \longrightarrow B \xrightarrow{\mathrm{id}_B} B \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0.$$

2.5 Injective and projective objects

Definition 2.5.1 (Injective and projective objects)**.** *Let* $\mathcal{H}$ *be a subclass of* $\mathrm{ob}(\mathcal{C})$*. We say that an object* B *is* $\mathcal{H}$***-injective*** *if for every* $A \in \mathcal{H}$ *we have*

$$\mathrm{Ext}^1_{\mathcal{C}}(A, B) = 0.$$

Dually we say that object A *is said to be* $\mathcal{H}$***-projective*** *if for every* $B \in \mathcal{H}$ *we have*

$$\mathrm{Ext}^1_{\mathcal{C}}(A, B) = 0.$$

If $\mathcal{H} = \mathrm{ob}(\mathcal{C})$ *the* $\mathcal{H}$*-injectives objects are simply called injective objects and the* $\mathcal{H}$*-projective objects are simply called projective objects. We will denote the class of injective objects of* $\mathcal{C}$ *as* $\mathrm{inj}(\mathcal{C})$ *and the class of projective objects as* $\mathrm{proj}(\mathcal{C})$*.*

From the definition it follows immediately that an object B is $\mathcal{H}$-injective if and only if for every $A \in \mathcal{H}$ every sequence

$$0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$$

splits. The dual result is similar.

Proposition 2.5.2. *An object* P *is projective if and only if for every epimorphism* $e \colon E \longrightarrow X$ *and every morphism* $f \colon P \longrightarrow X$ *there exists a morphism* $\bar{f} \colon P \longrightarrow E$ *such that*

$$\begin{tikzcd} & E \arrow[d, "e"] \\ P \arrow[ur, dashed, "\bar{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Proof. ($\Rightarrow$) Let P be a projective object and let $e\colon E \longrightarrow X$ be an epimorphism and $f\colon P \longrightarrow X$ be a morphism; by taking a pullback

$$\begin{tikzcd} P \times_X E \arrow[r, "f'"] \arrow[d, "e'"'] & E \arrow[d, "e"] \\ P \arrow[r, "f"'] & X, \end{tikzcd}$$

we get an epimorphism $e'\colon P \times_X E \longrightarrow P$ (see proposition 2.3.3). By considering the exact sequence associated to it

$$\begin{tikzcd} 0 \arrow[r] & \ker(e') \arrow[r] & P \times_X E \arrow[r, "e'"] & P \arrow[r] \arrow[l, bend left, "g"] & 0 \end{tikzcd}$$

we get a section $g\colon P \longrightarrow P \times_X E$, then $\bar{f} = f'g$.
($\Leftarrow$) We are given a SES

$$0 \longrightarrow A \longrightarrow B \xrightarrow{q} P \longrightarrow 0$$

and we want to find a section of q. The necessary morphism is found by noticing that q must be epic and applying the hypothesis to the following diagram:

$$\begin{tikzcd} & B \arrow[d, "q"] \\ P \arrow[ur, dashed] \arrow[r, "\mathrm{id}_P"'] & P. \end{tikzcd}$$

□

The dual result is also true.

Proposition 2.5.3. *An object I is injective if and only if for every monomorphism $e\colon X \longrightarrow E$ and every morphism $f\colon X \longrightarrow I$ there exists a morphism $\bar{f}\colon E \longrightarrow I$ such that*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e"'] & I \\ E \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

commutes.

One of the main use of injective and projective objects in abelian categories is defining derived functors, a concept that we will sometimes need; a good reference on the subject is [Wei94].

2.6 Grothendieck Categories

Definition 2.6.1. *Any abelian category $\mathcal{C}$ can also satisfy the following axioms:*

(AB3) $\mathcal{C}$ has all (small) coproducts (and therefore is cocomplete);

(AB4) $\mathcal{C}$ satisfies (AB3) and the coproduct of monomorphisms is a monomorphism;

(AB5) $\mathcal{C}$ satisfies (AB3) and filtered colimits in $\mathcal{C}$ are exact;

(AB3) $\mathcal{C}$ has all (small) products (and therefore is complete);*

(AB4) $\mathcal{C}$ satisfies (AB3*) and the product of epimorphisms is an epimorphism;*

(AB5) $\mathcal{C}$ satisfies (AB3*) and filtered limits in $\mathcal{C}$ are exact.*

It is clear that axiom (AB4) is equivalent to (AB3) plus the exactness of coproducts, while (AB4*) is equivalent to (AB3*) plus the exactness of products. Writing

$$\bigoplus_{i\in I} X_i = \varinjlim_{F\subseteq I \text{ finite}} \bigoplus_{j\in F} X_j$$

shows (AB5) $\Rightarrow$ (AB4); the dual argument also works.

Definition 2.6.2. *A **generator** $\mathcal{G}$ of a category $\mathcal{C}$ is an object G which satisfies the following property: for every pair of distinct morphisms $f, g : A \longrightarrow B$, there is a morphism $s\colon G \longrightarrow A$ such that $fs \neq gs$. A **family of generators** is a nonempty set $\mathcal{G} = \{G_i\}_i$ which satisfies the following property: for every pair of distinct morphisms $f, g : A \longrightarrow B$, there is an object $G_i \in \mathcal{G}$ and a morphism $s\colon G_i \longrightarrow A$ such that $fs \neq gs$.*

Remark 2.6.3. *It is easy to see that an object G is a generator if and only if the functor* $\operatorname{Hom}(G, -)$ *is faithful, therefore for any generator G we have that* $\operatorname{Hom}(G, X) = 0$ *implies $X = 0$. In an abelian category there is a slightly more interesting equivalent characterization: an object G is a generator if and only if* $\operatorname{Hom}(G, f) = f_*$ *being surjective implies that f is epic; similarly $\mathcal{G}$ is a family of generators if and only if* $\operatorname{Hom}(G, f) = f_*$ *being surjective for all $G \in \mathcal{G}$ implies that f is epic. If we are working in an (AB3) category, the existence of a family of generators implies the existence of a generator: we only need to set*

$$G = \bigoplus_{G_i\in\mathcal{G}} G_i.$$

Definition 2.6.4 (Grothendieck category)**.** *An abelian category $\mathcal{C}$ is called a **Grothendieck category** if it satisfies (AB5) and possesses a generator.*

We will now state some basic results on Grothendieck categories.

Theorem 2.6.5. *Every Grothendieck category is complete.*

Proof. See [Bor94]. □

Theorem 2.6.6. *Every Grothendieck category is well-powered.*

Proposition 2.6.7 (Baer's criterion)**.** *Let $\mathcal{C}$ be a Grothendieck category and let G be a generator of $\mathcal{C}$. An object I is injective if and only if for all subobjects $M \subseteq G$ and for all morphisms $f\colon G \longrightarrow I$, there exists a morphism $g'\colon M \longrightarrow I$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & I \\ G \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

commutes.

Theorem 2.6.8. *Every Grothendieck category has enough injectives.*

For the proofs of these last three theorems see [Sta25, Section 05AB].

Chapter 3

Model categories

We will introduce the concept of model categories and homotopy categories to provide a method to localize certain categories without recurring to the calculus of fractions. The main references we will be using are [DS95] and [Hov07].

3.1 Basic definitions

Let $\mathcal{C}$ be a category.

Definition 3.1.1 (Lift). *Given a commuting square in* $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] & Y, \end{tikzcd}$$

*a **lift** is a morphism* $f\colon B \longrightarrow X$ *such that*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, "f"] & Y \end{tikzcd}$$

commutes. In the case where a lift exists we will say that i *has the **left lifting property** (shortened as LLP) with respect to* p*; dually we will say that* p *has the **right lifting property** (shortened as RLP) with respect to* i*.*

Definition 3.1.2 (Retract). *Let* X, Y *be objects in* $\mathcal{C}$*, we say that* X *is a retract of* Y *if there exists arrows* $f\colon X \longrightarrow Y$ *and* $g\colon Y \longrightarrow X$ *such that* $gf = \mathrm{id}_X$*. In such cases we say that* f *is a **section** and* g *is a **retraction**.*

Remark 3.1.3. *The case we are most interested in is when we are taking retracts in a category of morphisms, unraveling the definition we get the following: if*

$\mathcal{C} = \mathrm{Ar}(\mathcal{D})$ *then* f *is a retract of* g *if and only if there exists* i, i', p, p' *such that*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"] & Y \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"] \arrow[rr, bend right, "\mathrm{id}_{X'}"'] & Y' \arrow[r, "p'"] & X' \end{tikzcd}$$

commutes.

We will now state a lemma that we will often use in dealing with model categories.

Lemma 3.1.4 (The retract argument)**.** *Let* $f\colon A \longrightarrow C$ *be an arrow of* $\mathcal{C}$ *that can be factored as* $f = pi$ *and suppose that* f *has the LLP with respect to* p*. Then* f *is a retract of* i*. Dually, if* f *has the RLP with respect to* i*, it is a retract of* p*.*

Proof. It suffices to prove the first case, the second case is dual to the one shown. Let us begin by observing that requiring that $f = pi$ is equivalent to ensuring the commutativity of the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"'] \arrow[d, "f"] & B \arrow[d, "p"'] \\ C \arrow[r, "\mathrm{id}_C"] & C. \end{tikzcd}$$

Applying the LLP to the square and expanding the diagram we get

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "f"] & A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "i"'] & A \arrow[d, "f"'] \\ C \arrow[r, "g"] & B \arrow[r, "p"] & C, \end{tikzcd}$$

where g is the morphism that we get from the LLP. This shows exactly the thesis. □

We can now introduce the main object that we will discuss.

Let us consider a triple $(W, \mathrm{Fib}, \mathrm{Cofib})$ of subclasses of $\mathrm{Ar}(\mathcal{C})$, called respectively weak equivalences ($\xrightarrow{\sim}$), fibrations ($\twoheadrightarrow$) and cofibrations ($\hookrightarrow$); we will name the class $W \cap \mathrm{Fib}$ as acyclic fibrations ($\overset{\sim}{\twoheadrightarrow}$) and $W \cap \mathrm{Cofib}$ as acyclic cofibrations ($\overset{\sim}{\hookrightarrow}$).

3.2 Model categories

Definition 3.2.1 (Model structure)**.** *They are said to form a* ***model structure*** *on $\mathcal{C}$ if they satisfy the following axioms:*

(MC1) *If f and g are composable morphisms of $\mathcal{C}$ and two out of three between f, g, gf are weak equivalences then also the third is.*

(MC2) *W,* Fib *and* Cofib *are closed under retractions.*

(MC3) *Trivial cofibrations have the LLP with respect to the fibrations, while the trivial fibrations have the RLP with the respect to the cofibrations.*

(MC4) *Every arrow f admits both of the following factorizations: (1) $f = pi$ where i is an acyclic cofibration and p is a fibration, and (2) $f = pi$ where i is a cofibration and p is an acyclic fibration.*

Definition 3.2.2 (Model category)**.** *A* ***model category*** *$\mathcal{C}$ is a complete and cocomplete category along with a model structure on it. We will denote its weak equivalences as $W(\mathcal{C})$, its fibrations as $\mathrm{Fib}(\mathcal{C})$ and its cofibrations as $\mathrm{Cofib}(\mathcal{C})$.*

Remark 3.2.3. *Sometimes in the definition of model category it is required that the factorization be functorial; we will not make such demand.*

We will from now on use $\mathcal{C}$ to refer to a model category.

Proposition 3.2.4. *In a model category, the cofibrations (resp. trivial cofibrations) are exactly the morphisms that have the LLP with respect to all trivial fibrations (resp. fibrations). Dually the fibrations (resp. trivial fibrations) are exactly the morphisms that have the RLP with respect to all trivial cofibrations (resp. cofibrations).*

Proof. We will only show the characterization of cofibrations, the other are analogous. By (**MC3**) cofibrations must have such property, therefore we need only to show the converse: let $f : A \longrightarrow B$ be a morphism that has the LLP with respect to all trivial fibrations, by applying (**MC4**) we can obtain a factorization $f = pi$ where p is a trivial fibration and i is a cofibration; by applying the retract argument and using (**MC2**) we conclude. □

Corollary 3.2.4.1. *Every isomorphism is both a trivial fibration and a trivial cofibration. This in particular is true for the identity morphisms.*

Proof. The corollary follows immediately from the proposition by using the inverse morphism to construct lifts. □

Corollary 3.2.4.2. *The class of weak equivalences, fibrations and cofibrations is closed under composition.*

Proof. The case of weak equivalences is a consequence of (**MC1**), we will prove the case of fibrations (the other is dual). Using the proposition, we need only to show that the composition of two fibrations p, p' has the RLP with respect to any acyclic cofibration i. To do so, it is sufficient to apply (**MC3**) twice as in the diagram below, first to get f and then again in the new square to obtain g.

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, hook, "\sim"', "i"] & X \arrow[d, two heads, "p"] \\
 & Y \arrow[d, two heads, "p'"] \\
B \arrow[r] \arrow[ur, dashed, "f"'] \arrow[uur, dashed, "g"] & Z
\end{tikzcd}$$

□

Corollary 3.2.4.3. *Being a fibration (resp. cofibration) is a property that is invariant with respect to taking pullbacks (resp. pushouts). The same is true for acyclic (co)fibrations.*

Proof. Let us only show the case with fibrations: let $p\colon A \twoheadrightarrow B$ be a fibration, then the universal property of the pullback implies that if p has the RLP with respect to a morphism, then we can say the same for the pullback (see the diagram below).

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook, "i"] & A \times_{B'} B \arrow[r] \arrow[d] & A \arrow[d, two heads, "p"] \\
Y \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & B \arrow[r] & B'
\end{tikzcd}$$

□

Definition 3.2.5 (Fibrant and cofibrant objects)**.** *An object A is said to be **fibrant** if the terminal morphism $A \longrightarrow *$ is a fibration; an object B is said to be **cofibrant** if the initial morphism $\emptyset \longrightarrow B$ is a cofibration.*

Remark 3.2.6. *Let A, B be cofibrant objects, given that the coproduct $A \amalg B$ can be seen as the pushout with respect to the initial object, we deduce that both* in_1 *and* in_2 *must be cofibrations.*

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{in}_1"] \\
B \arrow[r, "\mathrm{in}_2"'] & A \amalg B
\end{tikzcd}$$

The dual result is also true: if A, B are fibrant then both projections

$$\pi_1\colon A \times B \longrightarrow A$$
$$\pi_2\colon A \times B \longrightarrow B$$

must be fibrations.

Lemma 3.2.7 (Ken Brown's lemma)**.** *Let $\mathcal{D}$ be a category with a class of weak equivalences that satisfy (**MC1**). Suppose $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor which takes acyclic cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences.*
Dually, if F takes acyclic fibrations between fibrant objects to weak equivalences, then it takes weak equivalences between fibrant objects to weak equivalences.

Proof. We will only prove the case for acyclic cofibrations; the other case is dual. Let $f : A \longrightarrow B$ be a weak equivalence between cofibrant objects. Using (**MC4**) we can factor f as

$$A \amalg B \overset{j}{\hookrightarrow} C \overset{p}{\underset{\sim}{\twoheadrightarrow}} B,$$

where p is an acyclic fibration and j is a cofibration. By applying (**MC1**) we see that $j \circ \mathrm{in}_1$ and $j \circ \mathrm{in}_2$ are necessarily acyclic cofibrations.

$$\begin{tikzcd}
A \arrow[d, hook, "\mathrm{in}_1"'] \arrow[dr, "j\,\mathrm{in}_1", "\sim"'] \arrow[drr, bend left, "f"] & & \\
A \amalg B \arrow[r, hook, "j"] & C \arrow[r, two heads, "p", "\sim"'] & B \\
B \arrow[u, hook', "\mathrm{in}_2"] \arrow[ur, "j\,\mathrm{in}_2"', "\sim"] \arrow[urr, bend right, "\mathrm{id}_B"'] & &
\end{tikzcd}$$

Using our hypothesis we deduce that $F(j \circ \mathrm{in}_1)$ and $F(j \circ \mathrm{in}_2)$ are weak equivalences; applying (**MC1**) to $F(p) \circ F(j \circ \mathrm{in}_2) = \mathrm{id}_{F(B)}$ we get that $F(p)$ is a weak equivalence. We conclude by applying (**MC1**) to $F(p)F(j \circ \mathrm{in}_1) = F(f)$ to obtain the fact $F(f)$ is a weak equivalence. □

Lemma 3.2.8. *Let A be an object of $\mathcal{C}$. If the functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ takes acyclic fibrations to surjections, A is cofibrant; if the functor $\mathrm{Hom}_{\mathcal{C}}(-, A)$ takes acyclic cofibrations to surjections, A is fibrant.*
If instead the functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ takes fibrations to surjections, A is cofibrant and the initial map is a weak equivalence; if the functor $\mathrm{Hom}_{\mathcal{C}}(-, A)$ takes cofibrations to surjections, A is fibrant and the terminal map is a weak equivalence.

Proof. Let us assume that the functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ takes acyclic fibrations to surjections and let $f : X \longrightarrow Y$ be an acyclic fibration. If we consider a square as follows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[d, two heads, "f", "\sim"'] \\
A \arrow[r, "h"'] & Y
\end{tikzcd}$$

we see $h \in \mathrm{Hom}(A, Y)$. Given that $\mathrm{Hom}(A, f) : \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, Y)$ must be surjective, we have deduced that $\mathrm{Hom}(A, X)$ must be non-empty, therefore a lift exists in such a diagram. We conclude by proposition 3.2.4 that $\emptyset \longrightarrow A$ is a cofibration. The other cases are analogous. □

3.3 Homotopy relations

Definition 3.3.1 (Cylinder objects)**.** *Let A be an object of $\mathcal{C}$; a **cylinder object** is an object $A \wedge I$ (when we consider multiple cylinders we can add superscripts/subscripts to I), and a diagram*

$$\begin{tikzcd} A \amalg A \arrow[r, "i"] \arrow[rr, bend right, "\nabla_A"'] & A \wedge I \arrow[r, "q", "\sim"'] & A. \end{tikzcd}$$

*A cylinder object is a **good cylinder object** if the morphism*

$$A \amalg A \xrightarrow{i} A \wedge I$$

is a cofibration.
*A cylinder object is a **very good cylinder object** if the morphism*

$$A \wedge I \xrightarrow[\sim]{q} A$$

is an acyclic fibration. We also denote the morphisms $A \longrightarrow A \wedge I$ as follows:

$$\begin{tikzcd} A \arrow[dr, "\mathrm{in}_1"'] \arrow[drr, bend left, "i_1"] & & \\ & A \amalg A \arrow[r, "i"] & A \wedge I. \\ A \arrow[ur, "\mathrm{in}_2"] \arrow[urr, bend right, "i_2"'] & & \end{tikzcd}$$

Definition 3.3.2 (Path objects)**.** *Let B be an object of $\mathcal{C}$; a **path object** is an object B^I (when we consider multiple paths we can add superscripts/subscripts to I), and a diagram*

$$\begin{tikzcd} B \arrow[r, "j", "\sim"'] \arrow[rr, bend right, "\Delta_B"'] & B^I \arrow[r, "p"] & B \times B. \end{tikzcd}$$

*A path object is a **good path object** if the morphism*

$$B^I \xrightarrow{p} B \times B$$

is a fibration.
*A path object is a **very good path object** if the morphism*

$$B \xrightarrow[\sim]{j} B^I$$

is an acyclic cofibration. We also denote the morphisms $B^I \longrightarrow B$ as follows:

$$\begin{tikzcd}
 & & B \\
B \arrow[r, "j"] \arrow[urr, bend left, "\mathrm{id}_B"] \arrow[drr, bend right, "\mathrm{id}_B"'] & B^I \arrow[r, "p"] \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & B \times B. \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & & B
\end{tikzcd}$$

It is important to observe that for every object A a very good cylinder object exists always: we only need to apply (**MC4**) to $\nabla_A \colon A \amalg A \longrightarrow A$; dually for every object B a very good path object always exists.

Remark 3.3.3. *Given a cylinder $A \wedge I$, we know that $q \circ i_1 = q \circ i \circ \mathrm{in}_1 = \nabla_A \circ \mathrm{in}_1 = \mathrm{id}_A$ (see the diagram). Both id_A and q are weak equivalences, therefore by (**MC1**) we assert that i_1 must also be a weak equivalence (and the same can be said for i_2, p_1, p_2).*

$$\begin{tikzcd}
A \arrow[d, "\mathrm{in}_1"'] \arrow[dr, "i_1"] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\
A \amalg A \arrow[r, "i"] & A \wedge I \arrow[r, "q", "\sim"'] & A \\
A \arrow[u, "\mathrm{in}_2"] \arrow[ur, "i_2"'] \arrow[urr, bend right, "\mathrm{id}_A"'] & &
\end{tikzcd}$$

If $A \wedge I$ is a good cylinder and A is cofibrant then $i_1 = i \circ \mathrm{in}_1$ is the composition of cofibrations, therefore i_1 (and i_2) must be acyclic cofibrations; dually if B^I is a good path object and B is fibrant then p_1, p_2 are acyclic fibrations.

Definition 3.3.4 (Homotopies)**.** *Two morphisms $f, g \colon A \longrightarrow B$ are said to be **left homotopic** (denoted $f \overset{l}{\sim} g$) if there exist a cylinder $A \wedge I$ and a morphism $H \colon A \wedge I \longrightarrow B$ such that*

$$\begin{tikzcd}
A \arrow[dr, "i_1"'] \arrow[drr, bend left, "f"] & & \\
 & A \wedge I \arrow[r, "H"] & B \\
A \arrow[ur, "i_2"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

*commutes. We will refer to the diagram as a **left homotopy**.*
*Similarly, two morphisms $f, g \colon A \longrightarrow B$ are said to be **right homotopic** (denoted*

$f \overset{r}{\sim} g$) if there exist a path B^I and a morphism $H\colon A \longrightarrow B^I$ such that

$$\begin{tikzcd}
& & B \\
A \arrow[r, "H"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & B^I \arrow[ur, "p_1"] \arrow[dr, "p_2"] & \\
& & B
\end{tikzcd}$$

*commutes. We will refer to the diagram as a **right homotopy**.*
*An homotopy is said to be **good** (resp. **very good**) if the cylinder/path is good (resp. very good).*
We say that f and g are homotopic (denoted $f \sim g$) if they are both left and right homotopic.

We will now state some preliminary lemmas, each contains a statement and its dual, so we will only prove one of them.

Lemma 3.3.5. *If $f \overset{l}{\sim} g$ (or $f \overset{r}{\sim} g$) and f is a weak equivalence then g is also a weak equivalence.*

Proof. Let $H : A \wedge I \longrightarrow B$ be a left homotopy between f and g.

$$\begin{tikzcd}
A \arrow[dr, "i_1", "\sim"'] \arrow[drr, bend left, "f", "\sim"'] & & \\
& A \wedge I \arrow[r, "H"] & B \\
A \arrow[ur, "i_2", "\sim"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

By (**MC1**) H is a weak equivalence, so g must also be a weak equivalence. ☐

Lemma 3.3.6. *Let $f, g\colon A \longrightarrow B$ then:*

1. *If $f \overset{l}{\sim} g$ then there exists a good left homotopy from f to g.*

2. *If $f \overset{l}{\sim} g$ and B is fibrant then there exists a very good left homotopy from f to g.*

3. *If $f \overset{r}{\sim} g$ then there exists a good right homotopy from f to g.*

4. *If $f \overset{r}{\sim} g$ and A is cofibrant then there exists a very good right homotopy from f to g.*

Proof. The first statement follows from applying (**MC4**) to the morphism $i\colon A \amalg A \longrightarrow A \wedge I$, where $A \wedge I$ is the cylinder object of a left homotopy between f and g.
Let now hypothesize that B is fibrant and let us take $H'\colon A \wedge I' \longrightarrow B$ to be a good homotopy between f and g. By applying (**MC4**) we can factor the morphism $q\colon A \wedge I' \longrightarrow A$ as

$$A \wedge I' \underset{\sim}{\overset{i''}{\hookrightarrow}} A \wedge I'' \underset{\sim}{\overset{p''}{\twoheadrightarrow}} A.$$

By then finding a lift H'' in the square

$$\begin{tikzcd} A \wedge I' \arrow[r, "H'"] \arrow[d, hook, "i''"] & B \arrow[d, two heads] \\ A \wedge I'' \arrow[r] \arrow[ur, dashed, "H''"'] & * \end{tikzcd}$$

we find our desired very good homotopy. □

Definition 3.3.7. *We define $\pi^l(A,B)$ as the quotient of $\mathrm{Hom}_{\mathcal{C}}(A,B)$ given by the equivalence relation generated by the relation $\overset{l}{\sim}$. We define $\pi^r(A,B)$ as the quotient of $\mathrm{Hom}_{\mathcal{C}}(A,B)$ given by the equivalence relation generated by the relation $\overset{r}{\sim}$. When this relations are the same we will simply write $\pi(A,B)$.*

It is important to notice that $\overset{l}{\sim}$ and $\overset{r}{\sim}$ aren't generally equivalence relations, therefore the word "generated" is of key importance in the definition.

Lemma 3.3.8. *If A is cofibrant, then the relation $\overset{l}{\sim}$ is an equivalence relations on $\mathrm{Hom}_{\mathcal{C}}(A,B)$. If B is fibrant, then the relation $\overset{r}{\sim}$ is an equivalence relations on $\mathrm{Hom}_{\mathcal{C}}(A,B)$.*

Proof. The fact that the relation $\overset{l}{\sim}$ is symmetric is easy, given that we can find an involution for a cylinder that swaps i_1 and i_2, meanwhile showing that it is reflexive necessitates only observing the existence of the trivial cylinder

$$\begin{tikzcd} A \amalg A \arrow[r, "\nabla_A"] \arrow[rr, bend right, "\nabla_A"'] & A \arrow[r, "\mathrm{id}_A", "\sim"'] & A. \end{tikzcd}$$

We now only need to prove that the relation is transitive. Let us suppose that $f \overset{l}{\sim} g$ and that $g \overset{l}{\sim} h$, given by good homotopies $H\colon A \wedge I \longrightarrow B$ and $H'\colon A \wedge I' \longrightarrow B$. We can take the pushout given by

$$A \wedge I' \xleftarrow{i_1'} A \xrightarrow{i_2} A \wedge I$$

to get an object $A \wedge I''$. Using the universal property of the pushout we get a morphism $q'' \colon A \wedge I'' \longrightarrow A$.

$$\begin{tikzcd}
A \arrow[d] \arrow[dr, hook, "i_1", "\sim"'] & & & \\
A \amalg A \arrow[r, "i"] & A \wedge I \arrow[dr] \arrow[drr, bend left, "q", "\sim"'] & & \\
A \arrow[u] \arrow[d] \arrow[ur, hook, "\sim", "i_2"'] \arrow[dr, hook, "i'_1", "\sim"'] & & A \wedge I'' \arrow[r, dotted, "q''"] & A, \\
A \amalg A \arrow[r, "i'"] & A \wedge I' \arrow[ur] \arrow[urr, bend right, "\sim", "q'"'] & & \\
A \arrow[u] \arrow[ur, hook, "\sim", "i'_2"'] & & &
\end{tikzcd}$$

Using (**MC1**) and the fact that acyclic cofibrations are inviariant under pushouts, it is clear that q'' must be a weak equivalence, therefore $A \wedge I''$ is a cylinder. We find a morphism H'' using again the universal property of the pushout.

$$\begin{tikzcd}
 & A \wedge I \arrow[dr] \arrow[drr, bend left, "H"] & & \\
A \arrow[ur, "i_2"'] \arrow[dr, "i'_1"] & & A \wedge I'' \arrow[r, dashed, "H''"] & B \\
 & A \wedge I' \arrow[ur] \arrow[urr, bend right, "H'"'] & &
\end{tikzcd}$$

We conclude the proof, because H'' is a left homotopy between f and h. $\square$

Let us consider a morphism $p \colon X \longrightarrow Y$ and take $f, g \colon A \longrightarrow X$ such that $f \overset{l}{\sim} g$. If we let H be an homotopy from f, g, by postcomposing H with p we get an homotopy between pf and pg.

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend left, "f"] & & & \\
 & A \wedge I \arrow[r, "H"] & X \arrow[r, "p"] & Y \\
A \arrow[ur] \arrow[urr, bend right, "g"'] & & &
\end{tikzcd}$$

Therefore p induces the following map between $\pi^l(A, X)$ and $\pi^l(A, Y)$:

$$\begin{aligned} p_* \colon \pi^l(A, X) &\longrightarrow \pi^l(A, Y) \\ [f] &\mapsto [pf]. \end{aligned}$$

By duality, given $i\colon A \longrightarrow B$, we get a map

$$\begin{aligned} i^*\colon \pi^l(B,X) &\longrightarrow \pi^l(A,X) \\ [f] &\mapsto [fi]. \end{aligned}$$

Lemma 3.3.9. *If A is cofibrant and $p\colon X \longrightarrow Y$ is an acyclic fibration, then $p_*\colon \pi^l(A,X) \longrightarrow \pi^l(A,Y)$ is a bijection. If X is fibrant and $i\colon A \longrightarrow B$ is an acyclic cofibration, then $i^*\colon \pi^l(B,X) \longrightarrow \pi^l(A,X)$ is a bijection.*

Proof. Let A be cofibrant and let $p\colon X \longrightarrow Y$ be an acyclic fibration. Now let $[f] \in \pi^l(A,Y)$ be a (left) homotopy class of morphisms between A and Y. Then the hypotheses we have are exactly the ones needed to show that there exists a lift in the following square.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

Given that $p_*([g]) = [f]$ we have that p_* is surjective.
To prove injectivity, we consider $[f],[g] \in \pi^l(A,X)$ such that $[pf] = [pg]$, therefore, using lemma 3.3.6, we have a good left homotopy $H\colon A \wedge I \longrightarrow X$ between pf and pg. We have defined a diagram where a lift exists by (**MC3**).

$$\begin{tikzcd} A \amalg A \arrow[r, "f+g"] \arrow[d, hook, "i"] & X \arrow[d, two heads, "p"] \\ A \wedge I \arrow[r, "H"'] \arrow[ur, dashed, "K"'] & Y, \end{tikzcd}$$

K is exactly a left homotopy between f and g, therefore $[f] = [g]$.

□

Lemma 3.3.10. *Let $f,g\colon A \longrightarrow B$ be morphisms, then:*

- *If B is fibrant and $h\colon A' \longrightarrow A$ is a morphism then $f \overset{l}{\sim} g$ implies $fh \overset{l}{\sim} gh$.*
- *If A is cofibrant and $h\colon B \longrightarrow B'$ is a morphism then $f \overset{r}{\sim} g$ implies $hf \overset{r}{\sim} hg$.*

Proof. By lemma 3.3.6 we can assume the existence of a very good homotopy $H\colon A \wedge I \longrightarrow B$ between f and g. We then take a good cylinder object $A' \wedge I$ of A':

$$A' \amalg A' \xrightarrow{i'} A' \wedge I \underset{\sim}{\overset{q'}{\twoheadrightarrow}} A'.$$

Using the fact that the following diagram commutes,

$$\begin{tikzcd} A' \amalg A' \arrow[r, "\mathrm{in}_1 h + \mathrm{in}_2 h"'] \arrow[d, "\nabla_{A'}"'] & A \amalg A \arrow[d, "\nabla_A"] \\ A' \arrow[r, "h"] & A \end{tikzcd}$$

we get a diagram

$$\begin{tikzcd}
A' \amalg A' \arrow[r, "\mathrm{in}_1 h + \mathrm{in}_2 h"'] \arrow[d, hook, "i'"] & A \amalg A \arrow[r, hook, "i"'] & A \wedge I \arrow[d, two heads, "q", "\sim"'] \\
A' \wedge I \arrow[r, "\sim", "q'"'] & A' \arrow[r, "h"'] & A
\end{tikzcd}$$

which admits a lift $K\colon A' \wedge I \longrightarrow A \wedge I$ by (**MC3**). The left homotopy we are looking for is no other than HK.

$$\begin{tikzcd}
A' \arrow[r, "h"'] \arrow[d] & A \arrow[d] \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\
A' \amalg A' \arrow[r, "\mathrm{in}_1 h + \mathrm{in}_2 h"'] & A \amalg A \arrow[r, "K"] & A \wedge I \arrow[r, "H"] & B \\
A' \arrow[r, "h"] \arrow[u] & A \arrow[u] \arrow[ur, "i_2"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

□

We would now like to see when composition between morphisms is well defined in the homotopy groups.

Lemma 3.3.11. *If C is fibrant, then morphism composition defines a function*

$$\pi^l(A,B) \times \pi^l(B,C) \longrightarrow \pi^l(A,C)$$
$$([f],[g]) \mapsto [gf].$$

If A is cofibrant, then morphism composition defines a function

$$\pi^r(A,B) \times \pi^r(B,C) \longrightarrow \pi^r(A,C)$$
$$([f],[g]) \mapsto [gf].$$

Proof. We assume that C is fibrant. To check that a map between quotient sets is well defined we only need to show that it is well defined for the generating relation. Let $f \overset{l}{\sim} f'$ and $g \overset{l}{\sim} g'$; we have already shown that $gf \overset{l}{\sim} gf'$ and the previous lemma implies that $g'f' \overset{l}{\sim} gf'$, therefore $gf \overset{l}{\sim} g'f'$. □

This previous results can be morally summarized as: "When all objects are fibrant and cofibrant morphism composition is well behaved with respect to homotopy".

Proposition 3.3.12. *Let $f, g\colon A \longrightarrow B$ be morphism, then:*

- *If A is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*
- *If B is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

Proof. Let $H\colon A \wedge I \longrightarrow B$ be a good left homotopy between f and g and let

$$B \xrightarrow{\ j\ } B^I \xrightarrow{\ p\ } B \times B$$

be a good path object for B. Given that A is cofibrant, we know that the morphism $i_0\colon A \longrightarrow A \wedge I$ must be an acyclic cofibration. By composing with the projections $\pi_1, \pi_2\colon B \times B \longrightarrow B$ and observing that we get f, we can check that the following diagram commutes (where i, p are the maps of the cylinder $A \wedge I$).

$$\begin{tikzcd}
A \arrow[r, "jf"] \arrow[d, hook, "i_0", "\sim"'] & B^I \arrow[d, two heads, "p"] \\
A \wedge I \arrow[r, "(fq,H)"'] & B \times B
\end{tikzcd}$$

By (**MC3**) this diagram admits a lift, let k be such a lift. By remembering that $qi_1 = \mathrm{id}_A$ we see that the following diagram commutes

$$\begin{tikzcd}
 & & & B \\
A \arrow[r, "i_1"] \arrow[urrr, bend left, "f"] \arrow[drrr, bend right, "g"'] & A \wedge I \arrow[r, "k"] \arrow[urr, bend left=10, "fq"] \arrow[drr, bend right=10, "H"] & B^I \arrow[r, "p"] & B \times B \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & & & B
\end{tikzcd}$$

and therefore ki_1 is a right homotopy between f and g. $\square$

It is important to remark that when the hypothesis of this proposition are satisfied the relation $\overset{l}{\sim}$, $\overset{r}{\sim}$ (which are the same) are an equivalence relation, which we will simply denote as $\sim$.

Corollary 3.3.12.1. *Let $f, g\colon A \longrightarrow B$ be morphisms between a cofibrant and a fibrant object and let $A \wedge I$ be a cylinder object for A. If $f \sim g$ then there is a left homotopy between f and g through $A \wedge I$. Similarly if B^I is a path object for B and $f \sim g$ then there is a right homotopy between f and g through B^I.*

Proof. It immediately follows form the fact that in the proof of the proposition we are free to chose which cylinder/path object to use. $\square$

Theorem 3.3.13. *Let $f\colon A \longrightarrow B$ be a morphism between objects that are both cofibrant and fibrant. We have that f is a weak equivalence if and only if there exists a morphism $g\colon B \longrightarrow A$ such that $gf \sim \mathrm{id}_A$ and $fg \sim \mathrm{id}_B$, which we will call an homotopy inverse of f.*

Proof. ($\Rightarrow$) Let f be a weak equivalence. By (**MC4**) we can factor it as

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[rr, bend left, "f"] & C \arrow[r, two heads, "p"] & B, \end{tikzcd}$$

where both i and p are weak equivalences (given that f is one). We can find a left inverse r for i by finding a lift in the following diagram.

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, hook, "i"', "\sim"] & A \arrow[d, two heads] \\ C \arrow[r] \arrow[ur, dashed, "r"'] & \bullet \end{tikzcd}$$

We can now observe that $i^*([ir]) = [iri] = [i] = i^*([\mathrm{id}_C])$; given that lemma 3.3.9 shows that i^* must be a bijection we deduce that $ir \sim \mathrm{id}_C$. By a similar argument we can find a morphism s to use as a homotopy inverse of p, therefore we conclude by setting $g = rs$ and checking that it must be an homotopy inverse of f.
($\Leftarrow$) Let f be a morphism that admits a homotopy inverse g. By invoking (**MC4**) we can find a factorization of f as follows.

$$\begin{tikzcd} \emptyset \arrow[r, hook] & A \arrow[r, hook, "i", "\sim"'] \arrow[rr, bend left, "f"] & C \arrow[r, two heads, "p"] & B \arrow[r, two heads] & \bullet \end{tikzcd}$$

It is immediate to see that C must be both fibrant and cofibrant; to prove that f is a weak equivalence we only need to show that p is one. Let now $H\colon B \wedge I \longrightarrow B$ be a good (left) homotopy between fg and id_B. We can find a lift in the following commutative diagram.

$$\begin{tikzcd} B \arrow[r, "ig"] \arrow[d, hook, "i_1"', "\sim"] & C \arrow[d, two heads, "p"] \\ B \wedge I \arrow[r, "H"'] \arrow[ur, dashed, "H'"'] & B \end{tikzcd}$$

Setting $s = H'i_1$ we get that $ps = \mathrm{id}_B$. Given that i must be a weak equivalence it must admit a homotopy inverse r by the previous case of this theorem. It is clear from the following diagram that H' is an homotopy between s and ig,

$$\begin{tikzcd} B \arrow[d, "i_1"'] \arrow[dr, "ig"] & \\ B \wedge I \arrow[r, "H'"] & C \\ B \arrow[u, "i_2"] \arrow[ur, "s"'] & \end{tikzcd}$$

therefore, using the fact that all objects involved are fibrant and cofibrant, we have

$$sp \sim igp \sim igfr \sim ir \sim \mathrm{id}_C.$$

Given that id_C is a weak equivalence, we have that sp must also be a weak equivalence. Given that p is a retract of sp we have proven thesis.

$$\begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, "p"] & C \arrow[r, "\mathrm{id}_C"] \arrow[d, "sp"] & C \arrow[d, "p"'] \\ B \arrow[r, "s"] & C \arrow[r, "p"] & B \end{tikzcd}$$

□

3.4 Homotopy categories

Our objective in this section is to use the model structure on a category to build another category where weak equivalences are invertible.

Definition 3.4.1. *Let $\mathcal{C}$ be a category and let W be a collection of morphisms of $\mathcal{C}$. A localization of $\mathcal{C}$ at W is a pair $(\mathcal{C}[W^{-1}], Q)$, where $\mathcal{C}[W^{-1}]$ is a category and Q is a functor between $\mathcal{C}$ and $\mathcal{C}[W^{-1}]$, that satisfies the following:*

- *For all $w \in W$, $Q(w)$ is an isomorphism.*
- *For any category A and any functor $F\colon \mathcal{C} \longrightarrow A$ such that $F(w)$ is an iso for all $w \in W$, there exists a functor $F_W\colon \mathcal{C}[W^{-1}] \longrightarrow A$ and a natural isomorphism $F \cong F_W \circ Q$.*
- *The map between functor categories*

$$(-) \circ Q\colon \mathrm{Funct}(\mathcal{C}[W^{-1}], A) \longrightarrow \mathrm{Funct}(\mathcal{C}, A)$$

is full and faithful for every category A.

From now on, $\mathcal{C}$ will be a model category and W will be $W(\mathcal{C})$. To construct $\mathcal{C}[W^{-1}]$ we will need to define the following categories:

- $\mathcal{C}_c$ - the full subcategory of $\mathcal{C}$ where the objects are the cofibrant objects of $\mathcal{C}$.
- $\mathcal{C}_f$ - the full subcategory of $\mathcal{C}$ where the objects are the fibrant objects of $\mathcal{C}$.
- $\mathcal{C}_{cf}$ - the full subcategory of $\mathcal{C}$ where the objects are the objects of $\mathcal{C}$ which are both fibrant and cofibrant.
- $\pi\mathcal{C}_c$ - the category whose objects are cofibrant objects of $\mathcal{C}$ and whose morphisms are right homotopy classes of morphisms.
- $\pi\mathcal{C}_f$ - the category whose objects are fibrant objects of $\mathcal{C}$ and whose morphisms are left homotopy classes of morphisms.

- $\pi\mathcal{C}_{cf}$ - the category whose objects are the objects of $\mathcal{C}$ which are both fibrant and cofibrant and whose morphisms are homotopy classes of morphisms.

It is important to observe that these categories are well defined by lemma 3.3.11 and proposition 3.3.12. It is also important to note that theorem 3.3.13 states that weak equivalences in $\mathcal{C}_{cf}$ are exactly morphisms that admit a homotopy inverse.

Remark 3.4.2. *Given an object A of $\mathcal{C}$, we can apply (**MC4**) to the initial morphism $\emptyset \longrightarrow A$ to get a diagram as follows*

$$\emptyset \hookrightarrow QA \overset{p_A}{\underset{\sim}{\twoheadrightarrow}} A,$$

where p_A is an acyclic fibration and QA is cofibrant. Similarly we can do the same with the terminal map $A \longrightarrow \bullet$ to get a diagram as below,

$$A \overset{i_A}{\underset{\sim}{\hookrightarrow}} RA \twoheadrightarrow \bullet$$

where i_A is an acyclic cofibration and RA is fibrant. It is also important to note that if A was already fibrant then QA must also be fibrant. Similarly if A was already cofibrant, RA will also be cofibrant.

Proposition 3.4.3. *Given a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ there exists a morphism $\tilde{f}\colon QA \longrightarrow QB$ such that the diagram*

$$\begin{tikzcd} QA \arrow[r, "\tilde{f}"] \arrow[d, two heads, "p_A"] & QB \arrow[d, two heads, "p_B"'] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commutes. The map $\tilde{f}$ depends up to left homotopy or up to right homotopy only on f, and is a weak equivalence if and only if f is. If B is fibrant, then $\tilde{f}$ depends up to left homotopy or up to right homotopy only on the left homotopy class of f.

Proof. We find $\tilde{f}$ by finding a lift in the following diagram.

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & QB \arrow[d, two heads, "\sim"', "p_B"] \\ QA \arrow[r, "f\circ p_A"'] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$

We now observe that $[f \circ p_A] = (p_B)_*([\tilde{f}])$ (where brackets indicate left homotopy), therefore by lemma 3.3.9 we have that $[\tilde{f}]$ depends only on f. The statement for right homotopies follows from proposition 3.3.12. The weak equivalence condition follows from observing that f is a weak equivalence if and only if the induced morphism $QA \longrightarrow B$ is. The final statement follows from lemma 3.3.11. $\square$

There dual of the previous lemma is as follows.

Proposition 3.4.4. *Given a morphism* $f\colon A \longrightarrow B$ *in* $\mathcal{C}$ *there exists a morphism* $\bar{f}\colon RA \longrightarrow RB$ *such that the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i_A"'] & B \arrow[d, hook, "i_B"] \\ RA \arrow[r, "\bar{f}"] & RB \end{tikzcd}$$

commutes. The map $\bar{f}$ *depends up to left homotopy or up to right homotopy only on* f*, and is a weak equivalence if and only if* f *is. If* A *is cofibrant, then* $\bar{f}$ *depends up to left homotopy or up to right homotopy only on the right homotopy class of* f*.*

Remark 3.4.5. *The previous two propositions imply that* Q *and* R *are actually functors:*

$$\begin{aligned} Q\colon \mathcal{C} &\longrightarrow \pi\mathcal{C}_c \\ A &\longmapsto QA \\ (f\colon A \to B) &\longmapsto [\tilde{f}\colon QA \to QB], \end{aligned}$$

$$\begin{aligned} R\colon \mathcal{C} &\longrightarrow \pi\mathcal{C}_f \\ A &\longmapsto RA \\ (f\colon A \to B) &\longmapsto [\bar{f}\colon RA \to RB]. \end{aligned}$$

They will respectively be called cofibrant replacement and fibrant replacement. The last statement of the previous proposition also implies that there are also the following functors:

$$\begin{aligned} Q'\colon \pi\mathcal{C}_f &\longrightarrow \pi\mathcal{C}_{cf} \\ A &\longmapsto QA \\ [f\colon A \to B] &\longmapsto [\tilde{f}\colon QA \to QB], \end{aligned}$$

$$\begin{aligned} R'\colon \pi\mathcal{C}_c &\longrightarrow \pi\mathcal{C}_{cf} \\ A &\longmapsto RA \\ [f\colon A \to B] &\longmapsto [\bar{f}\colon RA \to RB]. \end{aligned}$$

We now introduce the category that will be the candidate for the localization we are looking for.

Definition 3.4.6. *The homotopy category* $\mathrm{Ho}(\mathcal{C})$ *of a model category* $\mathcal{C}$ *is the (locally small) category whose objects are the objects of* $\mathcal{C}$ *and whose homsets are*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B) = \mathrm{Hom}_{\pi\mathcal{C}_{cf}}(R'QA, R'QB).$$

We have a functor $\gamma\colon \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ *that sends a morphism* $f\colon A \longrightarrow B$ *to the equivalence class of the morphism* $R'Q(f)\colon R'QA \longrightarrow R'QB$.

Lemma 3.4.7. *If each of the objects* A *and* B *is both fibrant and cofibrant, the map*

$$\gamma\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B)$$

is surjective and induces a natural bijection $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B) \cong \pi(A, B)$.

Proof. Each of the objects $A, B, QA, QB, R'QA, R'QB$ is both fibrant and cofibrant, therefore we can find (by theorem 3.3.13) homotopy inverses p'_A and i'_{QB} of p_A and i_{QB}.

$$\begin{tikzcd}
R'QA \arrow[r, "g"] & R'QB \\
QA \arrow[u, tail, "i_{QA}", "\sim"'] \arrow[r, dashed, "g'"] \arrow[d, two heads, "p_A", "\sim"'] & QB \arrow[u, tail, "i_{QB}", "\sim"'] \arrow[d, two heads, "p_B", "\sim"'] \\
A \arrow[r, dashed, "g''"] & B
\end{tikzcd}$$

After setting $g' := i'_{QB} \circ g \circ i_{QA}$ and $g'' := p'_B \circ g' \circ p'_A$, we want to check if $\gamma(g'') = R'Q(g'') = [g]$. By the uniqueness results in the two previous propositions and the fact the diagram above commutes up to homotopy we have that $R'(Q(g'')) = R'([g']) = [g]$. The last result also follows from the two previous propositions. □

It is important to remark that from the proof of the above lemma the following is clear: any morphism $f\colon A \longrightarrow B$, even when A, B are not necessarily fibrant/cofibrant, can be written as

$$f = \gamma(p_B)\gamma(i_{QB})^{-1}\gamma(f')\gamma(i_{QA})\gamma(p_A)^{-1},$$

where $f'\colon RQ(A) \longrightarrow RQ(B)$ is a morphism of $\mathcal{C}$.

Lemma 3.4.8. *If* f *is a morphism of* $\mathcal{C}$*, then* $\gamma(f)$ *is an isomorphism of* $\mathrm{Ho}(\mathcal{C})$ *if and only if* f *is a weak equivalence. Furthermore the morphisms of* $\mathrm{Ho}(\mathcal{C})$ *are generated by the images of the morphisms of* $\mathcal{C}$ *under* γ *and inverses of images of weak equivalences of* $\mathcal{C}$ *under* γ.

Proof. The first statement is proven by applying theorem 3.3.13 and observing that Q and R' preserve and reflect weak equivalences. The second statement follows by the factorization given above. □

The lemma above implies a key result: let F, G be functors from $\mathrm{Ho}(\mathcal{C})$ to a category $\mathcal{D}$ and let t be a natural transformation between $F\gamma$ and $G\gamma$, then t can be extended to a natural transformation between F and G. Given that γ acts as the identity on objects, we only need to check that the commutativity of the naturality diagram holds for every morphism in $\mathrm{Ho}(\mathcal{C})$, not only on the image of γ, but this is clear from the factorization in the lemma above.

Lemma 3.4.9. *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor taking weak equivalences in $\mathcal{C}$ to isomorphisms of $\mathcal{D}$. Let $f, g\colon A \longrightarrow B$, if $f \overset{l}{\sim} g$ or $f \overset{r}{\sim} g$, then $F(f) = F(g)$.*

Proof. We assume $f \overset{l}{\sim} g$. Let $H\colon A \wedge I \longrightarrow B$ be a homotopy, therefore we know that the cylinder map $q\colon A \wedge I \longrightarrow A$ must be a weak equivalence. If $i_1, i_2\colon A \longrightarrow A \wedge I$ are the canonical morphisms, we have that necessarily $qi_1 = qi_2 = \mathrm{id}_A$. Given that $F(q)$ is an isomorphism by hypothesis, it must be that $F(i_1) = F(i_2)$; therefore $F(f) = F(H)F(i_1) = F(H)F(i_2) = F(g)$. □

Proposition 3.4.10. *Let A be a cofibrant object of $\mathcal{C}$, and let B be a fibrant object of $\mathcal{C}$. The map*

$$\gamma\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B)$$

is surjective and induces a natural bijection $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B) \cong \pi(A, B)$.

Proof. Given that γ inverts weak equivalences, by the lemma we have just proven it must also identify homotopical morphisms; as a consequence, the functor map $\gamma\colon \mathrm{Hom}_{(\mathcal{C})}(A, B) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B)$ must factor through $\pi(A, B)$. We thus have a commutative diagram as follows:

$$\begin{array}{ccc} \pi(RA, QB) & \xrightarrow{(i_A)^*(p_B)_*} & \pi(A, B) \\ \gamma\downarrow & & \downarrow\gamma \\ \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(RA, QB) & \xrightarrow[(i_A)^*(p_B)_*]{} & \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B). \end{array}$$

The upper horizontal map is bijective by lemma 3.3.9, the left vertical map is bijective by lemma 3.4.7 and the lower horizontal map is bijective because is given by pre/post-composing with isomorphisms in $\mathrm{Ho}(\mathcal{C})$. We conclude that right vertical map is bijective (therefore its lift must be surjective). □

Remark 3.4.11. *Using the same strategy as above, it is immediate to show that for any A, B we can write*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, B) \cong \pi(QA, RB).$$

We are now ready to prove the main result of this section.

Theorem 3.4.12. *Let $\mathcal{D}$ be a category and let $G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor that takes weak equivalences to isomorphisms of $\mathcal{D}$, then there exists a unique functor $G'\colon \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ such that $G = G'\gamma$.*

Proof. It is immediate from the definition for every morphism f and every weak equivalence h we must have that $G'(\gamma(f)) = G(f)$ and $G'(\gamma(h)^{-1}) = G(h)^{-1}$. By lemma 3.4.8 this is enough to define G' on all morphisms of $\mathrm{Ho}(\mathcal{C})$ and by lemma 3.4.9 we know that G' is well defined. $\square$

Corollary 3.4.12.1. *The pair* $(\mathrm{Ho}(\mathcal{C}), \gamma)$ *is a localization of* $\mathcal{C}$ *with respect to* $W(\mathcal{C})$.

Proof. We have already seen that $\gamma(h)$ is an isomorphism for all $h \in W(\mathcal{C})$. The theorem above implies that the functor

$$(-) \circ \gamma \colon \mathrm{Funct}(\mathrm{Ho}(\mathcal{C}), A) \longrightarrow \mathrm{Funct}(\mathcal{C}, A)$$

is not only equivalence of categories, but also surjective on objects, therefore the remaining conditions are easily proven. $\square$

Chapter 4

Cotorsion pairs and model categories

The main objective in this chapter will be obtaining a method to define model structures on abelian categories, by identifying the objects that we want to be fibrant/cofibrant. The reason we do so is to have more control on (co)fibrant replacement. An in-depth treatment of the subject can be found in [Bul16].

4.1 Small object argument

One of the most common tools used to build a model structure on a given category is known as "Quillen's small object argument"; it provides a tool to obtain a (functorial) factoriziation as required in (**MC4**). We will not provide a proof, to see the full result an excellent resource is chapter 2 of [Hov07].
Let $\mathcal{C}$ be a category.

Definition 4.1.1 (Injective and projective morphisms)**.** *Let I be a class of morphisms of $\mathcal{C}$.*

- *A morphism is said to be* ***I-injective*** *if it has the RLP with respect to every morphism in I. The class of I-injective morphisms is denoted I-inj.*
- *A morphism is said to be* ***I-projective*** *if it has the LLP with respect to every morphism in I. The class of I-projective morphisms is denoted I-proj.*
- *A morphism is said to be* ***I-cofibration*** *if it has the LLP with respect to every morphism in I-inj. The class of I-cofibrations is (I-inj)-proj and is denoted I-cof.*
- *A morphism is said to be* ***I-fibration*** *if it has the RLP with respect to every morphism in I-proj. The class of I-fibrations is (I-proj)-inj and is denoted I-fib.*

As an example, if $\mathcal{C}$ is a model category and $I = \mathrm{Cofib}(\mathcal{C})$, proposition 3.2.4 says that I-inj is exactly the class of acyclic fibrations of $\mathcal{C}$ and that I-cof is exactly I. Dually, if $I = \mathrm{Fib}(\mathcal{C})$, I-proj is exactly the class of acyclic cofibrations of $\mathcal{C}$ and that I-fib is exactly I.

Remark 4.1.2. *In definition 2.5.1, we have defined injective and projective objects in an abelian category: if we let I be the class of monomorphisms of $\mathcal{C}$, then an object Y is injective if and only if the terminal morphism $Y \longrightarrow 0$ is in I-inj. Dually if we let I be the class of epimorphisms of $\mathcal{C}$, then an object X is projective if and only if the initial morphism $0 \longrightarrow X$ is in I-proj.*

It is easy to verify that $I \subseteq I$-cof and $I \subseteq I$-fib; as a consequence we have that $(I\text{-cof})\text{-inj} = I\text{-inj}$; furthermore it is also clear that if $I \subseteq J$ we have that $J\text{-inj} \subseteq I\text{-inj}$ and $J\text{-proj} \subseteq I\text{-proj}$. From these two results follows that $(I\text{-cof})\text{-inj} = I\text{-inj}$ and $(I\text{-fib})\text{-proj} = I\text{-proj}$.
Let us assume from now that $\mathcal{C}$ is cocomplete.

Definition 4.1.3 (λ-sequences)**.** *Let λ be an ordinal. A **λ-sequence** in $\mathcal{C}$ is a colimit preserving functor $X \colon \lambda \longrightarrow \mathcal{C}$ (where λ is seen as a partial order); it is commonly written as*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots .$$

*We refer to the colimit morphism $X_0 \longrightarrow \mathrm{colim}_{\beta<\lambda}(X_\beta)$ as the **transfinite compositions** of the morphisms $X_\beta \longrightarrow X_{\beta+1}$.*

Given that for every limit ordinal β we know that $\mathrm{colim}_{\alpha<\beta}(\alpha) = \beta$, for every λ-sequence X and for every limit ordinal $\beta < \lambda$ the canonical morphism

$$\operatorname*{colim}_{\alpha<\beta} X \longrightarrow X_\beta,$$

is an isomorphism.

Remark 4.1.4. *One important result about (AB5) categories is the fact that monomorphism are closed under transfinite composition; it can be proven by transfinite induction and the only non trivial step is the one for limit ordinals: if X is a λ-sequence made up of monomorphisms and β is a limit ordinal, using the fact that ordinals are filtered and colimits are exact in (AB5) categories, we have that*

$$\ker(X_0 \longrightarrow X_\beta) = \operatorname*{colim}_{\alpha<\beta} \ker(X_0 \longrightarrow X_\alpha) = \operatorname*{colim}_{\alpha<\beta} 0 = 0.$$

Definition 4.1.5 (Filtered ordinal)**.** *Let γ be a cardinal; an ordinal α is said to be **γ-filtered** if for every $A \subseteq \alpha$ such that $|A| \le \gamma$ we have sup $A < \alpha$.*

Definition 4.1.6 (Small objects)**.** *Let I be a class of morphisms of $\mathcal{C}$, A an object of $\mathcal{C}$ and κ a cardinal. We say that A is* ***κ-small relative*** *to I if, for all κ-filtered ordinals λ and all λ sequences X_β such that $X_\beta \longrightarrow X_{\beta+1}$ is in I for $\beta < \lambda$, the canonical map*

$$\operatorname*{colim}_{\beta<\lambda} \operatorname{Hom}_{\mathcal{C}}(A, X_\beta) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is bijective. We say that A is ***small relative*** *I if it is κ-small relative to I for some κ. We say that A is* ***small*** *if it is small relative to the class of morphisms of $\mathcal{C}$.*

To avoid dealing with set-theoretic issues, our setting will often be that of Grothendieck categories, due to the proposition below.

Proposition 4.1.7. *Every object in a Grothendieck category is small.*

Proof. See proposition 1.2 of [Hov99]. □

Definition 4.1.8 (Relative cell complex)**.** *Let I be a set of morphisms. A* ***relative I-cell complex*** *is a transfinite composition of pushouts of elements of I. We denote the class of relative I-cell complexes as I-cell. We say that an object $A \in \mathcal{C}$ is a* ***I-cell complex*** *if the initial morphism $\emptyset \longrightarrow A$ is a relative I-cell complex.*

Lemma 4.1.9. *Let I be a class of morphisms in $\mathcal{C}$. We have I-cell $\subseteq$ I-cof.*

Proof. We already know that $I \subseteq I$-cof; given that I-cof is defined by a lifting property it is easy to show that I-cof is closed under finite compositions and pushouts, by proceeding as we did in corollary 3.2.4.2 and 3.2.4.3. It reamains only to show that I-cof is closed under transfinite composition, to prove that it is as such we will argue by transfinite induction: let λ be an ordinal and let X_β be a λ sequence such that $X_\beta \longrightarrow X_{\beta+1}$ is in I-cof. Let $\beta + 1 \leq \lambda$ be a successor ordinal, if we assume that $X_0 \longrightarrow X_\beta$ is in I-cof then $X_0 \longrightarrow X_{\beta+1}$ must also be in I-cof (given that I-cof is closed under finite composition). Let now $\beta \leq \lambda$ be a limit ordinal and let's assume that for every $\alpha < \beta$ we have that $X_0 \longrightarrow X_\alpha$ is in I-cof; let $A \longrightarrow B$ be in I-inj, we want to show that $X_0 \longrightarrow X_\beta$ has the LLP with respect to it, but this follows from the universal property of the colimit.

$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[d] & & A \arrow[dd] \\
X_\alpha \arrow[d] \arrow[urr, dashed] & & \\
X_\beta \cong \operatorname{colim}_{\alpha<\beta} X_\alpha \arrow[rr] \arrow[uurr, dashed] & & B
\end{tikzcd}$$

□

The following theorem, due to Quillen, is one of the main tools used to obtain (functorial) factorizations of a morphism (i.e. a method to prove (**MC4**)), which is usually the hardest part of defining a model structure on a category.

Theorem 4.1.10 (The small object argument)**.** *Let $\mathcal{C}$ be a category with small colimits and let I be a set of morphisms of $\mathcal{C}$. If the domains of the elements of I are small with respect to I-cell, then there exists a (functorial) factorization (γ, δ) such that, for every morphism f, $\gamma(f)$ is in I-cell and $\delta(f)$ is in I-inj.*

Model categories constructed through the small object argument enjoy peculiar properties, we will collect them in the following definition.

Definition 4.1.11 (Cofibrantly generated model categories)**.** *Let $\mathcal{C}$ be a model category. $\mathcal{C}$ is **cofibrantly generated** if there are sets I, J of morphisms of $\mathcal{C}$ such that:*

1. *The domains of the morphisms in I are small relative to I-cell.*
2. *The domains of the morphisms in J are small relative to J-cell.*
3. *The class of fibrations of $\mathcal{C}$ is J-inj.*
4. *The class of acyclic fibrations of $\mathcal{C}$ is I-inj.*

We will refer to I as the set of generating cofibrations and to J as the set of generating acyclic cofibrations.

The main properties of cofibrantly generated model categories are stated in the following proposition.

Proposition 4.1.12. *Let $\mathcal{C}$ be cofibrantly generated model category, with generating cofibrations I and generating acyclic cofibrations J. The following holds:*

1. *The cofibrations are the class I-cof.*
2. *Every cofibration is a retract of a relative I-cell complex.*
3. *The domains of morphisms in I are small relative to the cofibrations.*
4. *The trivial cofibrations are the class J-cof.*
5. *Every trivial cofibration is a retract of a relative J-cell complex.*
6. *The domains of morphisms in J are small relative to the trivial cofibrations.*

The theorem below gives us a method to build cofibrantly generated model categories through the small object argument.

Theorem 4.1.13. *Let $\mathcal{C}$ be a category with (small) limits and colimits, let W be a class of morphism of $\mathcal{C}$ and let I, J be sets of morphism of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and W as the subcategory of weak equivalences if and only if the following conditions are satisfied:*

1. *The class W satisfy the two-out-of-three property (**MC1**) and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *J-cell $\subseteq W \cap I$-cof.*
5. *I-inj $\subseteq W \cap J$-inj.*
6. *Either $W \cap I$-cof $\subseteq J$-cof or $W \cap J$-inj $\subseteq I$-inj.*

4.2 Cotorsion pairs

Let $\mathcal{C}$ be an abelian category.

Definition 4.2.1 (orthogonal complement)**.** *Let $\mathcal{D}$ be a class of objects of $\mathcal{C}$. We define the **left orthogonal complement** of $\mathcal{D}$ as the class*

$$^{\perp}\mathcal{D} = \left\{E \,:\, \operatorname{Ext}^1(E, D) = 0, \text{ for every } D \in \mathcal{D}\right\}.$$

*Dually, we define the **right orthogonal complement** of $\mathcal{D}$ as the class*

$$\mathcal{D}^{\perp} = \left\{E \,:\, \operatorname{Ext}^1(D, E) = 0, \text{ for every } D \in \mathcal{D}\right\}.$$

The easiest example of orthogonal complements is the following: if $\mathcal{D} = \operatorname{ob}(\mathcal{C})$, then $^{\perp}\mathcal{D}$ is the class of projective objects of $\mathcal{C}$, while $\mathcal{D}^{\perp}$ is the class of injective objects.

Remark 4.2.2. *Given a class of objects $\mathcal{D}$, it is clear that $\mathcal{D} \subseteq {}^{\perp}(\mathcal{D}^{\perp})$ and $\mathcal{D} \subseteq \left({}^{\perp}\mathcal{D}\right)^{\perp}$. Also if $\mathcal{D} \subseteq \mathcal{D}'$ we have ${}^{\perp}\mathcal{D}' \subseteq {}^{\perp}\mathcal{D}$ and $\mathcal{D}'^{\perp} \subseteq \mathcal{D}^{\perp}$, therefore $\left({}^{\perp}(\mathcal{D}^{\perp})\right)^{\perp} = \mathcal{D}^{\perp}$ and ${}^{\perp}\left(\left({}^{\perp}\mathcal{D}\right)^{\perp}\right) = {}^{\perp}\mathcal{D}$.*

Definition 4.2.3 (Cotorsion pair)**.** *Let $(\mathcal{D}, \mathcal{E})$ be a pair of classes of objects of $\mathcal{C}$, they are said to be a **cotorsion pair** if $\mathcal{D} = {}^{\perp}\mathcal{E}$ and $\mathcal{E} = \mathcal{D}^{\perp}$.*

From the earlier remark, it immediately follows that, if $(\mathcal{D}, \mathcal{E})$ is a cotorsion pair, $\mathrm{proj}(\mathcal{C}) \subset \mathcal{D}$ and $\mathrm{inj}(\mathcal{C}) \subset \mathcal{E}$.
One common way to define a cotorsion pair is to cogenerate it from a set, i.e. to choose a set $\mathcal{D}'$ and let $\mathcal{E} = \mathcal{D}'^{\perp}$ and $\mathcal{D} = {}^{\perp}\mathcal{E}$; the above remark proves that it is actually a cotorsion pair. The dual notion, generating the cotorsion pair from a set $\mathcal{E}' \subseteq \mathcal{E}$, does not commonly arise in practice, due to the asymmetry of certain theorems that we will need later.

The example we used earlier allows us to construct two examples of cotorsion pairs: $(\mathrm{proj}(\mathcal{C}), \mathrm{ob}(\mathcal{C}))$ and $(\mathrm{ob}(\mathcal{C}), \mathrm{inj}(\mathcal{C}))$. Given these examples, we would like to generalize the concept of "enough injectives" and "enough projectives" to cotorsion pairs.

Definition 4.2.4 (Complete cotorsion pair). *A cotorsion pair $(\mathcal{D}, \mathcal{E})$ is said to* ***have enough injectives*** *if for every $X \in \mathrm{ob}(\mathcal{C})$ there exists an an exact sequence*

$$0 \longrightarrow X \longrightarrow E \longrightarrow D \longrightarrow 0,$$

where $E \in \mathcal{E}$ and $D \in \mathcal{D}$; if we do not require that $D \in \mathcal{D}$ we simply say that $(\mathcal{D}, \mathcal{E})$ has ***enough $\mathcal{E}$-objects*** *(i.e. every object is a subobject of an element of $\mathcal{E}$). Dually, a cotorsion pair $(\mathcal{D}, \mathcal{E})$ is said to* ***have enough projectives*** *if for every $X \in \mathrm{ob}(\mathcal{C})$ there exists an an exact sequence*

$$0 \longrightarrow E \longrightarrow D \longrightarrow X \longrightarrow 0,$$

where $E \in \mathcal{E}$ and $D \in \mathcal{D}$; if we do not require that $E \in \mathcal{E}$ we simply say that $(\mathcal{D}, \mathcal{E})$ has ***enough $\mathcal{D}$-objects*** *(i.e. every object is a quotient of an element of $\mathcal{D}$). A cotorsion pair that has both enough injectives and enough projectives is said to be* ***complete****.*

Definition 4.2.5. *Let $\mathcal{D}$ be a class of objects in $\mathcal{C}$, let λ be an ordinal and let $X\colon \lambda \longrightarrow \mathcal{C}$ be a λ-sequence. If for any $\beta < \lambda$ the morphism $X_\beta \longrightarrow X_{\beta+1}$ is monic and its cokernel is in $\mathcal{D}$, we say that $X_0 \longrightarrow \mathrm{colim}_{\beta<\lambda} X$ is a* ***transfinite extension*** *of X_0 by $\mathcal{D}$. If it is also true that $X_0 \in \mathcal{D}$, we simply refer to $\mathrm{colim}_{\beta<\lambda} X$ as a transfinite extension of $\mathcal{D}$.*

Lemma 4.2.6. *Let us assume that $\mathcal{C}$ is complete and cocomplete; if $(\mathcal{D}, \mathcal{E})$ is a cotorsion pair then $\mathcal{D}$ is closed under retracts, extensions and transfinite extensions.*

Proof. The statement about retract is obvious: given that $\mathrm{Ext}^1(-, Y)$ is an additive functor, if X is a retract of X' then $\mathrm{Ext}^1(X, Y)$ is a retract of $\mathrm{Ext}^1(X', Y)$. The statement about extensions follows from the long exact sequence associated to Ext. We will now prove that for any $Y \in \mathrm{ob}(\mathcal{C})$, the class of objects X such that $\mathrm{Ext}^1(X, Y) = 0$ is closed under transfinite composition. Let λ be an ordinal and let us consider a λ-sequence $X\colon \lambda \longrightarrow \mathcal{C}$ such that $\operatorname{Ext}^1(X_{\beta+1}/X_\beta, Y) = 0$ and $i_\alpha\colon X_\beta \longrightarrow X_{\beta+1}$ is monic for any $\beta < \lambda$. We will now use transfinite induction to prove that $\operatorname{Ext}^1(X_\beta, Y) = 0$ for any $\beta \leq \lambda$, where we define $X_\lambda := \operatorname{colim}_{\beta<\lambda} X$. The base step X_0 is trivial. If $\beta := \alpha + 1$ is a successor ordinal, we have the exact sequence

$$0 \longrightarrow X_\alpha \longrightarrow X_{\alpha+1} \longrightarrow X_{\alpha+1}/X_\alpha \longrightarrow 0,$$

that induces

$$0 \cong \operatorname{Ext}^1(X_{\alpha+1}/X_\alpha, Y) \longrightarrow \operatorname{Ext}^1(X_{\alpha+1}, Y) \longrightarrow \operatorname{Ext}^1(X_\alpha, Y) \cong 0.$$

The case where β is a limit ordinal is a little more involved: we want to show that all exact sequences of the form

$$0 \longrightarrow Y \xrightarrow{\ f\ } A \xrightarrow{\ g\ } X_\beta \longrightarrow 0$$

are splitting. By pulling back the extensions along the morphisms $X_\alpha \longrightarrow X_\beta$ and by setting $A_\alpha := A \times_{X_\beta} X_\alpha$, we get a family of compatible short exact sequences as follows.

$$\begin{tikzcd}
& 0 \arrow[rr] & & Y \arrow[rr, "f_\alpha"] \arrow[dl, "\mathrm{id}_Y"'] \arrow[dd, "\mathrm{id}_Y"] & & A_\alpha \arrow[rr, "g_\alpha"] \arrow[dl, "j_\alpha"'] \arrow[dd] & & X_\alpha \arrow[rr] \arrow[dl, "i_\alpha"'] \arrow[dd] & & 0 \\
0 \arrow[rr] & & Y \arrow[rr, "f_{\alpha+1}"] \arrow[dr, "\mathrm{id}_Y"'] & & A_{\alpha+1} \arrow[rr, "g_{\alpha+1}"] \arrow[dr] & & X_{\alpha+1} \arrow[rr] \arrow[dr] & & 0 & \\
& 0 \arrow[rr] & & Y \arrow[rr, "f"] & & A \arrow[rr, "g"] & & X_\beta \arrow[rr] & & 0
\end{tikzcd}$$

We know by inductive hypothesis that all of these sequences (except possibly the last) must split; we will now use another transfinite induction to show that we can choose all of the sections $s_\alpha\colon X_\alpha \longrightarrow A_\alpha$ in a compatible way (i.e. $j_\alpha s_\alpha = i_\alpha s_{\alpha+1}$), obtaining at last a splitting of the desired exact sequence. Given that there is no compatibility condition for $\alpha = 0$, we are free to choose any section for s_0, so the base step of the induction is trivial. If α is a limit ordinal, the section s_α will simply be the following colimit morphism.

$$\begin{tikzcd}
X_{\alpha'} \arrow[r, "s_{\alpha'}"] \arrow[dr] & A_{\alpha'} \arrow[r] & A \\
& \operatorname{colim}_{\alpha'<\alpha} X \cong X_\alpha \arrow[ur, dashed, "s_\alpha"] &
\end{tikzcd}$$

We now only need to show the successor step: given a section $s_\alpha\colon X_\alpha \longrightarrow A_\alpha$ of g_α, we want to find a section $s_{\alpha+1}\colon X_{\alpha+1} \longrightarrow A_{\alpha+1}$ such that $j_\alpha s_\alpha = s_{\alpha+1} i_\alpha$. Given that we already know that the sequence ending in $X_{\alpha+1}$ must split, there exists a section $t_{\alpha+1}$ of $g_{\alpha+1}$. Given that

$$g_{\alpha+1}(j_\alpha s_\alpha - t_{\alpha+1} i_\alpha) = i_\alpha - i_\alpha = 0$$

and that Y is the kernel of $g_{\alpha+1}$, there is $h\colon X_\alpha \longrightarrow Y$ that satisfies $f_{\alpha+1}h = j_\alpha s_\alpha - t_{\alpha+1}i_\alpha$. The fact that $\mathrm{Ext}^1(X_{\alpha+1}/X_\alpha, Y) = 0$ implies that the map

$$\mathrm{Hom}_{\mathcal{C}}(X_{\alpha+1}, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_\alpha, Y)$$

is surjective, therefore there exists $k\colon X_{\alpha+1} \longrightarrow Y$ such that $ki_\alpha = h$. It is easy to check that $s_{\alpha+1} := t_{\alpha+1} + f_{\alpha+1}k$ is the section we are looking for. $\square$

It is clear that the statements about retracts and extensions are also true for $\mathcal{E}$, and are proved identically; furthermore the following lemma also holds.

Lemma 4.2.7. *Let $(\mathcal{D}, \mathcal{E})$ is a cotorsion pair in $\mathcal{C}$; if $\mathcal{C}$ is complete then $\mathcal{E}$ is closed under products. Dually, if $\mathcal{C}$ is cocomplete, $\mathcal{D}$ is closed under coproducts.*

Proof. Let us consider a set $\{E_i\}_{i\in I}$ of elements of $\mathcal{E}$, we want to prove (assuming that $\mathcal{C}$ is complete) that, for all $D \in \mathcal{D}$,

$$\mathrm{Ext}^1_{\mathcal{C}}(D, \prod_{i\in I} E_i) = 0.$$

Let us continue by choosing an exact sequence

$$0 \longrightarrow \prod_{i\in I} E_i \longrightarrow X \longrightarrow D \longrightarrow 0$$

with $D \in \mathcal{D}$. The morphism $\pi_j\colon \prod_{i\in I} E_i \longrightarrow E_j$ (for any $j \in I$) induces an extension (see section 2.4) that is the second line of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \prod_{i\in I} E_i \arrow[r, "f"] \arrow[d, "\pi_j"] & X \arrow[r] \arrow[d, "\phi_j"] & D \arrow[r] \arrow[d, "\mathrm{id}_D"] & 0 \\
0 \arrow[r] & E_j \arrow[r, "f_j"] & X_j \arrow[r] & D \arrow[r] & 0,
\end{tikzcd}$$

where X_j is the pushout $E_j \amalg_{(\prod_{i\in I} E_i)} X$. By hypothesis there exists a retraction $\sigma_j\colon X_j \longrightarrow E_j$ of f_j, hence by considering the morphisms $\sigma_j\phi_j\colon X \longrightarrow E_j$ the universal property of the product defines a morphsm $\Phi\colon X \longrightarrow \prod_{i\in I} E_i$. The commuting diagram above yields for all $j \in I$

$$\pi_j\Phi f = \sigma_j\phi_j f = \sigma_j f_j\pi_j = \pi_j,$$

therefore $\Phi f = \mathrm{id}_{\prod_{i\in I} E_i}$ and thus the extension we started with was split. The case where $\mathcal{C}$ is cocomplete is dual. $\square$

4.3 Hovey correspondence

We would like to use cotorsion pairs to define model structures on abelian categories, in a way that makes such structure compatible with the usual tools we use to study abelian categories. To do so we must fist make explicit these compatibility conditions.

Definition 4.3.1 (Abelian model categories)**.** *Let $\mathcal{C}$ be a model category where the underlying category is abelian; we will refer to $\mathcal{C}$ as an* ***abelian model category*** *if the following hold:*

- *A morphism is a cofibration if and only if it is a monomorphism with cofibrant cokernel.*
- *A morphism is a fibration if and only if it is an epimorphism with fibrant kernel.*

In an abelian model category we can give the following definition.

Definition 4.3.2 (Acyclic object)**.** *An object A is said to be* ***acyclic*** *if the zero morphism $A \longrightarrow 0$ is a weak equivalence.*

Clearly, if A is acyclic $0 \longrightarrow A$ is also a weak equivalence and vice versa.

Lemma 4.3.3. *If $\mathcal{C}$ is an abelian model category, C its cofibrant objects, F its fibrant objects and W its acyclic objects, then $\mathrm{Ext}^1(A, B) = 0$ when $A \in C$ and $B \in F \cap W$ or $A \in C \cap W$ and $B \in F$*

Proof. Let us consider $A \in C$ and $B \in F \cap W$, we want to show that $\mathrm{Ext}^1(A, B) = 0$. An element of $\mathrm{Ext}^1(A, B)$ is the equivalence class of a short exact sequence

$$0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0.$$

Given that A is cofibrant and is the cokernel of f, f is a cofibration. We can therefore find a lift in the following diagram,

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B"] \arrow[d, hook, "f"'] & B \arrow[d, two heads, "\sim"] \\ X \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd}$$

which is a retraction of f; therefore the exact sequence must split and $\mathrm{Ext}^1(A, B) = 0$. The other case is analogous. □

The following proposition charcterizes acyclic fibrations/cofibrations in an abelian model category more explicitly.

Proposition 4.3.4. *Let $\mathcal{C}$ be both an abelian category and a model category (not necessarily an abelian model category!) where every cofibration is a monomorphisms and every fibration is an epimorphism, the following two results hold: fibrations coincide with epimorphisms with fibrant kernels if and only if acyclic cofibrations coincide with monomorphisms with acyclic cofibrant cokernels; similarly, acyclic fibrations coincide with epimorphisms with acyclic fibrant kernels if and only if cofibrations coincide with monomorphisms with cofibrant cokernels.*

Proof. We will only consider the case where we assume that acyclic fibrations coincide with epimorphisms with acyclic fibrant kernels and we prove that cofibrations coincide with monomorphisms with cofibrant cokernels; the other three implications are analogous. We will proceed by considering a monomorphism $f\colon A \longrightarrow B$ and we will assume that $C = \operatorname{coker}(f)$ is cofibrant, therefore obtaining a ses as follows.

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

We can now consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & X \arrow[d, two heads, "g", "\sim"'] \\ B \arrow[r, "\beta"] & Y, \end{tikzcd}$$

where $g\colon X \longrightarrow Y$ is an acyclic fibration; our objective will be finding a lift in this diagram. If we set $Z = \ker(g)$ (and remember that it must be an acyclic fibrant object) we get another SES

$$0 \longrightarrow Z \longrightarrow X \xrightarrow{g} Y \longrightarrow 0.$$

By carefully applying the functor $\operatorname{Hom}_{\mathcal{C}}(-,-)$ to the exact sequences above, we get the following commuting diagram with exact rows and columns.

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(C,Z) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g_*"] \arrow[d] & \operatorname{Hom}_{\mathcal{C}}(C,Y) \arrow[r, "\delta"] \arrow[d] & \operatorname{Ext}^1(C,Z) \arrow[d] \\ \operatorname{Hom}_{\mathcal{C}}(B,Z) \arrow[r] \arrow[d, "f^*"] & \operatorname{Hom}_{\mathcal{C}}(B,X) \arrow[r, "g_*"] \arrow[d, "f^*"] & \operatorname{Hom}_{\mathcal{C}}(B,Y) \arrow[r, "\delta"] \arrow[d, "f^*"] & \operatorname{Ext}^1(B,Z) \arrow[d, "f^*"] \\ \operatorname{Hom}_{\mathcal{C}}(A,Z) \arrow[r] \arrow[d, "\delta"] & \operatorname{Hom}_{\mathcal{C}}(A,X) \arrow[r, "g_*"] \arrow[d, "\delta"] & \operatorname{Hom}_{\mathcal{C}}(A,Y) \arrow[r, "\delta"] \arrow[d, "\delta"] & \operatorname{Ext}^1(A,Z) \arrow[d] \\ \operatorname{Ext}^1(C,Z) \arrow[r] & \operatorname{Ext}^1(C,X) \arrow[r, "g_*"] & \operatorname{Ext}^1(C,Y) \arrow[r] & \cdots \end{tikzcd}$$

Our current objective is finding $\gamma \in \mathrm{Hom}_{\mathcal{C}}(B, X)$ such that $f^*\gamma = \alpha$ and $g_*\gamma = \beta$, we will do so through some diagram chasing. We begin by noticing that

$$g_*\delta\alpha = \delta g_*\alpha = \delta f^*\beta = 0.$$

By lemma 4.3.3 we know that $\mathrm{Ext}^1(C, Z) = 0$, therefore $g_* \colon \mathrm{Ext}^1(C, X) \longrightarrow \mathrm{Ext}^1(C, Y)$ is injective, thus $\delta\alpha = 0$ and by the exactness of the column we know that there exists $\gamma' \in \mathrm{Hom}_{\mathcal{C}}(B, X)$ such that $f^*\gamma' = \alpha$. Given that $f^*(\beta - g_*\gamma') = f^*\beta - g_*\alpha = 0$, the morphism $\beta - g_*\gamma'$ must have a preimage F in $\mathrm{Hom}_{\mathcal{C}}(C, Y)$; using again lemma 4.3.3, we know that $g_* \colon \mathrm{Hom}_{\mathcal{C}}(C, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, Y)$ is surjective, and as a result there must be a morphism $G \in \mathrm{Hom}_{\mathcal{C}}(C, X)$ such that $g_*G = F$. If we denote as G' the image of G in $\mathrm{Hom}_{\mathcal{C}}(B, X)$, it is easy to check that $\gamma = \gamma' - G'$ is the morphism we are looking for.

□

The objective of this section will be to prove "Hovey's Correspondence theorem" (see section 2 of [Hov02]), that shows a correspondence between abelian model structures and certain pairs of cotorsion pairs. We will fix an abelian category $\mathcal{C}$.

Definition 4.3.5 (Thick class)**.** *A class of objects W of $\mathcal{C}$ is said to be* ***thick*** *if it is closed under retracts and whenever two out of three objects in a SES are in W the third also is.*

Definition 4.3.6 (Hovey triple)**.** *Let (C, W, F) be a triple of classes of objects in $\mathcal{C}$. We say that (C, W, F) is a* ***Hovey triple*** *if W is thick and $(C \cap W, F)$ and $(C, F \cap W)$ are complete cotorsion pairs.*

Theorem 4.3.7 (Hovey's Correspondence - Part 1)**.** *If $\mathcal{C}$ is an abelian model category, C its cofibrant objects, F its fibrant objects and W its acyclic objects, then (C, W, F) is a Hovey triple.*

Proof. We begin by showing that $(C, F \cap W)$ is a complete cotorsion pair, given that proving it for $(C \cap W, F)$ is analogous. From lemma 4.3.3, it is immediate that $F \cap W \subseteq C^{\perp}$ and $C \subseteq {}^{\perp}(F \cap W)$, therefore to conclude the proof that $(C, F \cap W)$ is a cotorsion pair, we will also need to show that $C^{\perp} \subseteq F \cap W$ and ${}^{\perp}(F \cap W) \subseteq C$; we will only prove the first of these two results (the other is dual). Let us now consider an object B such that $\mathrm{Ext}^1(A, B) = 0$ for every $A \in C$; then if we take any cofibration $f \colon X \longrightarrow Y$ we get an exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{coker}(f) \longrightarrow 0,$$

where $\mathrm{coker}(f)$ must be a cofibrant object. Using the long exact sequence associated to the functor $\mathrm{Hom}_{\mathcal{C}}(-, B)$ and the SES above, we get

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, B) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(Y, B) \longrightarrow \mathrm{Ext}^1(\mathrm{coker}(f), B) \longrightarrow \cdots;$$

given that $\mathrm{Ext}^1(\mathrm{coker}(f), B)$ must be 0, f^* is surjective and therefore, by lemma 3.2.8, B must be in $F \cap W$. The last step we need is showing that this cotorsion pair is actually complete, but this is simply a consequence of (**MC4**): to show that $(C, F \cap W)$ has enough projectives, we can factor any initial morphism $0 \longrightarrow X$ using an acyclic fibration $B \longrightarrow X$, where B is cofibrant; taking the kernel of this morphism defines the exact sequence we need. Since the proof that shows that our cotorsion pair has enough injectives is identical, it must be complete.
The last step that we need to complete this proof, is verifying that W is a thick class. We know from axiom (**MC2**) that W is closed under retracts, so we will only check the condition on SESes. We begin by considering a SES as follows,

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

then we can factor g as

$$B \underset{\sim}{\overset{j}{\hookrightarrow}} B' \overset{g'}{\twoheadrightarrow} C$$

to get a commuting diagram with exact rows as follows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, hook, "j"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, two heads, "g'"] & C \arrow[r] & 0,
\end{tikzcd}$$

where $A' = \mathrm{coker}(g')$ and i is the morphism induced by the cokernel functor. By the snake lemma $\mathrm{coker}(i) = \mathrm{coker}\, j$ and so it is cofibrant, therefore i is also an acyclic cofibration by proposition 4.3.4; we can also observe that A' must be fibrant, given that it is the kernel of a fibration. If we now assume that $A \in W$ it is immediate that $A' \in W$, thus g' must be a trivial fibration; by (**MC1**) g must be a weak equivalence, therefore $B \in W$ if and only if $C \in W$. The remaining case ($B, C \in W$ implies $A \in W$) is simply the same argument in reverse. □

Theorem 4.3.8 (Hovey's Correspondence - Part 2). *If $\mathcal{C}$ is an abelian model category and (C, W, F) is a Hovey triple, then there exists a unique abelian model structure on $\mathcal{C}$ where C are the cofibrant objects, F are the fibrant objects and W are the acyclic objects.*

Proof. We will not provide the proof, given that it is rather technical and long. The main idea is defining the model structure in the following way:

- A morphism is a cofibration if and only if it is a monomorphism with cofibrant cokernel.
- A morphism is a fibration if and only if it is an epimorphism with fibrant kernel.

- A morphism is an acyclic cofibration if and only if it is a monomorphism with acyclic cofibrant cokernel.
- A morphism is an acyclic fibration if and only if it is an epimorphism with acyclic fibrant kernel.
- A morphism is a weak equivalence if it can be written as composition of an acyclic fibration and an acyclic cofibration.

The proof that these classes of morphisms satisfy axioms (MC1), (MC2), (MC3) and (MC4) is found in [Hov02]. □

We proceed with one important example.

Definition 4.3.9 (Frobenius categories). *An abelian category $\mathcal{C}$ is said to be a Frobenius category if it has enough projectives and enough injectives and where the classes of projectives and injectives coincide.*

Proposition 4.3.10. *Let $\mathcal{C}$ be a Frobenius category, there exists an abelian model structure on $\mathcal{C}$ where the cofibrations are the monomorphisms and the fibrations are the epimorphisms.*

Proof. Let us set C and F to be the class $\mathrm{ob}(\mathcal{C})$ and W to be the class of injective/projective objects. We have remarked already that (by definition) $\mathrm{ob}(\mathcal{C})^{\perp} = \mathrm{inj}(\mathcal{C})$ and ${}^{\perp}\mathrm{ob}(\mathcal{C}) = \mathrm{proj}(\mathcal{C})$ and vice versa; this satisfies the hypothesis needed to apply Hovey's correspondence. Given that $F = \mathrm{ob}(\mathcal{C}) = C$, there is no condition needed on the kernels and the cokernels. □

4.4 Small cotorsion pairs

The main obstacle in proving that certain classes form a Hovey triple, which we need if we want to use Hovey's correspondence theorem, is proving the completeness of the cotorsion pairs. This section will show that, if we are operating under certain hypothesis, this condition is free; in these particular cases we will also see that the model structure induced by the Hovey triple is cofibrantly generated.

Definition 4.4.1 (Small cotorsion pair). *Let $\mathcal{C}$ be an abelian category that is complete and cocomplete, let $(\mathcal{D}, \mathcal{E})$ be a cotorsion pair of $\mathcal{C}$. We say that $(\mathcal{D}, \mathcal{E})$ is **small** if all the following properties are satisfied.*

1. *$\mathcal{D}$ contains a family of generators $\mathcal{G}$.*
2. *$(\mathcal{D}, \mathcal{E})$ is cogenerated by a set $\mathcal{U}$.*
3. *for each $U \in \mathcal{U}$, there is a monomorphism $i_U \colon B_U \longrightarrow A_U$ with cokernel U such that, if $i_U^* \colon \mathrm{Hom}_{\mathcal{C}}(B_U, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A_U, X)$ is surjective for all $U \in \mathcal{U}$, then $X \in \mathcal{E}$.*

We refer to the set I containing all of the morphisms i_U for $U \in \mathcal{U}$ and all of the initial morphisms $0 \longrightarrow G$ for $G \in \mathcal{G}$ as the ***generating monomorphisms*** *of the cotorsion pair.*

It is important to remark that if $(\mathcal{D}, \mathcal{E})$ is a small cotorsion pair, the converse of condition 3 always holds: if $X \in \mathcal{E}$, then for all $U \in \mathcal{U}$ we have $\mathrm{Ext}^1(U, X) = 0$ and therefore by the long exact sequence associated to Ext we have that i_U^* is surjective. We can use this fact to recover from the set I of generating monomorphism the entire cotorsion pair: $\mathcal{E}$ is exactly the class of all objects X such that $i^* = \mathrm{Hom}_{\mathcal{C}}(i, X)$ is surjective for all $i \in I$ (the condition is trivial for morphisms of the form $0 \longrightarrow G$), while $\mathcal{D} = {}^{\perp}\mathcal{E}$.
From now on we will assume $\mathcal{C}$ to be a Grothendieck category.

Theorem 4.4.2. *Let I be a set of monomorphisms of $\mathcal{C}$, then I is the set of generating monomorphisms of a small cotorsion pair $(\mathcal{D}, \mathcal{E})$ if and only if the following conditions hold:*

1. *I contains the morphisms $0 \longrightarrow G_i$ for a generating set $\mathcal{G} = \{G_i\}$ of $\mathcal{C}$.*
2. *For every $E \in \mathrm{ob}(\mathcal{C})$ such that $i^* = \mathrm{Hom}_{\mathcal{C}}(i, E)$ is surjective for all $i \in I$ and for every $j \in I$, we have $\mathrm{Ext}^1(\mathrm{coker}(j), E) = 0$*

In this case, $\mathcal{E}$ is the class of all objects E such that i^* (as above) is surjective for all $i \in I$, while $\mathcal{D}$ is the smallest class closed under summands and transfinite extensions that contains all the cokernels of morphism in I. Furthermore $(\mathcal{D}, \mathcal{E})$ is complete.

Proof. If I is a set of generating monomorphisms, both conditions are easily verified. We will now prove that, given a set I that satisfies both condition, the pair $(\mathcal{D}, \mathcal{E})$ as in the statement of the theorem is a complete cotorsion pair.
We begin by showing that this pair has enough injectives: by applying the small object argument to $X \longrightarrow 0$ we get a factorization of the form $X \xrightarrow{f} Y \xrightarrow{g} 0$, where f is in I-cell and g is in I-inj. This last result implies that $\mathrm{Hom}_{\mathcal{C}}(i, Y)$ is surjective for all $i \in I$, therefore $Y \in \mathcal{E}$. Since monomorphisms are closed under pushouts and transfinite composition (we are in a Grothendieck category, which is AB3), we only need to show that $\mathrm{coker}(f)$ is in $\mathcal{D}$; it follows from the fact that $\mathrm{coker}(f)$ is a transfinite extension of cokernels of morphisms in I (pushouts preserve cokernels) and the definition of $\mathcal{D}$.
We will continue by showing that $(\mathcal{D}, \mathcal{E})$ has enough projectives: this time we factor $0 \longrightarrow X$ as $0 \xrightarrow{h} Z \xrightarrow{k} X$, where h is in I-cell and k is in I-inj. By proceeding as we did above, we get that $\mathrm{coker}(h) \cong Z$ must be in $\mathcal{D}$. Given that $0 \longrightarrow G_i$ for $G_i \in G$ are in I and k is in I-inj, k must be epic. The morphism $\ker(k) \longrightarrow 0$ is a pullback of of a morphism (k) in I-inj, therefore it is also in I-inj (see the proof of corollary 3.2.4.3) and therefore, as we did earlier, $\ker(k)$ is in $\mathcal{E}$.

The last step is proving that $(\mathcal{D}, \mathcal{E})$ is actually a cotorsion pair. Let us consider $D \in \mathcal{D}$ and $E \in \mathcal{E}$, given that $\mathrm{Ext}^1(\mathrm{coker}(i), E) = 0$ for all $i \in I$, by applying what we have learned by proving lemma 4.2.6, it follows that $\mathrm{Ext}^1(D, E) = 0$; therefore $\mathcal{D} \subseteq {}^{\perp}\mathcal{E}$ and $\mathcal{E} \subseteq \mathcal{D}^{\perp}$. Let us take $X \in \mathcal{D}^{\perp}$, given that $\mathrm{coker}(i) \in \mathcal{D}$ for all $i \in I$ we can say that $\mathrm{Ext}^1(\mathrm{coker}(i), X) = 0$ for all $i \in I$, thus $X \in \mathcal{E}$. Now we will assume that $X \in {}^{\perp}\mathcal{E}$, by using the fact that there are enough projectives we can find a short exact sequence

$$0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0,$$

where $Z \in \mathcal{E}$ and $Y \in \mathcal{D}$. This is a representative of an element of $\mathrm{Ext}^1(X, Z) = 0$, therefore it splits; given that $\mathcal{D}$ is closed under summands, $X \in \mathcal{D}$. □

Corollary 4.4.2.1. *Every small cotorsion pair of $\mathcal{C}$ is complete.*

Proposition 4.4.3. *Let us assume that $\mathcal{C}$ has an abelian model structure; if the cotorsion pairs $(C, F \cap W)$ and $(C \cap W, F)$ are small, then this model structure is cofibrantly generated.*

Proof. Let I be a set of generating monomorphisms for $(C, F \cap W)$, while letting J be a set of generating monomorphisms for $(C \cap W, F)$. We want to show that I is the set of generating cofibrations while J is the set of generating acyclic cofibrations; the smallness condition is trivial given that we are working in a Grothendieck category. Let f be a morphism in I-inj, given that $0 \longrightarrow G_h$ is in I for a generating family $\mathcal{G} = \{G_h\}_h$, then $\mathrm{Hom}_{\mathcal{C}}(G_h, f)$ is surjective for all h and thus f is epic. The map $\ker(f) \longrightarrow 0$ is a pullback of f and hence it is also in I-inj (see corollary 3.2.4.3); thus $\mathrm{Hom}_{\mathcal{C}}(i, \ker f)$ is surjective for all $i \in I$ and $\ker(f) \in F$. We conclude that f is a fibration; to show that if g is in J-inj then g is an acyclic fibration we can proceed analogously. □

Corollary 4.4.3.1. *If $\mathcal{C}$ has enough projectives, then every cotorsion pair that is cogenerated by a set is small, and hence complete.*

Proof. Let us suppose that $(\mathcal{D}, \mathcal{E})$ is a cotorsion pair in $\mathcal{C}$ and that $\mathcal{E} = \mathcal{U}^{\perp}$ for some set $\mathcal{U}$ (i.e. the cotorsion pair is cogenerated by $\mathcal{U}$). Given that $\mathcal{C}$ has enough projectives, there exists an epimorphism from a projective P to the generator G, therefore P must be a projective generator and is also in $\mathcal{D}$. Now, for all $U \in \mathcal{U}$, we consider an epimorphism from a projective $Q_U \longrightarrow U$, with kernel $i_U \colon K_U \longrightarrow Q_U$. The set $I = \{0 \longrightarrow P\} \cup \{i_U\}_{U \in \mathcal{U}}$ is the set of generating monomorphisms of $(\mathcal{D}, \mathcal{E})$. □

Chapter 5

Cotorsion pairs and chain complexes

In the previous chapter we have seen the various interaction between cotorsion pairs and model structures on Grothendieck categories, we will now see how we can use this knowledge to study chain complexes in these categories. The theory has been developed by James Gillespie in [Gil04] and [Gil07] and by Xiaoyan Yang and Nanqing Ding in [YD14].

5.1 Initial remarks

We begin by setting a standard notation for this chapter.

Notation. *Let $\mathcal{C}$ be an abelian category, let $A \in \mathrm{ob}(\mathcal{C})$ and let $n \in \mathbb{Z}$. We denote as $S^n A$ the object of $\mathrm{Ch}(\mathcal{C})$ given by the complex whose only nonzero element is A in position n; we also denote as $D^n A$ the object of $\mathrm{Ch}(\mathcal{C})$ given by the complex whose only nonzero elements are A in position $n-1, n$ and whose only nonzero morphsim is $d_{n-1} = \mathrm{id}_A$.*
If $X = (X_n, d_n)$ is a chain complex in $\mathrm{Ch}(\mathcal{C})$, we denote as $Z_n X$ the object in $\mathcal{C}$ given by $\ker d_n$, while we denote as $B_n X$ the object in $\mathcal{C}$ given by $\mathrm{im}\, d_{n+1}$; we also denote $Z_n X / B_n X$, the n-th homology of the complex, as $H_n X$. Finally, letting Y be another complex, we denote as $\mathrm{Hom}(X, Y)$ the complex

$$\cdots \longrightarrow \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X_k, Y_{k+n-1}) \longrightarrow \cdots,$$

where, given $f \in \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X_k, Y_{k+n})$, we set $(\delta_n f)_k = d^Y_{n+k} f_k - (-1)^n f_{k-1} d^X_k$. Lastly, we define the complex $\Sigma^n X$ as the complex whose objects are $(\Sigma^n X)_k = X_{k-n}$ and whose differential is $d^{\Sigma^n X}_k = (-1)^n d^X_{k-n}$.

We continue by proving some standard results about categories of chain complexes.

Definition 5.1.1 (Degreewise Ext). *Let $\mathcal{C}$ be an abelian category, let X, Y be objects in $\mathrm{Ch}(\mathcal{C})$, we define the* ***degreewise Ext group*** $\mathrm{Ext}^1_{dw}(X,Y)$ *as the subgroup of* $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X,Y)$ *given by all extensions which are split degreewise.*

Lemma 5.1.2. *Let $\mathcal{C}$ be an abelian category, let X, Y be objects in $\mathrm{Ch}(\mathcal{C})$. We have that*

$$\mathrm{Ext}^1_{dw}(X, \Sigma^{-n-1}Y) \cong H_n\mathrm{Hom}(X,Y) \cong \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, \Sigma^{-n}Y)/\sim,$$

where $\sim$ denotes chain homotopy.

Proof. Let $f = (f_k)_k$ be an element of $\mathrm{Hom}(X,Y)_n$. We see that $f \in \ker \delta_n$ if and only if $d^Y_{n+k} f_k = (-1)^n f_{k-1} d^X_k$, but $(-1)^n d^Y_{n+k}$ is exactly $d^{\Sigma^{-n}Y}_k$, therefore the condition is equivalent to checking if f is a morphism between X and $\Sigma^{-n}Y$. Similarly, the elements of $\mathrm{im}\,\delta_{n+1}$ are of the form

$$d^Y_{k+n+1} f_k - (-1)^{n+1} f_{k-1} d^X_k = (-1)^n d^{\Sigma^{-n}Y}_{k+1} f_k + (-1)^n f_{k-1} d^X_k,$$

thus they are exactly null-homotopic maps from X to $\Sigma^{-n}Y$; therefore $H_n\mathrm{Hom}(X,Y) \cong \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, \Sigma^{-n}Y)/\sim$.
We proceed by considering the complex $W = \Sigma^{-n-1}Y$ and an extension

$$0 \longrightarrow W \longrightarrow Z \longrightarrow X \longrightarrow 0,$$

that is split degreewise. This implies that $Z_k \cong W_k \oplus X_k$, therefore by the commutativity of the diagram

$$\begin{tikzcd}
W_k \arrow[r] \arrow[d, "d^W_k"] & W_k \oplus X_k \arrow[r] \arrow[d, "d^Z_k"] & X_k \arrow[d, "d^X_k"] \\
W_{k-1} \arrow[r] & W_{k-1} \oplus X_{k-1} \arrow[r] & X_{k-1}
\end{tikzcd}$$

we know that

$$d^Z_k = \begin{bmatrix} d^W_k & \phi_k \\ 0 & d^X_k \end{bmatrix}$$

for some $\phi_k \colon X_k \longrightarrow W_{k-1} = \Sigma^{-n}Y$; thus we have found an element $\phi = (\phi_k)_k \in \prod_k \mathrm{Hom}(X_k, Y_{n+k})$. If we substitute the form of d^W_k we just derived in the condition $d^W \circ d^W = 0$ we get that $d^Y_{k+n}\phi_k - (-1)^n \phi_{k-1} d^X_k = 0$, i.e. $\phi \in Z_n\mathrm{Hom}(X,Y)$. Similarly, two extensions Z, Z' are equivalent iff there exists a chain isomorphism h such that

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\mathrm{id}_W"] & Z \arrow[r] \arrow[d, "h"] & X \arrow[d, "\mathrm{id}_X"] \\
W \arrow[r] & Z' \arrow[r] & X
\end{tikzcd}$$

commutes, hence

$$h_k = \begin{bmatrix} 1 & \psi_k \\ 0 & 1 \end{bmatrix}$$

for all k. If we denote as ϕ, ϕ' the elements of $Z_n\mathrm{Hom}(X,Y)$ induced by Z, Z', some quick matrix calculations show that h is actually a morphism of chain complexes iff

$$\phi'_k - \phi_k = (-1)^{n+1} d^Y_{k+n}\psi_k - \psi_{k-1} d^X_k.$$

We conclude by remarking that, up to sign, this is equivalent to asking that $\phi - \phi' \in B_n\mathrm{Hom}(X,Y)$.

□

Lemma 5.1.3. *Let $\mathcal{C}$ be an abelian category, let $A \in ob(\mathcal{C})$ and let $X, Y \in$ $\mathrm{ob}(\mathrm{Ch}(\mathcal{C}))$. The following natural isomorphisms hold:*

1. $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(D^nA, Y) = \mathrm{Hom}_{\mathcal{C}}(A, Y_n)$.
2. $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, D^nA) = \mathrm{Hom}_{\mathcal{C}}(X_{n-1}, A)$.
3. $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(S^nA, Y) = \mathrm{Hom}_{\mathcal{C}}(A, Z_nY)$.
4. $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, S^nA) = \mathrm{Hom}_{\mathcal{C}}(X/B_nX, A)$.
5. $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(D^nA, Y) = \mathrm{Ext}^1_{\mathcal{C}}(A, Y_n)$
6. $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X, D^{n+1}A) = \mathrm{Ext}^1_{\mathcal{C}}(X_n, A)$
7. $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(S^nA, Y) = \mathrm{Ext}^1_{\mathcal{C}}(A, Z_nY)$
8. $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X, S^nA) = \mathrm{Ext}^1_{\mathcal{C}}(Z_nX, A)$

Proof. See [Gil04]. □

Lemma 5.1.4. *If $\mathcal{C}$ is a Grothendieck category, then* $\mathrm{Ch}(\mathcal{C})$ *is also a Grothendieck category.*

Proof. Let G be a generator of $\mathcal{C}$, the adjunction

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(D^nG, X) = \mathrm{Hom}_{\mathcal{C}}(G, X_n)$$

shows that $\{D^nG\}_{n\in\mathbb{Z}}$ is a generating family for $\mathrm{Ch}(\mathcal{C})$. We conclude by remembering that colimits in $\mathrm{Ch}(\mathcal{C})$ are taken dimensionwise. □

5.2 Induced cotorsion pairs

Let $\mathcal{C}$ be an abelian category, let $(\mathcal{D}, \mathcal{E})$ be a cotorsion pair.

Definition 5.2.1 (Induced cotorsion pair)**.** *Let X be a chain complex.*

1. *X is called a $\mathcal{D}$* ***complex*** *if it is exact and $Z_nX \in \mathcal{D}$ for all $n \in \mathbb{Z}$.*

2. *X is called a $\mathcal{E}$ **complex** if it is exact and $Z_nX \in \mathcal{E}$ for all $n \in \mathbb{Z}$.*
3. *X is called a dg-$\mathcal{D}$ **complex** if $X_n \in \mathcal{D}$ for all $n \in \mathbb{Z}$ and $\mathrm{Hom}(X, B)$ is exact for all $\mathcal{E}$ complexes B.*
4. *X is called a dg-$\mathcal{E}$ **complex** if $X_n \in \mathcal{E}$ for all $n \in \mathbb{Z}$ and $\mathrm{Hom}(A, X)$ is exact for all $\mathcal{D}$ complexes A.*

We denote the class of all $\mathcal{D}$ complexes as $\widetilde{\mathcal{D}}$, while the class of all dg-$\mathcal{D}$ complexes is denoted as $\mathrm{dg}\text{-}\widetilde{\mathcal{D}}$. Similarly, we denote the class of all $\mathcal{E}$ complexes as $\widetilde{\mathcal{E}}$, while the class of all dg-$\mathcal{E}$ complexes is denoted as $\mathrm{dg}\text{-}\widetilde{\mathcal{E}}$. We also use $\mathfrak{E}$ to denote the class of all exact complexes.

We begin with a fundamental result on these kinds of complexes.

Proposition 5.2.2. *Every morphism of complexes from a $\mathcal{D}$-complex to an $\mathcal{E}$-complex is null-homotopic.*

Proof. Let us consider $D \in \mathcal{D}$ and $E \in \mathcal{E}$ and chose a morphism $f\colon D \longrightarrow E$. We will show that there exists an homotopically equivalent morphism g such that $g_n d^D_{n+1} = 0$ and $d^E_n g_n = 0$ for all $n \in \mathbb{Z}$, where the subscript indicates the degree of the component; we will also denote with $Z_n f$ the induced morphism between Z_nD and Z_nE. We begin by considering the exact sequence (given us by the exactness of E at homological position $n+1$)

$$0 \longrightarrow Z_{n+1}E \longrightarrow E_{n+1} \xrightarrow{d^E_{n+1}} Z_nE \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{\mathcal{C}}(Z_nD, -)$ to such a sequence yields

$$\mathrm{Hom}_{\mathcal{C}}(Z_nD, E_{n+1}) \xrightarrow{(d^E_{n+1})_*} \mathrm{Hom}_{\mathcal{C}}(Z_nD, Z_nE) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(Z_nD, Z_{n+1}E),$$

therefore, by observing that $Z_nD \in \mathcal{D}$ and $Z_{n+1}E \in \mathcal{E}$ implies the equality $\mathrm{Ext}^1_{\mathcal{C}}(Z_nD, Z_{n+1}E) = 0$, we know that $(d^E_{n+1})_*$ must be surjective; thus there exists a morphism $\alpha_n\colon Z_nD \longrightarrow E_{n+1}$ that satisfies $Z_nf = (d^E_{n+1})_*\alpha_n$. With a similar argument (the vanishing of Ext) we prove that the exactness of D at position n, in the form of the exact sequence

$$0 \longrightarrow Z_nD \longrightarrow D_n \xrightarrow{d^D_n} Z_{n-1}D \longrightarrow 0,$$

implies the surjectivity of the map $\mathrm{Hom}_{\mathcal{C}}(D_n, E_{n+1}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z_nD, E_{n+1})$, therefore there exists $\beta_n\colon D_n \longrightarrow E_{n+1}$ such that α_n is the restriction of β_n to Z_nD. We can thus define the morphism g of complexes (that will be homotopically equivalent to f by definition) component by component, as

$$g_n = f_n + (d^E_{n+1}\beta_n - \beta_{n-1}d^D_n).$$

By direct computation

$$\begin{aligned} d^E_n g_n &= d^E_n f_n + d^E_n d^E_{n+1}\beta_n - d^E_n \beta_{n-1} d^D_n = \\ &= d^E_n f_n - d^E_n \beta_{n-1} d^D_n = d^E_n f_n - f_{n-1} d^D_n = 0 \end{aligned}$$

and similarly $g_n d^D_{n+1} = 0$. We will now prove that this implies that g is null-homotopic (and hence f also is). We observe that $d^E_n g_n = 0$ implies that $\mathrm{im}(g_n) \subseteq Z_n E$ and $g_n d^D_{n+1} = 0$ implies that g_n is also well defined up to quotienting with $B_n D = Z_n D$, therefore g_n induces a morphism $\bar{g_n} D_n / Z_n D \longrightarrow Z_n E$, which makes the following diagram commute (the isomorphism is the first isomorphism theorem).

$$\begin{tikzcd} D_n \arrow[rr, "\mathrm{id}_{D_n}"] \arrow[d, "d^D_n"] & & D_n \arrow[r, "\mathrm{id}_{D_n}"] \arrow[d] & D_n \arrow[d, "g_n"] \\ B_{n-1}D = Z_{n-1}D & & D_n/Z_n D \arrow[ll, "\bar{d^D_n}", "\cong"'] \arrow[r, "\bar{g_n}"] & Z_n E \end{tikzcd}$$

If we set $h_n = \bar{g_n}\left(\bar{d^D_n}\right)^{-1} : Z_{n-1}D \longrightarrow Z_n E$, the argument used to define α shows that there exists $k_n \colon Z_{n-1}D \longrightarrow E_{n+1}$ such that $h_n = \left(d^E_{n+1}\right)_* k_n$. A quick calculation shows that $g = d^E_{n+1} k_n d^D_n - k_{n-1} d^D_{n-1} d^D_n$, therefore $k_n d^D_n$ is the homotopy between g and 0 that we are looking for. □

It is important to remark that if $X \in \widetilde{\mathcal{D}}$ then it must be exact, so X_n is an extension of $Z_n X$ and $Z_{n-1}X$; thus by lemma 4.2.6, $X_n \in \mathcal{D}$ for all $n \in \mathbb{Z}$. Similarly, if $X \in \widetilde{\mathcal{E}}$ then $X_n \in \mathcal{E}$ for all $n \in \mathbb{Z}$. This result, together with proposition 5.2.2, implies that $\widetilde{\mathcal{D}} \subseteq$ dg-$\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{E}} \subseteq$ dg-$\widetilde{\mathcal{E}}$.

Lemma 5.2.3. *Bounded below complexes (i.e. complexes X where there exists $n \in \mathbb{Z}$ such that if $k < n$ then $X_k = 0$) with entries in $\mathcal{D}$ are dg-$\mathcal{D}$ complexes. Bounded above complexes (i.e. complexes X where there exists $n \in \mathbb{Z}$ such that if $k > n$ then $X_k = 0$) with entries in $\mathcal{E}$ are dg-$\mathcal{E}$ complexes.*

Proof. We will prove the first statement. Let X be a bounded below complex with entries in $\mathcal{D}$, we need to show that $\mathrm{Hom}(X, E)$ is exact for any $\mathcal{E}$ complex E. Given that if E is an $\mathcal{E}$-complex then $\Sigma^k E$ is also an $\mathcal{E}$-complex, we only need to prove that all morphisms $f \colon X \longrightarrow E$ are null-homotopic. Let us assume without loss of generality that X_n is the zero object for all negative n. We want to build a family of morphisms $h_k \colon X_k \longrightarrow E_{k+1}$ such that $f_k = d^E_{k+1} h_k + h_{k-1} d^X_k$ for $k \in \mathbb{Z}$, as pictured below.

$$\begin{tikzcd} \cdots \arrow[r, "d^X_3"] & X_2 \arrow[r, "d^X_2"] \arrow[d, "f_2"] \arrow[dl, "h_2"'] & X_1 \arrow[r, "d^X_1"] \arrow[d, "f_1"] \arrow[dl, "h_1"'] & X_0 \arrow[r, "d^X_0"] \arrow[d, "f_0"] \arrow[dl, "h_0"'] & 0 \arrow[r] \arrow[d] \arrow[dl, "h_{-1}"'] & \cdots \\ \cdots \arrow[r, "d^E_3"'] & E_2 \arrow[r, "d^E_2"'] & E_1 \arrow[r, "d^E_1"'] & E_0 \arrow[r, "d^E_0"'] & E_{-1} \arrow[r] & \cdots s \end{tikzcd}$$

We start by setting $h_k = 0$ for all negative k, then we will prove by induction on k that there is such h_k for all $k \geq 0$.
We want to define h_k by assuming the existence of h_{k-1}: we start by setting $g_k := f_k - h_{k-1}d_k^E$, then h_k is (if it exists) exactly a lift of g_k through d_{k+1}

$$\begin{tikzcd} & X_k \arrow[dl, dashed, "h_k"'] \arrow[d, "g_k"] \\ E_{k+1} \arrow[r, "d_{k+1}^E"'] & E_k. \end{tikzcd}$$

We notice that, remembering the inductive hypothesis,

$$d_k^E g_k = d_k^E(f_k - h_{k-1}d_k^E) = d_k^E f_k - f_{k-1}d_k^X + h_{k-2}d_{k-1}^X d_k^X = 0,$$

therefore g_k factors $Z_kE = B_kE$. By applying the functor $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X_k, -)$ to the exact sequence

$$0 \longrightarrow Z_{k+1}E \longrightarrow E_{k+1} \longrightarrow B_kE \longrightarrow 0$$

and remembering that $\mathrm{Ext}^1(X_k, Z_kE) = 0$, we get a surjection

$$(d_{k+1})_* \colon \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X_k, E_{k+1}) \longrightarrow \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X_k, B_k),$$

therefore a lift of g_k must exist.

□

Proposition 5.2.4. *If $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{D}$-objects and enough $\mathcal{E}$-objects (in particular when it is complete), then $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ and $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ are cotorsion pairs.*

Proof. We will only prove the first statement; we begin by showing that $(\text{dg-}\widetilde{\mathcal{D}})^{\perp} \subseteq \widetilde{\mathcal{E}}$. Let $X \in (\text{dg-}\widetilde{\mathcal{D}})^{\perp}$, by applying lemmas 5.1.3 and 5.2.3, we get that, for every $D \in \mathcal{D}$,

$$\mathrm{Ext}^1_{\mathcal{C}}(D, Z_nX) = \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(S^nD, X) = 0,$$

thus we have proven that $Z_nX \in {}^{\perp}\mathcal{D} = \mathcal{E}$ for all n. Let us now consider an epimorphism $f \colon D \longrightarrow Z_nX$ for some $n \in \mathbb{Z}$ and some $D \in \mathcal{D}$ (it must exists given that $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{D}$-objects); as before $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(S^nD, X) = 0$, therefore the induced morphism $f' \colon S^nD \longrightarrow X$ must be null-homotopic and hence $f' = d_{n+1}^X h_n$ for some homotopy $\{h_i \colon (S^nD)_i \longrightarrow X_{i+1}\}_i$. This result implies that $Z_nX = \mathrm{im}(f') \subseteq \mathrm{im}(d_{n+1}^X) = B_nX$. We have concluded that X must be ct, thus $X \in \widetilde{\mathcal{E}}$.
We continue by showing that ${}^{\perp}\widetilde{\mathcal{E}} \subseteq (\text{dg-}\widetilde{\mathcal{D}})$. Let $X \in {}^{\perp}\widetilde{\mathcal{E}}$ and let Y be a $\mathcal{E}$-complex, then $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X, Y) = 0$ and in particular $\mathrm{Ext}^1_{dw}(X, Y) = 0$; so $\mathrm{Hom}(X, Y)$ is exact. To show that $X_n \in \mathcal{D}$ we can apply lemma 5.1.3 to write, for all $E \in \mathcal{E}$,

$$\mathrm{Ext}^1_{\mathcal{C}}(X_n, E) \cong \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X, D^{n+1}E) = 0.$$

The opposite inclusions are immediate consequences of lemma 5.1.2 □

Definition 5.2.5 (Induced cotorsion pairs)**.** *Whenever the pairs* $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ *and* $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ *are cotorsion pairs, they are said to be the* ***induced cotorsion pairs*** *(from* $(\mathcal{D}, \mathcal{E})$*). We say that the induced cotorsion pairs are compatible if* $\widetilde{\mathcal{D}} = \text{dg-}\widetilde{\mathcal{D}} \cap \mathfrak{E}$ *and* $\widetilde{\mathcal{E}} = \text{dg-}\widetilde{\mathcal{E}} \cap \mathfrak{E}$*.*

Lemma 5.2.6. *If* $(\text{dg-}\widetilde{\mathcal{D}},\ \mathfrak{E},\ \text{dg-}\widetilde{\mathcal{E}})$ *is an Hovey triple, then a morphism is a weak equivalence in the induced model structure if and only if it is a quasi-isomorphism.*

Proof. Let $f\colon X \longrightarrow Y$ be a weak equivalence in the induced model structure, so by definition it can be written as qi, where $q\colon Z \longrightarrow Y$ is an acyclic fibration (i.e. an epi with acyclic fibrant kernel) and $i\colon X \longrightarrow Z$ is an acyclic cofibration (i.e. a mono with acyclic cofibrant cokernel). We can therefore write exact sequences as follows,

$$0 \longrightarrow X \xrightarrow{\ i\ } Z \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow B \longrightarrow Z \xrightarrow{\ p\ } Y \longrightarrow 0$$

with $A \in \widetilde{\mathcal{D}}$ and $B \in \widetilde{\mathcal{E}}$. If we consider the induced exact sequences

$$\cdots \longrightarrow H_{k-1}(A) \longrightarrow H_k(X) \xrightarrow{H_k(i)} H_k(Z) \longrightarrow H_k(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_k(B) \longrightarrow H_k(Z) \xrightarrow{H_k(p)} H_k(Y) \longrightarrow H_{k+1}(B) \longrightarrow \cdots,$$

the exactness of A and B implies that $H_k(i)$ and $H_k(p)$ are isomorphisms for all $k \in \mathbb{Z}$, therefore $H_k(f)$ must also be.
Let us now suppose that f is a quasi-isomorphism, by applying (MC4) we can write $f = pi$, where p is an acyclic fibration and i is a cofibration. Given that $\ker(p) \in \widetilde{\mathcal{E}} \subseteq \mathfrak{E}$, by applying the long exact sequence associated to p, it is clear that $H_k(p)$ is an iso for all $k \in \mathbb{Z}$; by hypothesis we know that $H_k(f)$ is also an iso for all $k \in \mathbb{Z}$, so we can conclude that $H_k(i)$ must also be invertible, which implies $\text{coker}(i) \in \mathfrak{E}$. We already know that $\text{coker}(p) \in \text{dg-}\widetilde{\mathcal{D}}$, so, if we remember that by hypothesis $(\text{dg-}\widetilde{\mathcal{D}},\ \mathfrak{E},\ \text{dg-}\widetilde{\mathcal{E}})$ is an Hovey triple, we can conclude that $\text{coker}(i) \in \text{dg-}\widetilde{\mathcal{D}} \cap \mathfrak{E} = \widetilde{\mathcal{D}}$; therefore i is an acyclic cofibration and f is a weak equivalence.

□

Definition 5.2.7 (Derived category)**.** *Let* $\mathcal{C}$ *be an abelian category and let* H *be the class of quasi-isomorphisms of* $\text{Ch}(\mathcal{C})$*; the* ***derived category*** *of* $\mathcal{C}$ *is the localization* $\text{Ch}(\mathcal{C})[H^{-1}]$*.*

From lemma 5.2.6 it is clear that, if we find any Hovey triple on $\mathcal{C}$ by using the induced cotorsion pairs, the homotopy category $\text{Ho}(\text{Ch}(\mathcal{C}))$ must exactly be the derived category of $\mathcal{C}$.

5.3 Hereditary cotorsion pairs

To reach our objective of obtaining a model structure by applying Hovey's correspondence, we need the induced cotorsion pairs to be compatible and complete. The work of Xiaoyan Yang and Nanqing Ding (see [YD14]) has found a complete characterization of which cotorsion pairs in certain complete and bicomplete abelian category induce compatible and complete cotrsion pairs in the category of complexes. We must remark that the paper originally claimed to settle the question in all complete and cocomplete abelian categories, as shown in [Her+25] by Dolors Herbera, Wolfgang Pitsch, Manuel Saorí and Simone Virili, this characterization only works in categories which are both (AB4) and (AB4*).

Let $\mathcal{C}$ be an abelian category and let $(\mathcal{D}, \mathcal{E})$ be a cotorsion pair in $\mathcal{C}$. We begin by proving a rather technical proposition that we will need later to prove the completeness (in under certain hypothesis) of the induced cotorsion pair; to do so we need some preliminary algebraic results.

Lemma 5.3.1. *Let us assume $\mathcal{C}$ to be complete and let's consider an inverse system in* $\mathrm{Ch}(\mathcal{C})$

$$\cdots \longrightarrow P_3 \xrightarrow{\mu_3} P_2 \xrightarrow{\mu_2} P_1 \xrightarrow{\mu_1} P_0.$$

We can define a morphism $\nu\colon \prod_{i\in\mathbb{N}} P_i \longrightarrow \prod_{i\in\mathbb{N}} P_i$ induced by

$$\begin{tikzcd}
\prod_{i\in\mathbb{N}} P_i \arrow[r, "\nu"] \arrow[d, "\pi_{n+1}"'] & \prod_{i\in\mathbb{N}} P_i \arrow[d, "\pi_n"] \\
P_{n+1} \arrow[r, "\mu_{n+1}"] & P_n.
\end{tikzcd}$$

If we assume each μ_i to be epic and dimensionwise split, the following sequence is split exact in each degree.

$$0 \longrightarrow \varprojlim P_n \xrightarrow{\lambda} \prod_{n\geq 0} P_n \xrightarrow{1-\nu} \prod_{n\geq 0} P_n \longrightarrow 0$$

Proof. See [Mur06b], definition 29, lemma 67 and proposition 85. □

Lemma 5.3.2. *Let's consider an inverse system of abelian groups*

$$\cdots \longrightarrow P_3 \xrightarrow{\mu_3} P_2 \xrightarrow{\mu_2} P_1 \xrightarrow{\mu_1} P_0.$$

If each μ_i is surjective, the following sequence is exact, where ν is defined by $\pi_n \circ \nu = \mu_{n+1} \circ \pi_{n+1}$ for all $n \geq 0$ (as in the previous lemma).

$$0 \longrightarrow \varprojlim P_n \xrightarrow{\lambda} \prod_{n\geq 0} P_n \xrightarrow{1-\nu} \prod_{n\geq 0} P_n \longrightarrow 0$$

Proof. See [Wei94], lemma 3.5.3. □

Before proving the following proposition, it is important to remark that results on filtered colimits/injective limits which are dual to the above lemmas also hold (in particular filtered coliimits are exact).

Proposition 5.3.3. *Let us assume that $\mathcal{C}$ is complete, cocomplete, (AB4) and (AB4*); let $X \in \mathrm{Ch}(\mathcal{C})$.*

1. *If $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{E}$-objects, then there exists an exact sequence in $\mathrm{Ch}(\mathcal{C})$*

$$0 \longrightarrow E \longrightarrow P \longrightarrow X \longrightarrow 0,$$

where $E \in \text{dg-}\widetilde{\mathcal{E}}$ and $P \in \mathfrak{E}$.

2. *If $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{D}$-objects, then there exists an exact sequence in $\mathrm{Ch}(\mathcal{C})$*

$$0 \longrightarrow X \longrightarrow P \longrightarrow D \longrightarrow 0,$$

where $D \in \text{dg-}\widetilde{\mathcal{D}}$ and $P \in \mathfrak{E}$.

Proof. We will only prove the first statement, given that the other is dual. Let us assume that $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{E}$-objects and let $X \in \mathrm{Ch}(\mathcal{C})$. We set $C_n := \mathrm{coker}(d^X_{n+1})$ and let $\rho_n := X_n \longrightarrow C_n$ be the natural epimorphism. We will now define a new complex Y^0 by performing the so-called "killing boundaries construction": let us choose an integer $n_0 \in \mathbb{Z}$ and a monomorphism

$$\iota \colon C_{n_0} \longrightarrow E_{n_0-1}$$

for $E_{n_0-1} \in \mathcal{E}$ (there are enough $\mathcal{E}$-objects), so that the objects of the complex are $Y^0_{n_0-1} := X_{n-1} \oplus E_{n_0-1}$ and $Y^0_k := X_k$ $(k \neq n_0)$, while the boundary is given by

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d^X_{n+1}} X_n \xrightarrow{(d^X_n, \iota\rho_n)} X_{n-1} \oplus E_{n_0-1} \xrightarrow{d^X_{n-1}} X_{n-2} \longrightarrow \cdots$$

It is clear that we get a morphism $Y^0 \longrightarrow X$ which is a degreewise split epi and whose kernel is $E^0 := S^{n_0-1}E_{n_0-1} \in \text{dg-}\widetilde{\mathcal{E}}$. It is easy to see that Y^0 is exact at n_0 and that the induced morphism $B_kY^0 \longrightarrow B_kX$ is an isomorphism for $k \neq n_0$. Given a sequence $(n_0, n_1, n_2, \dots)$ such that every integer appears in it infinitely many times we can repeat the killing boundaries construction to obtain a sequence of complexes Y^l and degreewise split epis $Y^{l+1} \longrightarrow Y^l$ such that Y^l is exact at position n_l, $B_kY^{l+1} \longrightarrow B_kY^l$ is an isomorphism for $k \neq l+1$ and

$$\ker(Y^{l+1} \longrightarrow Y^l) = S^{n_l-1}E_{n_l-1}$$

is in dg-$\widetilde{\mathcal{E}}$
We continue by defining, just like we did for $l = 0$, the complex

$$E^l := \ker(Y^l \longrightarrow Y^{l-1} \longrightarrow \cdots \longrightarrow Y^0 \longrightarrow X)$$

and remarking that the induced morphism $E^{l+1} \longrightarrow E^l$ must be degreewise split epis (ker is a functor, so it sends split epis in the arrow category to split epis); it is also important to remark that E^{l+1} is an extension of $S^{n_l-1}E_{n_l-1}$ and E^l, therefore by induction it must also be in dg-$\widetilde{\mathcal{E}}$. Furthermore for all $l \in \mathbb{N}$ we obtain the following degreewise split exact sequences (composition of split epis is a split epi):

$$0 \longrightarrow E^l \longrightarrow Y^l \longrightarrow X \longrightarrow 0.$$

Glueing the sequences associated to each degree yields

$$0 \longrightarrow \prod_{l=0}^{\infty} E^l_k \longrightarrow \prod_{l=0}^{\infty} Y^l_k \longrightarrow \prod_{l=0}^{\infty} X_k \longrightarrow 0,$$

which is split exact for all $k \in \mathbb{Z}$, therefore

$$0 \longrightarrow \prod_{l=0}^{\infty} E^l \longrightarrow \prod_{l=0}^{\infty} Y^l \longrightarrow \prod_{l=0}^{\infty} X \longrightarrow 0,$$

is exact. We continue by setting $E := \varprojlim E^l$ and $P := \varprojlim Y^l$ (and remarking that, trivially, $X = \varprojlim X$) and applying lemma 5.3.1 to obtain a commutative diagram with exact columns and where the last two rows are also exact;

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{l=0}^{\infty} E^l \arrow[r] \arrow[d, "1-\nu"] & \prod_{l=0}^{\infty} Y^l \arrow[r] \arrow[d, "1-\nu"] & \prod_{l=0}^{\infty} X \arrow[r] \arrow[d, "1-\nu"] & 0 \\
0 \arrow[r] & \prod_{l=0}^{\infty} E^l \arrow[r] \arrow[d] & \prod_{l=0}^{\infty} Y^l \arrow[r] \arrow[d] & \prod_{l=0}^{\infty} X \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

from the nine lemma follows the exactness of the sequence

$$0 \longrightarrow E \longrightarrow P \longrightarrow X \longrightarrow 0.$$

To prove that this sequence is the one we are looking for, it remains to prove that $E \in \text{dg-}\widetilde{\mathcal{E}}$ and $P \in \mathfrak{E}$, we begin by investigating E. The first step is proving that $E_k \in \mathcal{E}$ for all k, but if we set $K^l := \ker(E^{l+1} \longrightarrow E^l)$, the following diagram of kernels

$$\begin{tikzcd}
0 \arrow[r] & K^l \arrow[r] \arrow[d] & S^{n_l-1}E_{n_l-1} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E^{l+1} \arrow[r] \arrow[d] & Y^{l+1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^l \arrow[r] & Y^l \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

shows that $K^l \cong S^{n_l-1}E_{n_l-1} \in \text{dg-}\widetilde{\mathcal{E}}$. Given that the morphism in the projective system used to compute E are degreewise split epis, we can compute E_k degreewise to find

$$E_k = \big(\varprojlim E^l\big)_k = E^0_k \oplus \prod_{l=0}^{\infty} K^l_k,$$

thus $E_k \in \mathcal{E}$, given that $\mathcal{E}$ is part of a cotorsion pair and therefore closed under products (see lemma 4.2.7). We can apply the $\text{Hom}(D, -)$ functor (for a certain $D \in \mathcal{D}$) to the short exact sequence

$$0 \longrightarrow K^l \longrightarrow E^{l+1} \longrightarrow E^l \longrightarrow 0$$

to obtain

$$\cdots \longrightarrow \text{Hom}_{\mathcal{C}}(D, E^{l+1}) \longrightarrow \text{Hom}_{\mathcal{C}}(D, E^l) \longrightarrow \text{Ext}^1_{\mathcal{C}}(D, K^l) \longrightarrow \cdots.$$

Given that $K^l \in \text{dg-}\widetilde{\mathcal{E}}$ and $D \in \mathcal{D}$, we know that $\text{Ext}^1_{\mathcal{C}}(D, K^l) = 0$, therefore the morphism

$$\text{Hom}_{\mathcal{C}}(D, E^{l+1}) \longrightarrow \text{Hom}_{\mathcal{C}}(D, E^l)$$

must be epic for all l. Applying lemma 5.3.2, we obtain the following exact sequence:

$$0 \to \varprojlim \text{Hom}_{\mathcal{C}}(D, E^l) \to \prod_{l=0}^{\infty} \text{Hom}_{\mathcal{C}}(D, E^l) \to \prod_{l=0}^{\infty} \text{Hom}_{\mathcal{C}}(D, E^l) \to 0,$$

hence, given that $\prod_{l=0}^{\infty} \text{Hom}_{\mathcal{C}}(D, E^l)$ is exact ($E^l \in \text{dg-}\widetilde{\mathcal{E}}$ and the product of exact complexes of abelian groups is exact), $\varprojlim \text{Hom}_{\mathcal{C}}(D, E^l) \cong \text{Hom}_{\mathcal{C}}(D, E)$ is exact

($\mathrm{Hom}(D, -)$ is a continuous functor).
All that remains is showing that $P \in \mathfrak{E}$, we will show that it is exact at each homological position $k \in \mathbb{Z}$. Let us choose a cofinal subsequence $(n_{m_0}, n_{m_1}, \dots)$ of $(n_0, n_1, \dots)$ such that $n_{m_t} = k$ for all t; such a subsequence must exist given that each integer appears infinitely many times in the original sequence, furthermore the cofinality condition implies that we can compute the limit using only complexes that are exact in position k:

$$P = \varprojlim_{i \in \mathbb{N}} Y^i = \varprojlim_{i \in \mathbb{N}} Y^{m_i}.$$

Applying lemma 5.3.1 allows us to write the following

$$0 \longrightarrow P \longrightarrow \prod_{i=0}^{\infty} Y^{m_i} \xrightarrow{1-\nu} \prod_{i=0}^{\infty} Y^{m_i} \longrightarrow 0,$$

which is split exact in each degree and yields the following exact sequence:

$$0 \longrightarrow H_k(P) \longrightarrow H_k\left(\prod_{i=0}^{\infty} Y^{m_i}\right) \xrightarrow{H_k(1-\nu)} H_k\left(\prod_{i=0}^{\infty} Y^{m_i}\right) \longrightarrow 0.$$

If we denote as μ_j the morphism $Y^{m_j} \longrightarrow Y^{m_{j-1}}$, from the definition of ν follows immediately that

$$\pi_{m_j}(1-\nu) = \pi_{m_j} - \mu_{j+1}\pi_{m_{j+1}}$$

for all j; from the exactness of Y^l and the fact that for abelian groups products are exact, by denoting as $\mu_{t,i}$ and $\pi_{t,i}$ the degree i component of μ_t and π_t, we deduce

$$\begin{aligned}
\mu_{j+1,k}\pi_{m_{j+1},k}\left(Z_k \prod_{i=0}^{\infty} Y^{m_i}\right) &= \mu_{j+1,k}\pi_{m_{j+1},k}\left(\prod_{i=0}^{\infty} Z_k Y^{m_i}\right) = \\
&= \mu_{j+1,k}\left(Z_k Y^{m_{j+1}}\right) = \\
&= \mu_{j+1,k}\left(B_k Y^{m_{j+1}}\right) = \\
&= \mu_{j+1,k} d_{k+1}^{Y^{m_{j+1}}}\left(Y_{k+1}^{m_{j+1}}\right) = \\
&= d_{k+1}^{Y^{m_j}} \mu_{j+1,k+1}\left(Y_{k+1}^{m_{j+1}}\right) \subseteq B_k Y^{m_j}.
\end{aligned}$$

Given that the (AB4*) hypothesis allows us to write

$$\prod_{j=0}^{\infty} B_k Y^{m_j} = B_k \prod_{j=0}^{\infty} Y^{m_j},$$

we can conclude that $H_k(\mu_{j+1}\pi_{m_{j+1}})$ is trivial, therefore $H_k(\pi_{m_j}(1-\nu)) = H_k(\pi_j)$ for all j, thus $1-\nu$ acts as the identity on the k-th homology groups and hence $H_k(P) = 0$. Given that this holds for all k, P must be exact.

□

Definition 5.3.4 (hereditary cotorsion pairs). *We say that the cotorsion pair $(\mathcal{D},\mathcal{E})$ is **hereditary** if $\mathrm{Ext}^i_{\mathcal{C}}(D,E)=0$ for all $i\geq 1$ and $D\in\mathcal{D}, E\in\mathcal{E}$.*

We will additionally need two technical definitions.

Definition 5.3.5 (Covers and envelopes). *Let $\mathcal{F}$ be a subclass of $\mathrm{ob}(\mathcal{C})$, we say that a morphism $\phi\colon F\longrightarrow X$ is an **$\mathcal{F}$-precover** of X if the sequence of abelian groups*

$$\mathrm{Hom}_{\mathcal{C}}(F',F)\xrightarrow{\phi_*}\mathrm{Hom}_{\mathcal{C}}(F',X)\longrightarrow 0$$

*is exact for all $F'\in\mathcal{F}$. An $\mathcal{F}$-precover is said to be **special** if ϕ is an epimorphism and $\ker(\phi)\in\mathcal{F}^{\perp}$. If ϕ is an $\mathcal{F}$-precover such that any endomorphisms $f\colon F\longrightarrow F$ that satisfies $\phi f=\phi$ is necessarily an automorphism of F, we say that ϕ is an **$\mathcal{F}$-cover** of X.*
*Dually, we say that a morphism $\psi\colon X\longrightarrow F$ is an **$\mathcal{F}$-preenvelope** of X if the sequence of abelian groups*

$$\mathrm{Hom}_{\mathcal{C}}(F,F')\xrightarrow{\psi^*}\mathrm{Hom}_{\mathcal{C}}(X,F')\longrightarrow 0$$

*is exact for all $F'\in\mathcal{F}$. An $\mathcal{F}$-preenvelope is said to be **special** if ψ is a monomorphism and $\mathrm{coker}(\psi)\in{}^{\perp}\mathcal{F}$. If ψ is an $\mathcal{F}$-preenvelope such that any endomorphism $f\colon F\longrightarrow F$ that satisfies $f\psi=\psi$ is necessarily an automorphism of F, we say that ψ is an **$\mathcal{F}$-envelope** of X.*

It is clear that requiring every object to have a special $\mathcal{D}$-precover is equivalent to asking that $(\mathcal{D},\mathcal{E})$ has enough projectives, while requiring every object to have a special $\mathcal{E}$-preenvelope is equivalent to asking that $(\mathcal{D},\mathcal{E})$ has enough injectives.

Lemma 5.3.6. *Let $(\mathcal{D},\mathcal{E})$ be a hereditary cotorsion pair in $\mathcal{C}$, the class of objects that admit a special $\mathcal{D}$-precover and the class of objects that admit a special $\mathcal{E}$-preenvelope are closed under extensions, i.e. for any exact sequence*

$$0\longrightarrow X_1\longrightarrow X\longrightarrow X_2\longrightarrow 0$$

in $\mathcal{C}$ the following holds:

1. *If X_1,X_2 have a special $\mathcal{D}$-precover then X has a special $\mathcal{D}$-precover.*
2. *If X_1,X_2 have a special $\mathcal{E}$-preenvelope then X has a special $\mathcal{E}$-preenvelope.*

Proof. We will just prove the first statement since the second follows by duality. Let $\phi_1\colon D_1\longrightarrow X_1$ and $\phi_2\colon D_2\longrightarrow X_2$ be special $\mathcal{D}$-precovers, we want to use these to define a special $\mathcal{D}$-precover of X. The extension ξ

$$0\longrightarrow X_1\longrightarrow X\longrightarrow X_2\longrightarrow 0$$

induces through the morphism ϕ_2 another extension $\xi' := \phi_2^*\xi$ (see section 2.4) which fits the commutative diagram with exact rows below

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & E_2 \arrow[r, "\mathrm{id}_{E_2}"] \arrow[d] & E_2 \arrow[d] & \\
\xi' : \quad 0 \arrow[r] & X_1 \arrow[r] \arrow[d, "\mathrm{id}_{X_1}"] & Y \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d, "\phi_2"] & 0 \\
\xi : \quad 0 \arrow[r] & X_1 \arrow[r] & X \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $Y := X \times_{X_2} D_2$ and $E_2 := \ker(\phi_2)$. If we now set $E_1 := \ker(\phi_1)$, we get an exact sequence

$$0 \longrightarrow E_1 \longrightarrow D_1 \xrightarrow{\phi_1} X_1 \longrightarrow 0$$

from which, by applying $\mathrm{Hom}_{\mathcal{C}}(D_2, -)$, we get

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(D_2, D_1) \xrightarrow{(\phi_1)_*} \mathrm{Ext}^1_{\mathcal{C}}(D_2, X_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{C}}(D_2, E_1) \longrightarrow \cdots .$$

Given that $E_1 \in \mathcal{E}$ (ϕ_1 is special) and $(\mathcal{D}, \mathcal{E})$ is hereditary, $\mathrm{Ext}^2_{\mathcal{C}}(D_2, E_1) = 0$ and so $(\phi_1)_*\colon \mathrm{Ext}^1_{\mathcal{C}}(D_2, D_1) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(D_2, X_1)$ is surjective, therefore there exists an extension ξ''

$$\xi'' : \qquad 0 \longrightarrow D_2 \longrightarrow D \longrightarrow D_1 \longrightarrow 0$$

such that $(\phi_1)_*(\xi'') = \xi'$; given that $\mathcal{D}$ is closed under extensions, $D \in \mathcal{D}$. By writing explicitly the action of $(\phi_1)^*$ we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E_1 \arrow[r, "\mathrm{id}_{E_1}"] \arrow[d] & E_1 \arrow[d] & & \\
\xi'' : \quad 0 \arrow[r] & D_1 \arrow[r] \arrow[d, "\phi_1"] & D \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d, "\mathrm{id}_{D_2}"] & 0 \\
\xi' : \quad 0 \arrow[r] & X_1 \arrow[r] & Y \arrow[r] & D_2 \arrow[r] & 0.
\end{tikzcd}$$

By joining together the two diagrams (and using the snake lemma with the fact that ϕ_1 is epic) we get

$$
\begin{tikzcd}
 & 0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & 0 \\
\xi'' : & 0 \arrow[r] & D_1 \arrow[r] \arrow[d, "\phi_1"] \arrow[dd, bend right, "\phi_1"'] & D \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d, "\mathrm{id}_{D_2}"] \arrow[dd, bend right, "\phi_2"'] & 0 \\
\xi' : & 0 \arrow[r] & X_1 \arrow[r] \arrow[d, "\mathrm{id}_{X_1}"] & Y \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d, "\phi_2"] & 0 \\
\xi : & 0 \arrow[r] & X_1 \arrow[r] & X \arrow[r] & X_2 \arrow[r] & 0,
\end{tikzcd}
$$

where E is the kernel of the composition $\phi\colon D \longrightarrow Y \longrightarrow X$; given that $\mathcal{E}$ is closed under extensions, $E \in \mathcal{E}$ and therefore ϕ_* is surjective (the obstructing Ext vanishes because $(\mathcal{D}, \mathcal{E})$ is a cotorsion pair). We conclude by remarking that both $D \longrightarrow Y$ and $Y \longrightarrow X$ are epic (they are pullbacks/pushouts of epic morphisms), therefore $\phi\colon D \longrightarrow X$ is a special $\mathcal{D}$-precover. $\square$

Remark 5.3.7. *It is important to remark that in the proof above we have actually shown something stronger that the thesis: given special $\mathcal{D}$-precovers $\phi_1\colon D_1 \longrightarrow X_1$ and $\phi_2\colon D_2 \longrightarrow X_2$ we can actually choose a special $\mathcal{D}$-precover $\phi\colon D \longrightarrow X$ such that the following diagram with exact rows and columns commutes*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_1 \arrow[r] \arrow[d, "\phi_1"] & D \arrow[r] \arrow[d, "\phi"] & D_2 \arrow[r] \arrow[d, "\phi_2"] & 0 \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}
$$

where E_1, E_2, E denote the appropriate kernels. The case of preenvelopes is dual.

Lemma 5.3.8. *If $(\mathcal{D}, \mathcal{E})$ is complete and hereditary, then every exact complex admits a special $\widetilde{\mathcal{D}}$-precover and a special $\widetilde{\mathcal{E}}$-preenvelope.*

Proof. Let $X \in \mathfrak{C}$ be an exact complex, therefore for any $k \in \mathbb{Z}$ there exist an exact sequence

$$0 \longrightarrow Z_k X \longrightarrow X_k \longrightarrow Z_{k-1} X \longrightarrow 0.$$

Given that $(\mathcal{D},\mathcal{E})$ is complete we can find special $\mathcal{D}$-precovers $\phi'_i \colon D'_i \longrightarrow Z_iX$ for all $i \in \mathbb{Z}$, so we can apply lemma 5.3.6 and the observation following it to find a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\phi'_k) \arrow[r] \arrow[d] & \ker(\phi_k) \arrow[r] \arrow[d] & \ker(\phi'_{k-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D'_k \arrow[r, "\delta_k"] \arrow[d, "\phi'_k"] & D_k \arrow[r, "\varepsilon_k"] \arrow[d, "\phi_k"] & D'_{k-1} \arrow[r] \arrow[d, "\phi'_{k-1}"] & 0 \\
0 \arrow[r] & Z_kX \arrow[r] \arrow[d] & X_k \arrow[r] \arrow[d] & Z_{k-1}X \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that ϕ_k is a special $\mathcal{D}$-precover. Setting $d^D_k = \delta_{k-1}\varepsilon_k \colon D_k \longrightarrow D_{k-1}$ allows us to define the complex $D = \{(D_k, d^D_k)\}_k$, along with the morphism of complexes $\phi = \{\phi_k\}_k \colon D \longrightarrow X$. We can see for all k that $D_k \in \mathcal{D}$, that

$$\ker(d^D_k) = \ker(\delta_{k-1}\varepsilon_k) = \ker(\varepsilon_k) = D'_k$$

and that

$$\operatorname{im}(d^D_{k+1}) = \operatorname{im}(\delta_k\varepsilon_{k+1}) = \operatorname{im}(\delta_k) = D'_k,$$

therefore $D \in \widetilde{\mathcal{D}}$; the same exact argument shows that the complex $\ker(\phi)$ is in $\widetilde{\mathcal{E}}$. The construction above makes also clear that ϕ_k must be epic, therefore ϕ is the special $\widetilde{\mathcal{D}}$-precover we are looking for. The case of special $\widetilde{\mathcal{E}}$-preenvelopes is dual. □

We are finally ready to prove some of the main result of this thesis: a characterization of completeness and compatibility of the induced cotorsion pair.

Theorem 5.3.9. *Let $(\mathcal{D},\mathcal{E})$ be complete and hereditary and let $\mathcal{C}$ be complete, cocomplete, (AB4) and (AB4*); then the induced cotorsion pairs $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ are complete.*

Proof. Let $X \in \mathrm{Ch}(\mathcal{C})$, by applying proposition 5.3.3 we can find an exact sequence

$$0 \longrightarrow E \longrightarrow P \longrightarrow X \longrightarrow 0$$

with $E \in \text{dg-}\widetilde{\mathcal{E}}$ and $P \in \mathfrak{E}$. Applying lemma 5.3.8 to the exact complex P, we find a special $\widetilde{\mathcal{D}}$ precover ϕ of P as in the sequence below:

$$0 \longrightarrow E' \longrightarrow D \xrightarrow{\phi} P \longrightarrow 0.$$

By defining $E'' := E \times_P D$ we obtain a pullback commutative diagram as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E' \arrow[r, "\mathrm{id}_{E'}"] \arrow[d] & E' \arrow[d] & & \\
0 \arrow[r] & E'' \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, "\phi"'] & X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Given that E'' is an extension of E and E', $E'' \in \text{dg-}\widetilde{\mathcal{E}}$ and that $D \in \widetilde{\mathcal{D}}$, the second row of this diagram shows that $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ has enough projectives.
We now continue by using the completeness of $(\mathcal{D}, \mathcal{E})$ to find a monomorphism $\iota_k \colon X_k \longrightarrow E'_k$ for all $k \in \mathbb{Z}$ with $E'_k \in \mathcal{E}$. By lemma 5.2.3 the complexes $D^{k+1}E'_k$ are in dg-$\widetilde{\mathcal{E}}$, therefore, by lemma 4.2.7, $\prod_{k\in\mathbb{Z}} D^{k+1}E'_k$ also is. Thus, by defining the monomorphism $\psi \colon X \longrightarrow \prod_{k\in\mathbb{Z}} D^{k+1}E'_k$ through

$$\psi_k \colon X_k \xrightarrow{(\iota_k, \iota_{k-1}d_k^X)} \left(\prod_{k\in\mathbb{Z}} D^{k+1}E'_k\right)_k = E'_k \times E'_{k-1}$$

and by setting $Y := \operatorname{coker}(\psi)$, we can find a short exact sequence

$$0 \longrightarrow X \xrightarrow{\psi} \prod_{k\in\mathbb{Z}} D^{k+1}E'_k \longrightarrow Y \longrightarrow 0.$$

If we now apply to Y the fact that $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ has enough projectives (which we have already proven), we can find another exact sequence

$$0 \longrightarrow E'' \longrightarrow D \longrightarrow Y \longrightarrow 0,$$

hence by defining $E := \prod_{k\in\mathbb{Z}} D^{k+1}E'_k \times_Y D$ we get a pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & E'' \arrow[r, "\mathrm{id}_{E''}"] \arrow[d] & E'' \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & E \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "\psi"] & \prod_{k\in\mathbb{Z}} D^{k+1}E'_k \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

whose first exact row is the proof that $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ has enough injectives. The other statements follow by duality.

□

Theorem 5.3.10. *Let $(\mathcal{D}, \mathcal{E})$ be hereditary. If $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{D}$-objects, then* dg-$\widetilde{\mathcal{E}} \cap \mathfrak{E} = \widetilde{\mathcal{E}}$ *and if $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{E}$-objects, then* dg-$\widetilde{\mathcal{D}} \cap \mathfrak{E} = \widetilde{\mathcal{D}}$. *In particular, if $(\mathcal{D}, \mathcal{E})$ is complete then the induced cotorsion pairs are compatible.*

Proof. We will only prove dg-$\widetilde{\mathcal{E}} \cap \mathfrak{E} = \widetilde{\mathcal{E}}$, the other case is dual.
We begin by remarking that we have already proven that $\widetilde{\mathcal{E}} \subseteq$ dg-$\widetilde{\mathcal{E}} \cap \mathfrak{E}$, so we only need to prove the opposite containment. Let X be an exact complex in dg-$\widetilde{\mathcal{E}}$, we want to prove that $Z_kX \in \mathcal{E}$ for all $k \in \mathbb{Z}$, i.e. that $\mathrm{Ext}^1_{\mathcal{C}}(D_0, Z_kX) = 0$ for all $D_0 \in \mathcal{D}$. Given that the exactness of X allows us to write the short exact sequence

$$0 \longrightarrow Z_kX \longrightarrow X_k \longrightarrow Z_{k-1}X \longrightarrow 0,$$

the vanishing of this Ext functor is equivalent to asking that the morphism

$$\mathrm{Hom}_{\mathcal{C}}(D_0, X_k) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(D_0, Z_{k-1}X)$$

is surjective, i.e. that all morphisms $f\colon D_0 \longrightarrow Z_{k-1}X$ lift to X_k. By setting $Y_1 := X_k \times_{Z_{k-1}X} D_0$ we get a pullback diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & Z_kX \arrow[r] \arrow[d, "\mathrm{id}_{Z_kX}"] & Y_1 \arrow[r] \arrow[d] & D_0 \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & Z_kX \arrow[r] & X_k \arrow[r] & Z_{k-1}X \arrow[r] & 0
\end{tikzcd}$$

If we now use explicitly the fact that $(\mathcal{D}, \mathcal{E})$ has enough $\mathcal{D}$-objects, there must be an epimorphism $D_1 \longrightarrow Y_1$, so by composing the necessary morphisms with the diagram above we can find the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker\left(d_1^D\right) \arrow[r] \arrow[d] & D_1 \arrow[r, "d_1^D"] \arrow[d] & D_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_k X \arrow[r] & X_k \arrow[r, "d_k^X"] & X_{k-1}. &
\end{tikzcd}$$

This process can be repeated inductively: from a diagram

$$0 \longrightarrow Z_{n+1}X \longrightarrow X_{n+1} \longrightarrow Z_n X \longrightarrow 0$$

and a morphism $\ker\left(d_{n-k+1}^D\right) \longrightarrow Z_n X$ we can get a pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}X \arrow[r] \arrow[d] & Y_{n-k+2} \arrow[r] \arrow[d] & \ker\left(d_{n-k+1}^D\right) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n+1}X \arrow[r] & X_{n+1} \arrow[r] & Z_n X \arrow[r] & 0
\end{tikzcd}$$

which induces, by choosing an epimorphism $D_{n-k+2} \longrightarrow Y_{n-k+2}$ with $D_{n-k+2} \in \mathcal{D}$ and by taking the appropriate compositions, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\left(d_{n-k+2}^D\right) \arrow[r] \arrow[d] & D_{n-k+2} \arrow[r, "d_{n-k+2}^D"] \arrow[d] & D_{n-k+1} \arrow[d] \\
0 \arrow[r] & Z_{n+1}X \arrow[r] & X_{n+1} \arrow[r] & X_n.
\end{tikzcd}$$

Given that by construction

$$\ker(d_n^D) = \mathrm{im}(Y_{n+1} \longrightarrow \ker(d_n^D)) = \mathrm{im}(d_{n+1}^D),$$

we have thus found an exact complex D

$$\cdots \longrightarrow D_2 \xrightarrow{d_2^D} D_1 \xrightarrow{d_1^D} D_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

and a morphism $D \longrightarrow X$ such that its component in degree 0 is exactly f. We will now prove by induction that $Z_n D \in \mathcal{D}$ for all n (and thus $D \in \widetilde{\mathcal{D}}$): the base step is trivial, $Z_0 D = D_0 \in \mathcal{D}$; by applying $\mathrm{Hom}(-, E)$, for some $E \in \mathcal{E}$, to the short exact sequence

$$0 \longrightarrow Z_n D \longrightarrow D_n \longrightarrow Z_{n-1}D \longrightarrow 0$$

we obtain the exact sequence

$$\cdots \to \mathrm{Ext}^1_{\mathcal{C}}(D_n, E) \to \mathrm{Ext}^1_{\mathcal{C}}(Z_n D, E) \to \mathrm{Ext}^2_{\mathcal{C}}(Z_{n-1}D, E) \to \cdots,$$

but $D_n \in \mathcal{D}$ implies that $\operatorname{Ext}^1_{\mathcal{C}}(D_n, E) = 0$ and the inductive hypothesis plus the fact that the cotorsion pair is hereditary implies that $\operatorname{Ext}^2_{\mathcal{C}}(Z_{n-1}D, E) = 0$, therefore $\operatorname{Ext}^1_{\mathcal{C}}(Z_n D, E) = 0$ and $Z_n D \in \mathcal{D}$. What we have just shown (and the fact that shifting degree preserves membership in $\widetilde{\mathcal{D}}$) implies that $\operatorname{Hom}(\Sigma^{k-1}D, X)$ must be exact and hence the morphism we defined (which is a degree $k-1$ chain map) is nullhomotopic: this, in degree zero, implies exactly that the lift we are looking for exists (if h is the homotopy then $f = d^X_k h$). □

Given that the need to check the vanishing of all Ext^k makes the definition of hereditary cotorsion pair unwieldy, we need to find a simpler equivalent condition to more easily check the hypothesis of the theorems above; luckily the following proposition holds.

Proposition 5.3.11. *Let $(\mathcal{D}, \mathcal{E})$ be a complete cotorsion pair in $\mathcal{C}$, the following are equivalent:*

1. *$(\mathcal{D}, \mathcal{E})$ is hereditary, i.e. $\operatorname{Ext}^i_{\mathcal{C}}(D, E) = 0$ for all $i \geq 1$, $D \in \mathcal{D}$ and $E \in \mathcal{E}$;*
2. *$\operatorname{Ext}^2_{\mathcal{C}}(D, E) = 0$ for all $D \in \mathcal{D}$ and $E \in \mathcal{E}$;*
3. *$\mathcal{D}$ is closed under taking kernels of epimorphisms;*
4. *$\mathcal{E}$ is closed under taking cokernels of monomorphisms.*

Proof. It is clear that (1) ⇒ (2).
To prove (2) ⇒ (3), we begin by letting $D_1, D_2 \in \mathcal{D}$ and letting $\phi\colon D_1 \longrightarrow D_2$ be an epimorphism, then we set $X := \ker(\phi)$; the exact sequence

$$0 \longrightarrow X \longrightarrow D_1 \longrightarrow D_2 \longrightarrow 0$$

induces for all $E \in \mathcal{E}$

$$\cdots \longrightarrow \operatorname{Ext}^1_{\mathcal{C}}(D_1, E) \longrightarrow \operatorname{Ext}^1_{\mathcal{C}}(X, E) \longrightarrow \operatorname{Ext}^2_{\mathcal{C}}(D_2, E) \longrightarrow \cdots,$$

given that $\operatorname{Ext}^1_{\mathcal{C}}(D_1, E) = 0 = \operatorname{Ext}^2_{\mathcal{C}}(D_2, E)$, it is clear that $\operatorname{Ext}^1_{\mathcal{C}}(X, E) = 0$ and $X \in \mathcal{D}$.
Now, to prove (3) ⇒ (1), we consider a representative ξ of an element of $\operatorname{Ext}^n_{\mathcal{C}}(D, E)$ for $n \geq 1, D \in \mathcal{D}$ and $E \in \mathcal{E}$

$$0 \longrightarrow E \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow D \longrightarrow 0$$

We now define another exact complex by repeating the same process of successives pullbacks we used in the proof of theorem 5.3.10 to the complex X given by

$$\cdots \longrightarrow 0 \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow 0 \longrightarrow \cdots,$$

starting from an epimorphism $P_0 \longrightarrow X_0$ with $P_0 \in \mathcal{D}$ (it exists thanks to the completeness of the cotorsion pair), to obtain

$$\begin{tikzcd}
0 \arrow[r] & \ker(d_1^P) \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1^P"] \arrow[d] & P_0 \arrow[d] \\
0 \arrow[r] & Z_1X \arrow[r] & X_1 \arrow[r] & X_0
\end{tikzcd}$$

where the morphisms from $P_1 \in \mathcal{D}$ are defined from an epimorphsim $P_1 \longrightarrow X_1 \times_{X_0} P_0$. By adding preceding zeroes to ξ to form a complex X, we can continue the process to get a semi-infinite complex P and a morphism $P \longrightarrow X$,

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "d_n^P"] \arrow[d] & P_{n-1} \arrow[r, "d_{n-1}^P"] \arrow[d] & P_{n-2} \arrow[r, "d_{n-2}^P"] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & X_{n-1} \arrow[r] & X_{n-2} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & 0
\end{tikzcd}$$

By construction it is clear that $H_0(P) = \mathrm{coker}(X_1 \longrightarrow X_0) = D = H_0(X)$ and that $H_{n-1}(P) = \ker(X_{n-1} \longrightarrow X_{n-2}) = E = H_{n-1}(X)$, hence the morphism is a quasi-isomorphism and induces the following commutative diagram

$$\begin{tikzcd}
\xi'\colon & 0 \arrow[r] & E \arrow[r] \arrow[d, "\mathrm{id}_E"] & \mathrm{coker}(d_n^P) \arrow[r, "d_{n-1}^P"] \arrow[d] & P_{n-2} \arrow[r, "d_{n-2}^P"] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, "\mathrm{id}_D"] & 0 \\
\xi\colon & 0 \arrow[r] & E \arrow[r] & X_{n-1} \arrow[r] & X_{n-2} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & D \arrow[r] & 0,
\end{tikzcd}$$

which is an equivalence between extensions ξ and ξ'. The same inductive argument we used in theorem 5.3.10 shows that $\mathrm{im}(d_i^P) \in \mathcal{D}$ for all i, thus

$$0 \longrightarrow E \longrightarrow \mathrm{coker}(d_n^P) \longrightarrow \mathrm{im}(d_n^P) \longrightarrow 0$$

splits ($\mathrm{Ext}^1_{\mathcal{C}}(\mathrm{im}(d_n^P), E) = 0$); it allows us to find a retraction $\rho\colon \mathrm{coker}(d_n^P) \longrightarrow E$. We can hence write the diagram

$$\begin{tikzcd}
\xi'\colon & 0 \arrow[r] & E \arrow[r] \arrow[d, "\mathrm{id}_E"] & \mathrm{coker}(d_n^P) \arrow[r, "d_{n-1}^P"] \arrow[d, "\rho"] & P_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, "\mathrm{id}_D"] & 0 \\
\xi''\colon & 0 \arrow[r] & E \arrow[r, "\mathrm{id}_E"] & E \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & D \arrow[r, "\mathrm{id}_D"'] & D \arrow[r] & 0
\end{tikzcd}$$

that shows that ξ' is equivalent to ξ'', but it is clear that ξ'' represents 0 in $\mathrm{Ext}^n_{\mathcal{C}}(D, E)$, thus $\mathrm{Ext}^n_{\mathcal{C}}(D, E) = 0$.
The remaining implications (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) follow from duality.

□

From this proposition follow some of the core results.

Corollary 5.3.11.1. *Let $(\mathcal{D}, \mathcal{E})$ be complete, the following holds:*

1. *If $(\mathcal{D}, \mathcal{E})$ is hereditary, then the induced cotorsion pairs $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ in $\mathrm{Ch}(\mathcal{C})$ are also hereditary.*
2. *If either of the induced cotorsion pairs $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ is hereditary, then $(\mathcal{D}, \mathcal{E})$ is hereditary.*
3. *If $\mathcal{C}$ is complete and cocomplete, the following statement are equivalent:*
   (a) $(\mathcal{D}, \mathcal{E})$ is hereditary.
   (b) $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ are hereditary.
   (c) $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ are compatible.

Proof. (1) Let's begin by proving that $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ is hereditary by showing that $\text{dg-}\widetilde{\mathcal{D}}$ is closed under taking kernels of epimorphism: we consider an exact sequence

$$0 \longrightarrow D \longrightarrow D' \longrightarrow D'' \longrightarrow 0$$

where $D', D'' \in \text{dg-}\widetilde{\mathcal{D}}$; given that $(\mathcal{D}, \mathcal{E})$ is hereditary and D_k is the kernel of the morphism between $D'_k \in \mathcal{D}$ and $D''_k \in \mathcal{D}$, it is clear that $D_k \in \mathcal{D}$ for all k. If we choose a complex $E \in \widetilde{\mathcal{E}}$, we get an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(D_{k+n}, E_n) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(D'_{k+n}, E_n) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(D''_{k+n}, E_n) \arrow[dll] \\ & \mathrm{Ext}^1_{\mathcal{C}}(D_{k+n}, E_n) \arrow[r] & \mathrm{Ext}^1_{\mathcal{C}}(D'_{k+n}, E_n) \arrow[r] & \cdots, \end{tikzcd}$$

given that for all $k, n \in \mathbb{Z}$ we know that $\mathrm{Ext}^1_{\mathcal{C}}(D_{k+n}, E_n) = 0$ and in the category of abelian groups products are exact, we get an exact sequence

$$0 \to \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(D_{k+n}, E_n) \to \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(D'_{k+n}, E_n) \to \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(D''_{k+n}, E_n) \to 0,$$

thus the following sequence is exact.

$$0 \longrightarrow \mathrm{Hom}(D, E) \longrightarrow \mathrm{Hom}(D', E) \longrightarrow \mathrm{Hom}(D'', E) \longrightarrow 0$$

From the associated long exact sequence is clear that $\mathrm{Hom}(D, E)$ is exact, therefore $D \in \text{dg-}\widetilde{\mathcal{D}}$. The other case is dual.
(2) Let us assume that $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ is hereditary; we consider an exact sequence in the base category $\mathcal{C}$ as follows

$$0 \longrightarrow X \longrightarrow X' \longrightarrow X'' \longrightarrow 0,$$

with $X', X'' \in \mathcal{D}$, we need to show that $X \in \mathcal{D}$. It is clear that the complexes D^0X' and D^0X'' are in $\widetilde{\mathcal{D}}$, therefore using the following exact sequence

$$0 \longrightarrow D^0X \longrightarrow D^0X' \longrightarrow D^0X'' \longrightarrow 0$$

and the hereditariness of $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ (in the form of proposition 5.3.11), we know that $D^0X \in \widetilde{\mathcal{D}}$; this implies $X \in \mathcal{D}$.
(3) The fact that condition (a) implies condition (c) is exactly theorem 5.3.10, while the fact that condition (a) is equivalent to condition (b) is given by points (1) and (2) of this corollary; thus we only need to show that condition (c) implies condition (a). We will show that $\widetilde{\mathcal{D}} = \text{dg-}\widetilde{\mathcal{D}} \cap \mathfrak{E}$ implies that $(\mathcal{D}, \mathcal{E})$ is hereditary. Let us consider an exact sequence

$$0 \longrightarrow D \longrightarrow D' \longrightarrow D'' \longrightarrow 0,$$

with $D', D'' \in \mathcal{D}$. Given that $(\mathcal{D}, \mathcal{E})$ is complete (in particular it has enough projectives) we can find an exact complex

$$X\colon \qquad \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow D' \longrightarrow D'' \longrightarrow 0 \longrightarrow \cdots$$

with $X_k \in \mathcal{D}$ for all k. By applying lemma 5.2.3, we find that $X \in \text{dg-}\widetilde{\mathcal{D}}$. By construction $X \in \mathfrak{E}$, so the compatibility condition implies $X \in \widetilde{\mathcal{D}}$; it follows that $Z_0X = \ker(D' \longrightarrow D'') = D \in \mathcal{D}$. □

We can finally put together all that we have proven thus far.

Corollary 5.3.11.2. *If $\mathcal{C}$ is complete, cocomplete, (AB4) and (AB4*) and if $(\mathcal{D}, \mathcal{E})$ is complete and hereditary then there exists a model structure on* $\mathrm{Ch}(\mathcal{C})$ *where:*

- $\mathfrak{E}$ *is the class of trivial objects.*
- $\text{dg-}\widetilde{\mathcal{D}}$ *is the class of cofibrant objects.*
- $\widetilde{\mathcal{D}}$ *is the class of acyclic cofibrant objects.*
- $\text{dg-}\widetilde{\mathcal{E}}$ *is the class of fibrant objects.*
- $\widetilde{\mathcal{E}}$ *is the class of acyclic fibrant objects.*

5.4 Some examples

We would like to produce some standard examples of Hovey triples on categories of chain complexes.

Example 5.4.1. *Let R be a commutative ring and let $\mathcal{C} = R$-Mod. Given that R-Mod is complete, cocomplete, (AB4) and (AB4*) we can use corollary 5.3.11.2 to find many different model structures to describe the derived category. One example is as follows: the cotorsion pair $(\mathrm{proj}(\mathcal{C}), \mathrm{ob}(\mathcal{C}))$ is complete given that R-Mod has enough projectives; furthermore it is a standard result of homological algebra that, for all $X \in \mathrm{ob}(\mathcal{C}))$, $P \in \mathrm{proj}(\mathcal{C})$ and $k \geq 1$, we have $\mathrm{Ext}^k_{\mathcal{C}}(P, X)$ is 0 and thus $(\mathrm{proj}(\mathcal{C}), \mathrm{ob}(\mathcal{C}))$ is hereditary. The model structure we have defined is called the projective model structure on R-Mod. A similar argument with $(\mathrm{ob}(\mathcal{C}), \mathrm{inj}(\mathcal{C}))$ defines the injective model structure on R-Mod.*

In a more general context (e.g. Grothendieck categories), the hypothesis of theorem 5.3.9 (and thus of corollary 5.3.11.2) may not hold, therefore we need some other tool.

Definition 5.4.2 (Locally cogenerated class)**.** *Let κ be a cardinal and let $\mathcal{C}$ be a category, we say that an object X of $\mathcal{C}$ is **κ-generated** if the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ commutes with κ-filtered colimits of monomorphism (i.e. it is κ-small relative to monomorphism, if we generalize the definition to allow κ-filtered colimits that are not indexed on ordinals). For any class $\mathcal{F} \subseteq \mathrm{ob}(\mathcal{C})$, we denote with $Gen_\kappa\, \mathcal{F}$ the class of all κ-generated elements of $\mathcal{F}$.*
*Let now κ be a regular cardinal and let $\mathcal{F} \subseteq \mathrm{ob}(\mathcal{C})$, we say that $\mathcal{F}$ is a **κ-locally cogenerated class** if every nonzero $F \in \mathcal{F}$ has a nonzero subobject $S \subseteq F$ such that $S \in Gen_\kappa\, \mathcal{F}$ and $F/S \in \mathcal{F}$. We say that $\mathcal{F}$ is a **locally cogenerated class** if it is a κ-locally cogenerated class for any regular cardinal κ.*

Proposition 5.4.3. *Let $\mathcal{C}$ be a Grothendieck category. If $\mathcal{F}$ is a locally cogenerated class containing a generator, closed under transfinite extensions and retracts, then the cotorsion pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by $Gen_\kappa\, \mathcal{F}$ and is a small cotorsion pair; in particular it must be complete.*

Proof. See proposition 4.8 of [Gil07]. $\square$

We continue with the so called injective model structure on $\mathrm{Ch}(\mathcal{C})$, where $\mathcal{C}$ is a Grothendieck category.

Example 5.4.4. *We note that the cotorsion pair $(\mathrm{ob}(\mathcal{C}), \mathrm{inj}(\mathcal{C}))$ is small by Baer's criterion (see proposition 2.6.7), therefore it must also be complete. It is also a standard result of homological algebra that, for all $X \in \mathrm{ob}(\mathcal{C}))$, $I \in \mathrm{inj}(\mathcal{C})$ and $k \geq 1$, we have $\mathrm{Ext}^k_{\mathcal{C}}(X, I)$ is 0, so this cotorsion pair is hereditary. By applying theorem 5.3.10, we know that $(\text{dg-}\widetilde{\mathrm{ob}(\mathcal{C})}, \widetilde{\mathrm{inj}(\mathcal{C})})$ and $(\widetilde{\mathrm{ob}(\mathcal{C})}, \text{dg-}\widetilde{\mathrm{inj}(\mathcal{C})})$ are compatible. We would like to check what exactly are these four classes of complexes.*

- *A complex is in $\widetilde{\mathrm{ob}(\mathcal{C})}$ if and only if it is exact and its cycles are in $\mathrm{ob}(\mathcal{C})$; given that the second condition is trivial, $\widetilde{\mathrm{ob}(\mathcal{C})} = \mathfrak{E}$.*

- *A complex is in* $\widetilde{\mathrm{inj}(\mathcal{C})}$ *if and only if it is exact and its cycles are in* $\mathrm{inj}(\mathcal{C})$*; the fact that* Z_nX *is injective implies that* $\mathrm{Ext}^1_{\mathcal{C}}(Z_{n-1}X, Z_nX) = 0$*, thus the exact sequence*

$$0 \longrightarrow Z_nX \longrightarrow X_n \longrightarrow Z_{n-1}X \longrightarrow 0$$

  splits. We conclude that the $\mathrm{inj}(\mathcal{C})$*-complexes are exactly the complexes that can be written as countable products of disks* D^kI*, where* I *is injective. It is known that these objects are exactly the elements of* $\mathrm{inj}(\mathrm{Ch}(\mathcal{C}))$*, see section 2.3 of [Gar99] for a proof in the category of modules; the argument easily generalizes.*

- *A complex* Y *is in* $\mathrm{dg}\text{-}\widetilde{\mathrm{inj}(\mathcal{C})}$ *if* Y_k *is injective for all* $k \in \mathbb{Z}$ *and* $\mathrm{Hom}(X, Y)$ *is exact for all exact complexes* X*. The complexes in* $\mathrm{dg}\text{-}\widetilde{\mathrm{inj}(\mathcal{C})}$ *are said to be K-injective.*

- *A complex is in* $\mathrm{dg}\text{-}\widetilde{\mathrm{ob}(\mathcal{C})}$ *if and only if it is in* ${}^{\perp}\widetilde{\mathrm{inj}(\mathcal{C})} = {}^{\perp}\mathrm{inj}(\mathrm{Ch}(\mathcal{C}))$*; therefore* $\mathrm{dg}\text{-}\widetilde{\mathrm{ob}(\mathcal{C})} = \mathrm{ob}(\mathrm{Ch}(\mathcal{C}))$*.*

Given that $\mathrm{Ch}(\mathcal{C})$ *is Grothendieck just like* $\mathcal{C}$*, we don't need to check if the pair* $(\mathrm{dg}\text{-}\widetilde{\mathrm{ob}(\mathcal{C})}, \widetilde{\mathrm{inj}(\mathcal{C})}) = (\mathrm{ob}(\mathrm{Ch}(\mathcal{C})), \mathrm{inj}(\mathrm{Ch}(\mathcal{C})))$ *is complete; it remains only to verify the completeness of the pair* $(\widetilde{\mathrm{ob}(\mathcal{C})}, \mathrm{dg}\text{-}\widetilde{\mathrm{inj}(\mathcal{C})}) = (\mathfrak{E}, K\text{-}inj)$*. It is clear that* $\mathfrak{E}$ *is closed under retractions and the fact that* $\mathfrak{E}$ *is closed under transfinite extensions follows from the fact that every Grothendieck category is (AB5), therefore we can apply proposition 5.4.3, since the result that* $\mathfrak{E}$ *is locally cogenerated follows from standard results on cardinality in Grothendieck categories and the fact that a quotient of exact complexes is exact. We have hence constructed the injective model structure on* $\mathrm{Ch}(\mathcal{C})$*.*

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