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High Probability Logic through Conditional Logic: a proof-theoretic approach

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Introduction

Conditional logics play a fundamental role both in natural language and in formal reasoning. Under this name we group all the logics that aim to represent some kind of conditionality - namely a dependence between two propositions - which is more fine-grained than classical material implication. For this reason a different interpretation than the one of classical logic is called for. The wide variety of conditionals found in everyday language - ranging from factual, to non-monotonic, and to counterfactual conditionals - makes their analysis and classification a complex task.

In this thesis we explore logical systems in which conditionality is represented by enriching the classical logic propositional language with operators apt to represent conditional sentences. Among the several alternative approaches that have been developed we analyze the possible-worlds account, first introduced by Stalnaker [30] and Lewis [23], which provides a semantic framework for the interpretation of the famous class of counterfactual conditionals, and Adams' probabilistic approach ([1], [3]), which suggests that conditionals should be evaluated in terms of probability rather than absolute truth. More specifically, we will discuss Lewis' conditional logic $\mathbb{V}$ and its extensions, and Adams' logic **HPA** for high probability.

Proof theory is the discipline that studies proofs, treating them as mathematical objects and analyzing their properties in a formal system. In this context, we focus on sequent calculus, a powerful and expressive formalism introduced by Gerhard Gentzen in [13]. If compared with other proof systems, sequent calculus offers, among others, the advantage that it is well-suited for automated proof search: to determine whether a formula is derivable or not in a logical system, starting from the formula itself, inference rules are systematically applied until either an instance of an initial sequent is reached or no further rules can be applied. To this aim, sequent calculi pursue the desirable property of analyticity: all that is needed to prove a formula has to be contained in the formula itself. A key result is then to prove that certain rules like *cut* - that depend from analyticity - can in fact be eliminated. We focus in this thesis on the proof-theoretic properties of the conditional logics examined. Lewis' conditional logic $\mathbb{V}$ is already inserted in a rich apparatus of proof systems, of which we report the main approaches. High probability logic **HPA**, instead, is well-formalized in a (Hilbert-style) axiomatic system, but lacks of a proof system with better analytic properties.

Our objective is to explore the connection between these two logics, exploiting the good properties of the sequent calculi for the better-known logic $\mathbb{V}$ and assessing its compatibility with **HPA** and the implications that follow.

Another relevant feature of this study is that it considers the two different ways to define the logics in exam, namely the semantic account, relying on models, and the (axiomatic) syntactic account, pointing out which are the relations between them. We relate these two different approaches in two different types of sequent calculi. *Labelled sequent calculi* are built starting from the semantics and feature a close relationship with the models; *internal sequent calculi*, instead, are characterized by a "formula interpretation": every sequent of a derivation has a meaning in the language on which the logic is based. The sequent calculi on which we are mainly interested are the labelled calculus **G3V**, presented in [16] and the internal calculus $\mathcal{I}_{\mathbb{V}}$, introduced in [15]. In the paper [16] an interesting map relating these two calculi is presented as well.

In conclusion, the primary objective of this thesis is to determine the relationship between Adams's logic **HPA** and the (much better-known) logics of Lewis, further developing observations made by Adams himself (cf. [5]). Specifically, to achieve this goal, we have employed the internal sequent calculus IV.

The structure of this thesis is as follows: Chapter 1 introduces Lewis' conditional logic, analyzing possible-world semantics and in particular sphere models. Chapter 2 addresses proof theory for conditional logics, focusing on labelled and internal sequent calculi. Chapter 3 is dedicated to Adams' high probability logic, while Chapter 4 examines the relationships between logic $\mathbb{V}$ (and its extensions) and logic **HPA**, evaluating their theoretical implications.

Chapter 1

Lewis' conditional logic

A conditional is, in everyday natural language, an expression relating a proposition (called the antecedent) to another proposition (the consequent). In the English language, for example, the conditional with antecedent A and consequent B is often expressed by the sentence:

If A, *then* B.

The variety of conditionals in the natural language is wide. For instance, factual conditionals express cause-effect relation, non-normal conditionals express a relation that is *usually* true, deontic conditionals express obligation, counterfactual conditionals express consequences of a *state of affairs* that did not obtain. Many other types of conditionals exist, and with the most varied and diverse features, making it hard to find general criteria to classify them.

Also in mathematical reasoning conditionals play a key role: they make it possible to deduct a conclusion from certain hypotheses, at different levels of reasoning. For instance, the sentence

If the natural number n is divisible by four, *then* n is even.

is a conditional sentence. The usual interpretation of conditionals in mathematics is the one of classical logic, in which a conditional is interpreted by *material implication*, here denoted with $\supset$. Material implication is truth-functional, in the sense that the truth value of the conditional statement $A \supset B$ is determined by the truth values of its components A and B. In particular, a conditional $A \supset B$ is classically true if either its antecedent A is false, or its consequent B is true. This interpretation comes along with the so-called "paradoxes of material implication", namely the cases in which sentences that - to our intuition - shouldn't be true, or at least sound far-fetched, are instead validated. For instance, "If $2 + 2 = 5$, then there would be no even numbers" is classically true, since its antecedent is false. Also conditional sentences with true consequent are always true, leading to analogous problematic examples.

Given the large variety of conditionals in the natural language, this paradoxical behaviour is even emphasized when it comes to everyday language. A very problematic and relevant class of conditionals is the one of counterfactuals, that can be characterized as conditional sentences in which the antecedent is false (in the actual world). In the classical interpretation counterfactuals are true no matter what, since their antecedent is false. In everyday reasoning we would like to give different truth values to them, though. For example the following two counterfactual statements - adapted from [14] - are true, according to the classical interpretation:

(1) If Harris had won the elections, Trump would not be president.

(2) If Harris had won the elections, she would have refused the position.

Even if both are counterfactuals, we want to be able to tell that (1) is true, while (2) is false, at least assuming Harris' willingness to become president.

Simple sentences like the previous examples make it clear that there is a mismatch between the classical, truth-functional interpretation of implication and the meaning we want to give to some classes of conditionals. Counterfactuals are surely the most notable example of this discrepancy; for this reason, several approaches have been proposed to deal with counterfactuals, the principal ones being the *possible worlds account*, developed notably by Lewis [23] and Stalnaker [30], and the *probabilistic account* of Adams (among the others [1], [3]). These approaches, eventually with little modifications, ended up to be fitting not only for counterfactuals, but for other classes of conditionals as well. We will discuss Adams' probabilistic account in Chapter 3, in particular as regards sentences expressing high probability, while we will deepen the possible world account and its developments in this chapter.

Among the many existing conditional logic systems we will mention logic $\mathbb{PCL}$ (*Preferential Conditional Logic*), introduced by Burgess in [8], and look more closely at logic $\mathbb{V}$ (logic of *Variably strict conditionals*), introduced by Lewis in [23]. The strength of these basic systems is that it is possible to embrace different families of conditionals by modifying them slightly.

To describe Lewis' system of logics arising out of $\mathbb{V}$ we will look at it from the two different points of view of semantics and syntax. A *semantic* characterization of the logic consists in the construction of a class of models fitting the logic, i.e. a class of structures endowed with an interpretation that assigns the truth values to formulas in the language. A *syntactic* characterization consists in determining a set of axioms and inference rules from which it is possible to derive true propositions from the axioms. These two characterizations are linked by the results of *soundness* and *completeness*, that prove that a formula is *valid* in the model if and only if it can be *derived* in the syntactic system.

We will mention in Section 1.1 some different semantic models for conditional logics, to finally define the models that Lewis found for conditional logic, namely *sphere models*. Then in section 1.2 we will present two different axiomatizations for Lewis' conditional logics, that are sound and complete with respect to sphere models. For a complete and detailed discussion over these topics we refer to [14].

1.1 Semantics

Many models have been conceived to describe conditional logics in the possible worlds account. In this section we we will give a general overview on the principal models of this kind designed for conditionals, namely selection function models, preferential models, neighbourhood models and sphere models.

A large part of the efforts in this direction has focused on counterfactual conditionals, which occupy a relevant position in this discussion for the evidence of the big gap from classical implications; some attempts ended up to fit other classes of conditionals though. In fact, selection function models have been devised by Stalnaker [30] to describe counterfactuals. Preferential models, studied by Lewis [23] and Burgess [8] among the others, can fit some systems of non-monotonic logics, i.e. logics allowing the conclusions to be revised in light of new information. Neighbourhood models are a generalization of sphere models, that according to Lewis cover not only counterfactuals, but also other classes of conditionals.

In particular, we will mention in Section 1.1.1 the main features of selection function models, preferential models and neighbourhood models. For the presentation of these models we follow Sections 1.3 and 1.4 of [14]. Given this general framework, we will introduce sphere models, which are our main focus, in Section 1.1.2. We will see in Chapter 2 that basing a proof systems on sphere models can simplify the structure of the rules; moreover, sphere models will play a role of primary importance in Chapter 4.

1.1.1 Possible worlds accounts

We want to showcase the models for conditional logic mentioned above and some of their main properties.

The language for conditional logics we will employ in this section is the extension of the classical propositional language with the binary conditional operator $>$:

Definition 1.1. Let **Prop** be a countably denumerable set of propositional variables. The set of well formed **formulas** $\mathcal{F}_>$ of a propositional conditional logic is generated by the following grammar, for p in **Prop** and A, B in $\mathcal{F}_>$:

$$A, B ::= p \mid \bot \mid A \,\&\, B \mid A \vee B \mid A \supset B \mid A > B$$

The connective $>$ is the conditional operator, which can be seen as expressing a non-monotonic conditional, a counterfactual, a deontic conditional or conditionals of other classes, depending on the system we are considering. We will call it *conditional*, or *non-material implication*, in opposition to the *(material) implication* $\supset$. Finally, for any formulas A, B in $\mathcal{F}_>$, the negation of A is written $\neg A$, as an abbreviation of $A \supset \bot$; the constant for true is denoted by $\top$ and is an abbreviation of $\neg\bot$; the biconditional, denoted $A \supset\subset B$, is an abbreviation for the formula $(A \supset B) \,\&\, (B \supset A)$.

We start from the observation that we want conditionals not to be truth-functional: not only conditionals of different kinds, but also conditionals from the same class (such as counterfactuals) may need to be evaluated in different ways, even if their compounds have the same truth values. In Kripke models for modal logics (cf. [7]) this requirement is met: truth of modal operators depend not only on the *actual* truth values of their components, but also on the richer structure of the model. Operators of this kind are called *intensional*. The modal operator $\Box$, for instance, features this property.

In order to draw inspiration from this perspective, we start this section with an overview on Kripke models for the modal logic **K**. For a detailed treatment on modal logic we refer to [7] and [26].

Definition 1.2. A **(modal) frame F** is a structure $\langle W, R\rangle$, where:

- W is a non-empty set, called the *set of possible worlds*;
- $R \subseteq W \times W$ is a binary relation on W and we will call it the *accessibility relation*.

A propositional evaluation establishes which propositional variables are true at a world x of the set W:

Definition 1.3. Let $\mathbf{F} = \langle W, R\rangle$ be a frame; a **propositional evaluation** is a map

$$[\![\ ]\!]\colon \mathbf{Prop} \longrightarrow \mathcal{P}(W)$$

Finally, building on the previous definitions, we define a Kripke model:

Definition 1.4. A **Kripke Model** $\mathcal{M}$ is a frame $\mathbf{F}{=}\langle W, R\rangle$ endowed with a valuation $[\![\ ]\!]$:

$$\mathcal{M} = \langle W, R, [\![\ ]\!]\rangle$$

We say that the model $\mathcal{M}{=}\langle W, R, [\![\ ]\!]\rangle$ is based on the frame $\mathbf{F}{=}\langle W, R\rangle$.

The propositional evaluation $[\![\ ]\!]$ is extended to all the formulas of the modal language as follows:

Definition 1.5. We call $\mathcal{F}_\Box$ the extension of the propositional language with the intensional modal operator $\Box$. Let $\mathbf{F}{=}\langle W, R\rangle$ be a frame. The propositional evaluation

$$[\![\ ]\!]\colon \mathbf{Prop} \longrightarrow \mathcal{P}(W)$$

can be inductively **extended** to any formula of the modal language $\mathcal{F}_\Box$ in the following "standard" way. Let $x \in W$:

- $[\![\bot]\!] = \emptyset$
- $x \in [\![A \,\&\, B]\!]$ *iff* $x \in [\![A]\!] \cap [\![B]\!]$
- $x \in [\![A \vee B]\!]$ *iff* $x \in [\![A]\!] \cup [\![B]\!]$
- $x \in [\![A \supset B]\!]$ *iff* $x \in [\![A]\!] \subseteq [\![B]\!]$
- $x \in [\![\Box A]\!]$ *iff* for all $y \in W$, if xRy then it holds that $y \in [\![A]\!]$

We can now set the standard definitions to describe different notions of validity of a formula:

Definition 1.6. Given a Kripke model $\mathcal{M}$, with underlying set W and evaluation function $[\![\ ]\!]$, we say that of a formula A is **valid** (or true) **at a world** x *iff* $x \in [\![A]\!]$, and we will equivalently write $x \Vdash A$. We say that A is **valid** (or true) **in the model** $\mathcal{M}$, and write $\mathcal{M} \models A$, *iff* A is true at all the worlds in the model. Finally, we say that A is **valid**, and write $\models A$, *iff* A is valid in all models. In this case, we say that A is a **theorem** of the logic.

Once we have given the basics for *possible worlds* Kripke semantics defined over formulas of the modal language, we are now ready to include the conditional in the discussion.

The framework of Kripke models looks suitable to describe conditionals. In fact, all the models we present in this section are classes of Kripke models enriched or modified with certain properties. We list hereafter some of the main approaches existing, to introduce in Section 1.1.2 the semantics of our main interest: sphere models.

A first attempt of describing (counterfactual) conditionals, by C. I. Lewis [22], is to interpret them as a *strict conditional*, namely:

$$A > B := \Box(A \supset B)$$

The conditional operator defined in this way is intensional, depending on the modal necessity operator, and captures well the meaning of simple counterfactual sentences. But this approach is problematic when we consider several counterfactuals together, as pointed out by Lewis' "party example", in Section 1.2 of [23].

The fact that no Kripke model can describe situations of this kind brings in the idea of enriching such models with additional properties, in order to capture more specific aspects of different classes of conditionals.

In **selection function models** (cf. [30]), instead of a binary relation (as in Kripke models) the models feature a selection function f. This function takes as arguments a world x and the set $[\![A]\!]$ of the worlds in which a certain formula A is true, returning the world y which is the most similar world to x, in which A holds.

The truth condition for a conditional $A > B$ at a selection function models is

$$x \Vdash A > B \textit{ iff } f(x, [\![A]\!]) \Vdash B$$

i.e. the conditional $A > B$ is true at the world x when, in the most similar world to the actual one in which the antecedent A holds, also the consequent B holds. This truth condition represents well counterfactuals. However, it was criticized by Lewis for being based on too strong assumptions; a detailed treatment on this topic is out of our scopes.

Preferential models are structures

$$\mathcal{M} = \langle W, \{W_x\}_{x \in W}, \{\preceq_x\}_{x \in W}, [\![\ ]\!] \rangle$$

where W is a non-empty set, $[\![\ ]\!]$ is a propositional evaluation and, for any world x, $W_x \subseteq W$ represents the set of the worlds accessible from x and $\preceq_x$ is a reflexive and transitive relation over W_x. In these models, the notation $y \preceq_x z$ is interpreted as "world y is more similar to world x than world z is". Intuitively a conditional formula $A > B$ is true at x when for every world w in W_x in which the antecedent A holds, there is a world y more similar to x than w such that A holds at y and $A \supset B$ holds at every world more similar to x than y.

Neighbourhood models are a generalization of sphere models (cf. Section 1.1.2) in which to each world x is associated a family of sets of worlds, that in general does not need to be nested. These sets are called *neighbourhoods*, and the worlds in each neighbourhood are interpreted as the ones sharing the same degree of similarity to the actual world x. The truth condition for conditional formulas is similar to the one of sphere models (see Definition 1.9), but the inclusion between spheres - neighbourhoods here - must be imposed.

On the other hand, this class of models is more general than sphere models, and can thus represent wider classes of conditionals; on the other hand, though, the further restriction that's needed to express the truth of a conditional makes the condition more complicated than the one of sphere models, and this is the reason why we have preferred sphere models in this study.

1.1.2 Sphere models

Sphere models were introduced by Lewis in [23]. This particular class of Kripke models is enriched with spheres, sets of worlds with the same degree of similarity to the actual world, characterized by the property of being *nested*. Lewis came to this idea in opposition to C. I. Lewis' *strict conditionals*, presented in section 1.1.1. For sphere models Lewis talks instead of *variably strict conditionals*, in the sense that not only the worlds related to the actual ones matter in determining the truth value of a conditional (as happened for the intensional strict conditional), but even how similar these worlds are to the actual one is a key factor. In this way the truth value of a formula does not depend just on the truth values of its constituents at a certain world, but also on how far that world is from the actual one.

In fact, conditional sentences that are classically evaluated to the same truth value can be evaluated differently in this framework. This makes particular classes of sphere models (namely *centered* and *weakly centered* sphere models) an optimal setting to work with counterfactuals; but sphere models can represent also other kinds of conditionals, as we will see in the following.

We can now give our definition of a *sphere model*.

Definition 1.7. A **sphere model** is a structure

$$\mathcal{M} = \langle W, S, [\![\ ]\!] \rangle$$

where:

- W is a non-empty set; we call its elements **(possible) worlds**;
- $S : W \to \mathcal{P}(\mathcal{P}(W))$ is a function that associates to every world x a family $S(x)$ of subsets of W; we call **spheres around** x the elements of $S(x)$;
- $[\![\ ]\!]$: **Prop** $\to \mathcal{P}(W)$ is a propositional evaluation, which associates to every atomic formula p the set of worlds at which p is true.

Moreover, S satisfies the following properties:

- *Non-emptiness*: For any α in $S(x)$, α in non-empty;
- *Nesting*: For any α, β in $S(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$;
- *Closure under non-empty union*: If $H \subseteq S(x)$ and $H \neq \emptyset$, then $\bigcup H$ is a sphere in $S(x)$;
- *Closure under non-empty intersection*: If $H \subseteq S(x)$ and $H \neq \emptyset$, then $\bigcap H$ is a sphere in $S(x)$.

We remark that Lewis' original definition of sphere models introduced in Chapter 1 of [23], deviates from Definition 1.7 in two points:

1) In the original definition the condition of non-emptiness is absent and the condition of closure under *non-empty* union is replaced by closure under union. Thus for any world x, setting in our condition of closure under non-empty union $H = \emptyset \subseteq S(x)$ we get by this last condition that also $\emptyset = \bigcup \emptyset$ is a sphere around x;

2) Lewis includes in the original definition of sphere model the following property:

Centering: For all α in $S(x)$ the set $\{x\}$ is a sphere in $S(x)$.

Thus, combining this with the nesting and the non-emptiness properties , it turns out that for any sphere α in $S(x)$, $\{x\} \subseteq \alpha$ and thus $x \in \alpha$.

Following the notations of [14] we chose to replace the technical and "counterintuitive" inclusion of the empty set of Lewis with the non-emptiness condition. Moreover, Lewis' choice of including the centering condition in [23] is due to the fact that the model with the centering property turns out to be the one that best represents counterfactuals. As we are not focused on counterfactual conditionals only, we choose (following [14], again) to present the basic version of a sphere model of Definition 1.7. Starting from it, with the modular addition of properties on the model, among which is the centering property, we will be able to encompass different classes of conditionals.

To define the extension of the evaluation function $[\![\ ]\!]$ to conditional formulas of language $\mathcal{F}_>$ we make use of the following notation, introduced in [27]:

Definition 1.8. Let $\mathcal{M} = \langle W, S, [\![\ ]\!] \rangle$ be a sphere model. Given a sphere $\alpha \subseteq W$ and a formula A in $\mathcal{F}_>$ we say that:

1) A **existentially satisfies** α, and write $\alpha \Vdash^{\exists} A$, *iff* there exists a world x in α such that $x \Vdash A$;

2) A **universally satisfies** α, and write $\alpha \Vdash^{\forall} A$, *iff* for all worlds x in α, $x \Vdash A$.

The notation above permits to express in a convenient way the extension of the propositional evaluation $[\![\ ]\!]$ to conditional formulas:

Definition 1.9. Let $\mathcal{M} = \langle W, S, [\![\ ]\!] \rangle$ be a sphere model. Given two formulas A, B in $\mathcal{F}_>$, the condition for a world x to belong to $[\![A > B]\!]$ is the following:

$$x \Vdash A > B \textit{ iff } \text{if there exists } \alpha \text{ in } S(x) \text{ such that } \alpha \Vdash^{\exists} A, \text{ then there exists } \beta \text{ in } S(x) \text{ such that } \beta \Vdash^{\exists} A \text{ and } \beta \Vdash^{\forall} A \supset B.$$

As we mentioned above, adding the centering condition to sphere models makes them a suitable model for counterfactuals; adding different conditions may make them fit other classes of conditionals. Among all the additional model properties proposed by Lewis in [23] we report the most interesting ones for our purposes:

Definition 1.10. Extensions of sphere models are defined by adding the following properties to the function S of Definition 1.7:

- N *Normality*: For all x in W it holds that $S(x) \neq \emptyset$;
- T *Total reflexivity*: For all x in W there exists α in $S(x)$ such that $x \in \alpha$;
- W *Weak centering*: For all x in W, for all α in $S(x)$, it holds that $x \in \alpha$;
- C *(Strong) Centering*: For all x in W, for all α in $S(x)$, it holds that $\{x\} \in S(x)$, so that $\{x\} \in \alpha$;
- U *Local uniformity*: For all x in W, α in $S(x)$ and y in α, it holds that $\bigcup S(x) = \bigcup S(y)$;
- A *Local absolutness*: For all x in W, α in $S(x)$ and y in α, it holds that $S(x) = S(y)$.

Remark 1.11. There are some (easy) relations between the conditions above: C implies W, W implies T and T implies N; A implies U.

As observed by Lewis, the basic sphere model is too weak, and we already discussed that adding condition C makes the system suitable to describe counterfactuals. Condition W is motivated by Lewis by the fact that in the smallest sphere there may be not only the actual world, but also other worlds so similar to the actual one that we cannot detect the difference. Hence the actual world must belong to all the spheres, but there may be other worlds with the same property. Condition T even permits the actual world to be not in the inner sphere, but just in some of the spheres. Lewis claims that this class of models could fit deontic conditionals: if the function S groups worlds according to their grade of morality, inner spheres may contain worlds that are morally better than the worlds that are just in the outer spheres. This way the innermost sphere would be the one containing the "morally perfect" worlds, and the actual world (definitely) does not belong to it. Condition N is even weaker, and only requires the system of spheres to be non-empty. Condition U imposes that the set of worlds belonging to some sphere is always the same; condition A is stronger and it even imposes that the set of the spheres of any two worlds is the same. Also models satisfying conditions U, A or them both are designed by Lewis as suitable to represent deontic conditionals.

We will present in the next section possible axiomatizations that are sound and complete with respect to these systems.

1.2 Axiomatizations

In this section we present two possible axiomatizations for Lewis' conditional logics, that are sound and complete with respect to the sphere models introduced in the previous section. One of them is based on the language $\mathcal{F}_>$ introduced in Definition 1.1. This axiomatization covers both the conditional logic families $\mathbb{PCL}$ and $\mathbb{V}$, as well as weaker systems. The other axiomatization relies on the comparative plausibility operator $\preccurlyeq$, introduced by Lewis in [23], which is interdefinable with $>$. This axiomatization covers logic $\mathbb{V}$ and its extensions. We will see in Chapter 2 that, regarding proof theory for conditional logics, each of the two axiomatizations presents some benefits respect to the other one.

Figure 1.1 shows the set of axiom schemata and inference rules based on the language $\mathcal{F}_>$, introduced in Definition 1.1.

This axiomatization covers not only the logics $\mathbb{PCL}$ and $\mathbb{V}$ mentioned above, but also the weaker system $\mathbb{CK}$. Logic $\mathbb{CK}$ is the smallest normal conditional logic, i.e. the smallest logic closed under rules (RCEA) and (RCK). It is sound and complete with respect to class-selection function models, as proved by Chellas in [9]. As showed in Figure 1.1, by adding axioms to $\mathbb{CK}$ one obtains logic $\mathbb{PCL}$ (*Preferential Conditional Logic*), introduced in [8]. $\mathbb{PCL}$ is sound and complete with respect to preferential models, and suitable to represent non-monotonic conditionals. Finally, Lewis' basic logic $\mathbb{V}$ (logic of *Variably strict conditionals*), is obtained by $\mathbb{PCL}$ by the addition of the axiom (CV) of "strengthening of the antecedent". Lewis himself proves in [23] that $\mathbb{V}$ is sound and complete with respect to sphere models (cf. Definition 1.7). As well as further model properties can be added to "basic" sphere models in order to obtain new classes of sphere models, it is possible to add axioms to the basic axiomatic system $\mathbb{V}$ in order to capture the logics that correspond to (i.e. are sound and complete w.r.t.) the extensions of sphere models described in Definition 1.10. In figure 1.2 the "additional" axioms mentioned above are reported next to the frame conditions to which they correspond. Both local uniformity and absoluteness are characterized by

$\mathbb{CK}$		Axiomatization of classical propositional logic
	(RCEA)	$\dfrac{A \supset\subset B}{(A > C) \supset\subset (B > C)}$
	(RCK)	$\dfrac{A \supset B}{(C > A) \supset (C > B)}$
	(R-And)	$(A > B) \,\&\, (A > C) \supset (A > (B \,\&\, C))$
$\mathbb{PCL}$		Axiomatization of $\mathbb{CK}$
	(ID)	$A > A$
	(CM)	$(A > B) \,\&\, (A > C) \supset ((A \,\&\, B) > C)$
	(RT)	$(A > B) \,\&\, ((A \,\&\, B) > C) \supset (A > C)$
	(OR)	$(A > C) \,\&\, (B > C) \supset ((A \vee B) > C)$
$\mathbb{V}$		Axiomatization of $\mathbb{PCL}$
	(CV)	$((A > C) \,\&\, \neg(A > \neg B)) \supset ((A \,\&\, B) > C)$

Figure 1.1: Axiom systems of conditional logics

(N)	$\neg(\top > \bot)$	*Normality*
(T)	$A \supset \neg(A > \bot)$	*Total reflexivity*
(W)	$(A > B) \supset (A \supset B)$	*Weak centering*
(C)	$(A \,\&\, B) \supset (A \supset B)$	*(Strong) Centering*
(U_1)	$(\neg A > \bot) \supset \neg(\neg A > \bot) > \bot$	*Uniformity (1)*
(U_2)	$\neg(A > \bot) \supset ((A > \bot) > \bot)$	*Uniformity (2)*
(A_1)	$(A > B) \supset (C > (A > B))$	*Absoluteness (1)*
(A_2)	$\neg(A > B) \supset (C > \neg(A > B))$	*Absoluteness (2)*

Figure 1.2: Axioms for extensions

two axioms, namely (U_1) and (U_2), and (A_1) and (A_2) respectively.

Analogously to what we did with $\mathbb{V}$, the same axioms can be added to logic $\mathbb{PCL}$, generating a version of the extensions in which the nesting condition does not hold. For a more complete discussion over this topic we include also this family of logics in Figure 1.3. We report in there logics $\mathbb{V}$, $\mathbb{PCL}$ and their extensions, whose names follow the notation that the letters added to the name of the basic logic stand for the axioms added to their axiomatic systems. The conditional logic cube has to be interpreted in the following way: if two systems are connected by a line, then the upper system is an extension of the lower one. Thus we retrieve the relations between the model conditions described in Remark 1.11:

Remark 1.12. In $\mathbb{VC}$, (C) implies (W); in $\mathbb{VW}$ (W) implies (T); in $\mathbb{VT}$ (T) implies (N). In $\mathbb{VA}$, both (U_1) and (U_2) are valid.

We present now a second axiomatic system, that is equivalent to the previous one but takes advantage of the comparative plausibility operator $\preccurlyeq$. The alternative axiomatization that we are presenting in this section is based in fact on the following language.

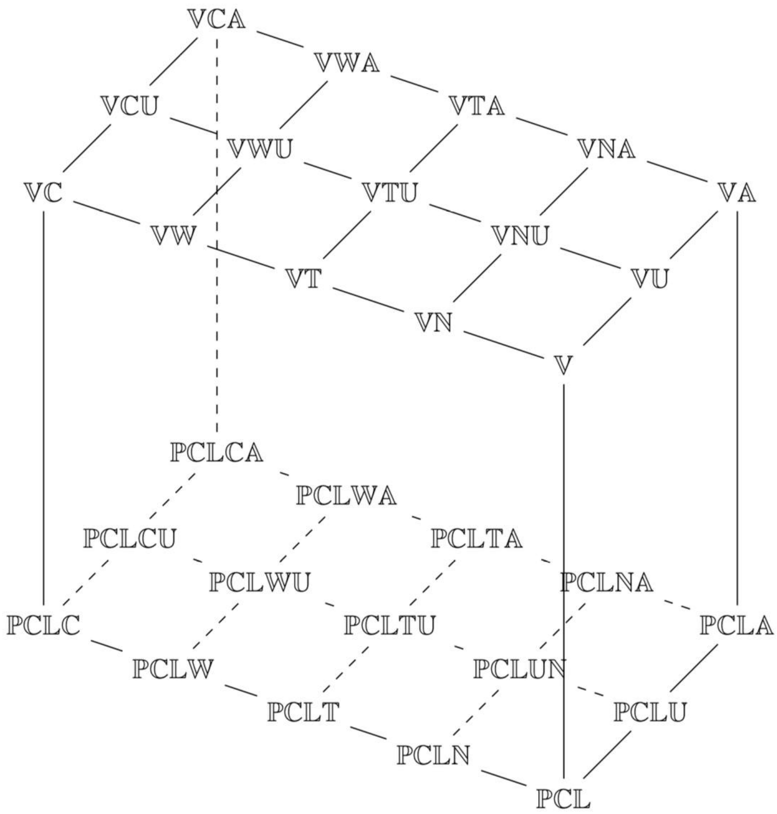


Figure 1.3: The conditional logic cube

Definition 1.13. Let **Prop** be a countably denumerable set of propositional variables. The language $\mathcal{F}_{\preccurlyeq}$, generated by the following grammar:

$$A, B ::= p \mid \bot \mid A \,\&\, B \mid A \vee B \mid A \supset B \mid A \preccurlyeq B$$

for p in **Prop** and A, B in $\mathcal{F}_{\preccurlyeq}$.

The symbol $\preccurlyeq$ denotes comparative plausibility, and the formula $A \preccurlyeq B$ is interpreted as "A is at least as plausible as B".

The big difference between the languages $\mathcal{F}_{>}$ and $\mathcal{F}_{\preccurlyeq}$ lies in the fact that the former assumes as primitive the conditional $>$, while the latter assumes as primitive the comparative plausibility operator $\preccurlyeq$. In logic $\mathbb{V}$ and its extensions the two operators are interdefinable via the following equivalences:

$$(1) \quad A > B \equiv (\bot \preccurlyeq A) \vee \neg((A \,\&\, \neg B) \preccurlyeq (A \,\&\, B)) \tag{1.1}$$

$$(2) \quad A \preccurlyeq B \equiv ((A \vee B) > \bot) \vee \neg((A \vee B) > \neg A) \tag{1.2}$$

Remark 1.14. As observed in [15], in logic $\mathbb{V}$ and its extensions the conditional can be equivalently defined in terms of comparative plausibility with the following simplified version of the expression 1.1:

$$A > B \equiv (\bot \preccurlyeq A) \vee \neg((A \,\&\, \neg B) \preccurlyeq A)$$

According to the equivalence 1.2, the truth condition of comparative plausibility in a sphere model $\mathcal{M} = \langle W, S, [\![\ ]\!] \rangle$ is, for any world x:

$$x \Vdash A \preccurlyeq B \textit{ iff } \text{for all } \alpha \text{ in } S(x), \text{ if } \alpha \Vdash^{\exists} B \text{ then } \alpha \Vdash^{\exists} A.$$

This condition is quite simple if compared with the truth condition in sphere models of the conditional $>$. We will see in Chapter 2 how to take advantage of this property of the comparative plausibility operator in sequent calculi for conditional logics.

As we showed for the conditional $>$, also for the comparative plausibility $\preccurlyeq$ together with the semantic characterization comes an axiomatic description of the logics in which we are interested. In Figure 1.4 the axioms for Lewis' logic $\mathbb{V}$ and its extensions are presented. The formulation therein displayed is indeed equivalent to the one of Figure 1.1 for $\mathbb{V}$, eventually complemented with the appropriate axioms from Figure 1.2 for the extensions.

$\mathbb{V}$		Axiomatization of classical propositional logic
	(CPR)	$\dfrac{\vdash B \supset A}{\vdash A \preccurlyeq B}$
	(CPA)	$(A \preccurlyeq (A \vee B)) \vee (B \preccurlyeq (A \vee B))$
	(TR)	$(A \preccurlyeq B)\,\&\,(B \preccurlyeq C) \supset (A \preccurlyeq C)$
	(CO)	$(A \preccurlyeq B) \vee (B \preccurlyeq A)$
Extensions of $\mathbb{V}$		Axiomatization of $\mathbb{V}$
	(N)	$\neg(\bot \preccurlyeq \top)$
	(T)	$(\bot \preccurlyeq \neg A) \supset A$
	(W)	$A \supset (A \preccurlyeq \top)$
	(C)	$(A \preccurlyeq \top) \supset A$

Figure 1.4: Lewis' logics and axioms for $\preccurlyeq$

Chapter 2

Proof theory for Lewis' logics

Labelled sequent calculi, as presented in [26], allow to obtain analytic sequent calculi starting from a relational Kripke-style semantics, internalizing the semantics into the language. In fact, the language is enriched with *relational atoms*, that represent relations between objects of the model, and the formulas are endowed with *labels* - thus the name of this class of sequent calculi - corresponding to possible worlds, that express the forcing relation (one for all, the formula $x\colon A$ corresponds to $x \Vdash A$ in the model). In this way, it is possible to set up a sequent calculus with an in-built strong connection with the model it ought to represent.

Internal sequent calculi on the other hand are calculi with the property that the language is not modified, but the structure is enriched in order to reduce the complexity of the rules. These calculi feature the property that every sequent in a derivation has an interpretation in terms of the language of the logic they represent. This makes this approach closer to the syntax we are interested in, even if the relation with the elements of the model becomes less clear.

In this section we will mention some labelled sequent calculi for conditional logics in Section 2.1, based on models of different kinds. Sequent calculi **G3P** and its extensions (introduced in [18]) are based on preferential models, while sequent calculi **G3CL** and its extensions (presented in [28]) are based on neighbourhood models. Both these families of calculi are sound and complete with respect to logic $\mathbb{PCL}$ and its extensions, including Lewis' logics of the family of $\mathbb{V}$. The conditional $>$ is taken as primitive in the axiomatizations we consider. Finally, labelled sequent calculi **G3V** and extensions are the main focus of Section 2.1: they are built on sphere models, representing logic $\mathbb{V}$ and some of its extensions. The axiomatization on which these last systems are based is the one featuring the comparative plausibility operator $\preccurlyeq$: this choice, together with the nesting property, results in a much more simple truth condition in the models for $\preccurlyeq$-formulas, than the one of the conditional $>$.

In Section 2.2 we introduce internal sequent calculi $\mathcal{I}_{\mathbb{V}}$ and its extensions, based on the comparative plausibility axiomatization of logic $\mathbb{V}$, in which the syntax is enriched with *blocks*, structures of the form $[A_1, \ldots, A_n \lhd B]$ with the meaning $A_1 \preccurlyeq B \vee \cdots \vee A_n \preccurlyeq B$. The calculus $\mathcal{I}_{\mathbb{V}}$, in particular, will be used in Chapter 4 to give a direct proof of the completeness of $\mathbb{V}$ w.r.t. Adams' logic of high probability (see Chapter 3). In section 2.2.3 we cite the versions of $\mathcal{I}_{\mathbb{V}}$ with invertible rules, namely the sequent calculi $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ and extensions, that feature better structural properties and better covers a larger number of logics of Lewis' family.

We conclude this general discussion observing that the two approaches of labelled and internal calculi, looking quite different for their structure and meaning, can be linked. A lot of studies have been done to investigate the relations between sequent calculi of the two kinds, since strict links between them could make it possible to exploit the good properties of one type of calculi to obtain the analogous results in the other one. We mention here the study conducted in this direction in [16], where the two directions of a mapping between the labelled calculus **G3V** and the internal calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ are displayed.

2.1 Labelled Calculi

In this section we present labelled sequent calculi for conditional logics based on different semantics. Given two examples of labelled calculi, we focus on calculus **G3V** and its extensions for Lewis' logics of the family of $\mathbb{V}$, based on sphere models.

The labelled sequent calculus **G3P**, introduced in [18], is based on preferential models, whereas the labelled sequent calculus **G3CL**, introduced in [28], is based on neighbourhood models. These calculi, despite arising from different semantics, have some features in common. First of all, they cover logic $\mathbb{PCL}$ and its extensions, including $\mathbb{V}$ (obtained by $\mathbb{PCL}$ by adding the axiom (CV), cf. Section 1.2) and its extensions. Then, both the calculi are based on the axiomatization of conditional logics in which the conditional > is primitive. Moreover these two families of calculi have good structural properties, they are sound and complete with respect to the corresponding logics, and feature finite proof-search. For more detailed treatments of this topic and the complete proofs of the properties of these families of calculi, we refer to [18] as for calculus **G3P** and to [28] and [17] as for calculus **G3CL**.

Finally, we want to stress that both calculi are modular on the logics they cover, in the sense that a labelled sequent calculus for a stronger conditional logic can be obtained by adding the appropriate rules to the calculus for a weaker conditional logic. Starting from this fact, we can introduce the sequent calculus **G3V** as a version of **G3CL** for sphere models.

The sequent calculus **G3V**, introduced in [16], enters in this framework presenting some differences from the calculi mentioned above. First of all, it is based on the $\preccurlyeq$ axiomatization of logic $\mathbb{V}$. Moreover it stems from sphere models, the class of neighbourhood models featuring the property of nesting. In fact, **G3V** arises from **G3CL**, but has a notably simplified structure with respect to the latter. Considering just some particular logics (namely the ones corresponding to sphere models), with this approach we lose the modularity over all of the conditional logics arising from $\mathbb{PCL}$, that was a strong property of **G3P** and **G3CL** instead; this is due to the fact that the conditional > and the comparative plausibility $\preccurlyeq$ are not interdefinable in logics weaker than $\mathbb{V}$. However, the benefit in terms of simplicity of the calculus is big enough to consider this calculus to describe Lewis' logics. Thus **G3V** is a notably simpler version of **G3CL**, but just for a specific class of conditional logics.

To justify our choice of the sequent calculus **G3V** to deal with Lewis' logics we give here an idea of its simplification with respect to **G3CL**. We remind the truth condition for a conditional formula in a sphere model, already presented in Definition 1.9:

$x \Vdash A > B$ *iff* if there exists α in $S(x)$ such that if $\alpha \Vdash^{\exists} A$, then there exists β in $S(x)$ suche that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \supset B$.

We already explained in Section 1.1.1 that the corresponding condition for neighbourhood models is very similar: it just adds the condition that β is contained in α, to retrieve the fact that in nested models any two neighbours are comparable. In fact, calculus **G3CL** is based on this semantic interpretation of conditional. However, we already noticed that this condition is too rich to be expressed by a single rule of inference, since it consists of an existential quantifier in the scope of a universal quantifier; for this reason sequent calculi based on such condition (like **G3CL**) require quite complex groups of rules to include the conditional $>$.

Consider now the axiomatization for Lewis' logics featuring the comparative plausibility operator $\preccurlyeq$. Its truth condition in neighbourhood models has a similar complexity than the conditional's. In sphere models, instead, as already mentioned in 1.2, its condition is remarkably simpler:

$$x \Vdash A \preccurlyeq B \textit{ iff } \text{for all } \alpha \text{ in } S(x), \text{ if } \alpha \Vdash^{\exists} B \text{ then } \alpha \Vdash^{\exists} A.$$

This makes the rules of the sequent calculus **G3V** much simpler than those of the calculi based on other semantics (and axiomatizations).

To conclude this general introduction, sequent calculi **G3P** and **G3CL**, arising respectively from preferential and neighbourhood models, are based on a language in which the conditional operator $>$ is primitive. For this reason the proof systems are quite complicated, but on the other hand they have a wide modularity over all the extensions of $\mathbb{PCL}$. When restricting to sphere models, it is convenient to adopt the language based on the comparative plausibility, as its semantic explanation in *nested* neighbourhood models is much simpler than the one of the conditional. For this reason we present in the following the labelled family of sequent calculi of **G3V** and outline some of their main properties. They cover only Lewis' logic $\mathbb{V}$ and some of its extensions, but have a very simple structure when compared to calculi based on the conditional. For a detailed treatment of this discussion we refer to [14].

As already mentioned, we call **G3V** the basic sequent calculus of this family, corresponding to $\mathbb{V}$. Its extensions $\mathbf{G3V^N}$, $\mathbf{G3V^T}$, $\mathbf{G3V^W}$, $\mathbf{G3V^C}$ cover respectively $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$. Finally, by $\mathbf{G3V^*}$ we denote any of the calculi above.

Consider the language $\mathcal{F}_{\preccurlyeq}$ introduced in section 1.2: it is generated by the following grammar, for p propositional variable and A, B formulas in $\mathcal{F}_{\preccurlyeq}$:

$$A, B ::= p \mid \bot \mid A \,\&\, B \mid A \vee B \mid A \supset B \mid A \preccurlyeq B$$

The labelled sequent calculus **G3V** is based on models characterized by the property of nesting, i.e. Lewis' sphere models. To give the rules of this calculus, built starting from sphere models semantics, we first enrich language $\mathcal{F}_{\preccurlyeq}$ as follows:

Definition 2.1. Let $x, y, z, \ldots$ be variables for worlds in a sphere model, $\alpha, \beta, \gamma, \ldots$ variables for spheres and, for any world x, let $S(x)$ denote the set of spheres around x. We add to the language $\mathcal{F}_{\preccurlyeq}$ the formulas of the following form (and meaning), called **relational atoms**:

- $\alpha \in S(x)$, "the sphere α belongs to the system of spheres around the world x";
- $x \in \alpha$, "the world x belongs to the sphere α";
- $\alpha \subseteq \beta$, "the sphere α is contained into the sphere β".

Let A in $\mathcal{F}_{\preccurlyeq}$. The set of **labelled formulas** is defined by the following clauses, for which we report the meaning, too:

- Relational atoms are labelled formulas;
- $x\colon A$, "formula A is true at world x", is a labelled formula;
- $\alpha \Vdash^{\exists} A$, "$A$ is true at some world belonging to the sphere α", is a labelled formula;
- $\alpha \Vdash^{\forall} A$, "$A$ is true at all worlds belonging to the sphere α", is a labelled formula.

Moreover, we will denote by $\{x\}$ the sphere consisting of the only world x and, for any world x, by $At(x)$ any of the formulas of the set $\{x\colon P, x \in \alpha, \alpha \in S(x), y \in \{x\}\}$, with P atomic formula. Notice that relational atoms expand the language describing the structure of the sphere model, while the other labelled formulas are defined to represent truth at a world ($x\colon A$ stands for $x \Vdash A$), and at a sphere ($\alpha \Vdash^{\exists} A$ and $\alpha \Vdash^{\exists} A$, with a uniform notation with Definition 1.8).

We remind that a *multiset* is a collection of elements that keeps track of the multiplicity of any element. We can thus extend the usual notion of sequent as follows:

Definition 2.2. A **sequent** of $\mathbf{G3V}^*$ is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are multisets of labelled formulas, and relational atoms may occur only in Γ.

We have displayed in Figure 2.1 the rules for the family of calculi $\mathbf{G3V}^*$. Propositional rules are the usual ones, i.e. the ones of the labelled sequent calculus **G3K** displayed in [26], p. 507. We write (x!) as the side condition of a rule to express the requirement that label x must not occur in the conclusion of the rule. As for the extensions of **G3V**, the system $\mathbf{G3V^N}$, corresponding to normal sphere models, is obtained by adding rules N and 0 to **G3V**. Rule 0 stands for the requirement of non-emptiness of spheres in the model, which is not necessary in the basic sequent calculus **G3V**. The side condition ($\star$) of this rule requires that no formulas of form $w \in \alpha$ are in Γ, and that at least one formula of form $\alpha \Vdash^{\exists} A$ or $\alpha \Vdash^{\forall} A$ is in $\Gamma \cup \Delta$; these conditions are added in order to avoid loops in the root-first proof-search and to ensure that the rule 0 is applied only when it is actually needed. System $\mathbf{G3V^T}$ is obtained by adding the rule T for total reflexivity to $\mathbf{G3V^N}$. $\mathbf{G3V^W}$, featuring weak centering, is obtained by adding the rule W to $\mathbf{G3V^T}$. Finally $\mathbf{G3V^C}$ covers the centering condition, by adding to $\mathbf{G3V^W}$ the four rules C, Single, Repl_1 and Repl_2.

To illustrate how the calculus works, we provide here an example of a derivation in $\mathbf{G3V}^*$ from [16].

Example 2.3. We show the derivation of axiom (T)

$$(\bot \preccurlyeq \neg A) \supset A$$

in the corresponding labelled calculus $\mathbf{G3V^T}$; we denote by "…" formulas that are inessential in the derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x\colon A, \ldots \Rightarrow x\colon \bot, x\colon A, \ldots}\,\text{init}}{\ldots, \Rightarrow, x\colon A, x\colon \neg A, \ldots}\,R\supset}{x \in \alpha, \alpha \in S(x), x\colon \bot \preccurlyeq \neg A \Rightarrow x\colon A, \alpha \Vdash^{\exists} \neg A}\,R\Vdash^{\exists} \qquad \dfrac{\dfrac{}{x \in \alpha, x\colon \bot, \ldots \Rightarrow \ldots}\,L\bot}{\alpha \Vdash^{\exists} \bot, x \in \alpha, \alpha \in S(x), x\colon \bot \preccurlyeq \neg A \Rightarrow x\colon A}\,L\Vdash^{\exists}}{\dfrac{x \in \alpha, \alpha \in S(x), x\colon \bot \preccurlyeq \neg A \Rightarrow x\colon A}{\dfrac{x\colon \bot \preccurlyeq \neg A \Rightarrow x\colon A}{\Rightarrow x\colon (\bot \preccurlyeq \neg A) \supset A}\,R\supset}\,\mathsf{T}}\,L\preccurlyeq$$

Initial sequents

$$\text{init}\ \frac{}{x{:}\,p,\Gamma \Rightarrow \Delta, x{:}\,p} \qquad L\bot\ \frac{}{x{:}\,\bot,\Gamma \Rightarrow \Delta}$$

Propositional rules

$$L\&\ \frac{x{:}\,A, x{:}\,B, \Gamma \Rightarrow \Delta}{x{:}\,A \,\&\, B, \Gamma \Rightarrow \Delta} \qquad R\&\ \frac{\Gamma \Rightarrow \Delta, x{:}\,A \quad \Gamma \Rightarrow \Delta, x{:}\,B}{\Gamma \Rightarrow \Delta, x{:}\,A \,\&\, B} \qquad L\vee\ \frac{x{:}\,A, \Gamma \Rightarrow \Delta \quad x{:}\,B, \Gamma \Rightarrow \Delta}{x{:}\,A \vee B, \Gamma \Rightarrow \Delta}$$

$$R\vee\ \frac{\Gamma \Rightarrow \Delta, x{:}\,A, x{:}\,B}{\Gamma \Rightarrow \Delta, x{:}\,A \vee B} \qquad L\supset\ \frac{\Gamma \Rightarrow \Delta, x{:}\,A \quad x{:}\,B, \Gamma \Rightarrow \Delta}{x{:}\,A \supset B, \Gamma \Rightarrow \Delta} \qquad R\supset\ \frac{x{:}\,A, \Gamma \Rightarrow \Delta, x{:}\,A}{\Gamma \Rightarrow \Delta, x{:}\,A \supset B}$$

Rules for local forcing

$$(x!)\ L\Vdash^{\exists}\ \frac{x \in \alpha, x{:}\,A, \Gamma \Rightarrow \Delta}{\alpha \Vdash^{\exists} A, \Gamma \Rightarrow \Delta} \qquad R\Vdash^{\exists}\ \frac{x \in \alpha, \Gamma \Rightarrow \Delta, x{:}\,A, \alpha \Vdash^{\exists} A}{x \in \alpha, \Gamma \Rightarrow \Delta, \alpha \Vdash^{\exists} A}$$

Rules for comparative plausibility

$$(\alpha!)\ R\preccurlyeq\ \frac{\alpha \Vdash^{\exists} B, \alpha \in S(x), \Gamma \Rightarrow \Delta, \alpha \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x{:}\,A \preccurlyeq B}$$

$$L\preccurlyeq\ \frac{\alpha \in S(x), x{:}\,A \preccurlyeq B, \Gamma \Rightarrow \Delta, \alpha \Vdash^{\exists} B \quad \alpha \Vdash^{\exists} A, \alpha \in S(x), x{:}\,A \preccurlyeq B, \Gamma \Rightarrow \Delta}{\alpha \in S(x), x{:}\,A \preccurlyeq B, \Gamma \Rightarrow \Delta}$$

Rules for inclusion and nesting

$$L\subseteq\ \frac{x \in \alpha, \alpha \subseteq \beta, x \in \beta, \Gamma \Rightarrow \Delta}{x \in \alpha, \alpha \subseteq \beta, \Gamma \Rightarrow \Delta}$$

$$\text{Nes}\ \frac{\alpha \subseteq \beta, \alpha \in S(x), \beta \in S(x), \Gamma \Rightarrow \Delta \quad \beta \subseteq \alpha, \alpha \in S(x), \beta \in S(x), \Gamma \Rightarrow \Delta}{\alpha \in S(x), \beta \in S(x), \Gamma \Rightarrow \Delta}$$

Rules for extensions

$$(\alpha!)\ \text{N}\ \frac{\alpha \in S(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (\star)\ (y!)\ 0\ \frac{y \in \alpha, \alpha \in S(x), \Gamma \Rightarrow \Delta}{\alpha \in S(x), \Gamma \Rightarrow \Delta} \qquad (\alpha!)\ \text{T}\ \frac{x \in \alpha, \alpha \in S(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{W}\ \frac{x \in \alpha, \alpha \in S(x), \Gamma \Rightarrow \Delta}{\alpha \in S(x), \Gamma \Rightarrow \Delta} \qquad \text{Single}\ \frac{x \in \{x\}, \{x\} \in S(x), \Gamma \Rightarrow \Delta}{\{x\} \in S(x), \Gamma \Rightarrow \Delta} \qquad \text{C}\ \frac{\{x\} \in S(x), \{x\} \subseteq \alpha, \alpha \in S(x), \Gamma \Rightarrow \Delta}{\alpha \in S(x), \Gamma \Rightarrow \Delta}$$

$$(*)\ \text{Repl}_1\ \frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(x), \Gamma \Rightarrow \Delta} \qquad (*)\ \text{Repl}_2\ \frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(y), \Gamma \Rightarrow \Delta}$$

($\star$) $w \in \alpha$ does not occur in Γ; $\alpha \Vdash^{\exists} A$ or $\alpha \Vdash^{\forall} A$ occur in $\Gamma \cup \Delta$.
($*$) $At(x) ::= x{:}\,p \mid x \in \alpha \mid \alpha \in S(x) \mid y \in \{x\}$, for p atomic formula.

Figure 2.1: Rules of **G3V***

We will outline now some properties of calculi $\mathbf{G3V}^*$. We will not report here the proofs for these results, since it is not the focus of this study. For a more detailed treatment of this topic we refer to Chapter 3.7 of [14]. In particular, most of the following results are proved adapting the proofs of the corresponding results for the $>$-based sequent calculus **G3CL** and its extensions.

Theorem 2.4 (Soundness)**.** If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{G3V}^*$, then it is valid in all sphere models.

Besides the above result, these sequent calculi present a full apparatus of the desirable structural properties. Initial sequents with arbitrary formulas put in place of the atom p - that we call *generalized initial sequents* - are derivable, as well as the generalized versions of the replacement rules Repl_1 and Repl_2 (i.e. their instances in which an arbitrary formula with label x replaces $At(x)$, and the same formula with label y replaces $At(y)$). The rule of reflexivity Ref and the rule of transitivity Tr, that we report below, are respectively height-preserving admissible and admissible.

$$\mathsf{Refl}\,\frac{\alpha \subseteq \alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \mathsf{Tr}\,\frac{\alpha \subseteq \gamma, \alpha \subseteq \beta, \beta \subseteq \gamma, \Gamma \Rightarrow \Delta}{\alpha \subseteq \beta, \beta \subseteq \gamma, \Gamma \Rightarrow \Delta}$$

Moreover, structural rules of weakening and contraction are admissible. Admissibility of the rule of cut holds as well: it can be proved following the general strategy proposed in [29] to prove cut-admissibility in labelled systems. We complete this list of results by stating the result of completeness of $\mathbf{G3V}^*$ with respect to logic $\mathbb{V}$ and its extensions:

Theorem 2.5 (Completeness)**.** If a formula A in $\mathcal{F}_{\preccurlyeq}$ is valid in $\mathbb{V}$ or in one of its extensions, then the sequent $\Rightarrow x{:}\,A$ is derivable in the corresponding calculus $\mathbf{G3V}^*$.

This result has been proved in [14] by showing that the inference rules and the axioms of $\mathbb{V}$ and its extensions from Figure 1.4 are respectively admissible and derivable in the corresponding calculus of family $\mathbf{G3V}^*$. Finally following the strategy described in Section 4 of [28] for **G3CL** and its extensions, it is possible to show that there is a terminating proof search for $\mathbf{G3V}^*$.

We want to conclude this section putting forward, in the next remark, an alternative proof strategy for the completeness of $\mathbf{G3V}^*$ with respect to the corresponding logic of Lewis':

Remark 2.6. An interesting - alternative - strategy to prove soundness and completeness of calculi $\mathbf{G3V}^*$ with respect to the corresponding extension of $\mathbb{V}$ is showing their equivalence with **G3CL** and its extensions, which we denote hereafter with $\mathbf{G3CL}^*$.

Our idea is to show a two-directions map between the two families of calculi, such that:

- on sequents, it translates $>$-formulas into $\preccurlyeq$-formulas and vice versa;
- on derivations, it is defined inductively on the height of the derivation, distinguishing cases according to the last rule applied.

This strategy is similar to the one adopted in [16] to show the equivalence of the calculi **G3V** and $\mathcal{I}_{\mathbb{V}}$. In the direction of the translation from $\mathbf{G3CL}^*$ to $\mathbf{G3V}^*$ we require $\mathbf{G3CL}^*$ derivation to be in a sort of "normal form", in which the formulas belonging to the language of **G3CL** that have no counterpart in the language of **G3V** are handled immediately after their introduction in the derivation; the converse translation is standard.

Finally, as already claimed, this result, together with the soundness and completeness of **G3CL*** with respect to $\mathbb{V}$ and its extensions - proved in [28] - would entail soundness and completeness og **G3V*** as well.

We leave the detail of the alternative, modular proof of the completeness of **G3V*** already here to future works. We want to highlight the constructivity of this result, that providing translations between derivations of different calculi goes in the direction paved in [16].

2.2 Internal Calculi

Internal sequent calculi are calculi in which complicated (semantic) conditions are included enriching the syntactic structure. Another characteristic property of this kind of calculi is that sequents have a formula-interpretation: every sequent, despite its enriched structure, is equivalent to a well-formed formula in the language.

The sequent calculi $\mathcal{I}_{\mathbb{V}}$ and extensions, that we consider in this section, are characterized by *blocks*, i.e. structures of the form $[\Sigma \lhd A]$, where Σ is a finite multiset of formulas and A is a formula of the language. They are based on the language in which the comparative plausibility operator $\preccurlyeq$ is primitive, and the formula-interpretation is the following:

$$[A_1, \ldots, A_n \lhd B] \qquad \text{means} \qquad A_1 \preccurlyeq B \vee \cdots \vee A_n \preccurlyeq B$$

for any formulas $A_1, \ldots, A_n, B$.

The sequent calculi we present cover logic $\mathbb{V}$ and some of its extensions: in particular, we will be interested in the versions for $\mathbb{V}$ and $\mathbb{VW}$ in Chapter 4; however, it is hard to find internal calculi for logics weaker than $\mathbb{V}$, i.e. for $\mathbb{PCL}$ and its extensions not including axiom (CV).

In this section we recall the standard sequent calculi introduced in [15] to describe a large part of Lewis' logics of counterfactuals, namely the logic $\mathbb{V}$ and its extensions $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$, $\mathbb{VA}$ and $\mathbb{VNA}$; we will hereafter generically denote any of them by $\mathcal{L}$. In fact, the calculi we are going to present are modular for the family of Lewis' logics, in the sense that a sequent calculus for stronger logics can be obtained by adding independent rules to the calculus for a weaker one. Calculi $\mathcal{I}_{\mathcal{L}}$ are *standard* in the sense that each connective is handled by a finite number of rules, and each rule has a fixed finite number of premisses.

We first introduce in Section 2.2.1 the family of calculi $\mathcal{I}_{\mathcal{L}}$, which are the first version of the calculi of this kind: they contain explicit contraction rules as well as some non-invertible rules. Still, each of them is sound with respect to the corresponding Lewis logic. Moreover, we show in Section 2.2.2 that the cut rule is eliminable for the calculus $\mathcal{I}_{\mathbb{V}}$, and this provides as a consequence the completeness of the calculus with respect to Lewis' logic $\mathbb{V}$. This is enough to prove, in the next chapter, the important result of completeness of $\mathcal{I}_{\mathbb{V}}$ with respect to Adams' logic of high probability **HPA**. We conclude this section by mentioning in 2.2.3 the alternative version $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ of the calculi $\mathcal{I}_{\mathcal{L}}$, in which contraction rules are admissible and all rules are invertible, i.e. if an instance of the conclusion of the rule is derivable, also the corresponding instance of premiss(es) is (are). The calculi $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ are in fact a refined version of $\mathcal{I}_{\mathcal{L}}$, being equivalent to the latter, but with a stronger apparatus of structural properties. This ensures completeness w.r.t. $\mathbb{V}$ and many of its extensions, namely $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$, and allows terminating proof-search, thus providing a decision procedure for the respective logics. Finally, we remark that completeness can be proved also for the logics $\mathbb{VA}$ and $\mathbb{VNA}$, the ones including the absoluteness condition among the logics mentioned above, but we will not discuss these cases here. For this and other details we refer to [15], where this class of calculi has been first introduced, and to [14] for a complete treatment of this subject.

2.2.1 Sequent calculi $\mathcal{I}_{\mathcal{L}}$ for Lewis' logic $\mathbb{V}$ and extensions

The family of calculi we will present in this section is based on the comparative plausibility operator $\preccurlyeq$ and not directly on the conditional $>$; therefore we will use the language $\mathcal{F}_{\preccurlyeq}$, introduced in Definition 1.13.

Let **Prop** be a countably denumerable set of propositional variables. We recall that the set of conditional formulas $\mathcal{F}_{\preccurlyeq}$ is given by the following grammar, for A and B in $\mathcal{F}_{\preccurlyeq}$ and p in **Prop**:

$$A, B ::= p \mid \bot \mid A \,\&\, B \mid A \vee B \mid A \supset B \mid A \preccurlyeq B$$

Negation and true can be defined in terms of the other connectives as usual: for any conditional formula A, the negation $\neg A$ is defined as $A \supset \bot$; true ($\top$) is defined as $\neg\bot$.

We introduce now the internal sequent calculi $\mathcal{I}_{\mathcal{L}}$ for Lewis logic $\mathbb{V}$ and some of its extensions, namely $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$, $\mathbb{VA}$ and $\mathbb{VNA}$. Sequents are based on multisets of formulas in order to make contraction rules explicit. As usual, we denote the union of multisets Γ and Δ by Γ, Δ. Sequents for these calculi do not only consist of formulas: the blocks mentioned above enrich their structure, providing a compact way of representing disjunctions of $\preccurlyeq$-formulas.

Definition 2.7. A **conditional block** is a structure consisting of a multiset Σ of formulas and a single formula A, written $[\Sigma \lhd A]$. A **sequent** is an expression $\Gamma \Rightarrow \Delta$, where Γ is a multiset of conditional formulas and Δ is a multiset of conditional formulas and blocks. The formula interpretation of a sequent is given by:

$$\iota(\Gamma \supset \Delta', [\Sigma_1 \lhd A_1], \ldots, [\Sigma_n \lhd A_n]) := \bigwedge \Gamma \supset \bigvee \Delta' \vee \bigvee_{1 \leq i \leq n} \bigvee_{B \in \Sigma_i} (B \preccurlyeq A_i)$$

In Figure 2.2 we report the sequent calculi $\mathcal{I}_{\mathcal{L}}$ for Lewis' logic $\mathbb{V}$ and some of its extensions. Notice that initial sequents and propositional rules, as well as two of the contraction rules, are those of calculus **G3c**, the contraction-free sequent calculus for classical propositional logic (cf. [29], [32]). The two new contraction rules, namely $\mathsf{Ctr_B}$ and $\mathsf{Ctr_i}$, are needed to handle contraction of blocks and inside blocks, respectively. Given a sequent $\Gamma \Rightarrow \Delta$, we call a derivation of $\Gamma \Rightarrow \Delta$ in $\mathcal{I}_{\mathcal{L}}$ any tree in which $\Gamma \Rightarrow \Delta$ is the root, every leaf is an instance of the conclusion of rules init or $L\bot$, and every non-leaf node is the conclusion of an instance of a rule of $\mathcal{I}_{\mathcal{L}}$, having the premisses of that instance as children. We say that sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{I}_{\mathcal{L}}$, and write $\mathcal{I}_{\mathcal{L}} \vdash \Gamma \Rightarrow \Delta$, *iff* it has a derivation. As usual, given a formula $G \in \mathcal{L}$, we say that G is derivable in $\mathcal{I}_{\mathcal{L}}$ *iff* the sequent $\Rightarrow G$ is derivable in $\mathcal{I}_{\mathcal{L}}$. Finally, we recall that the height of a derivation is the maximum length of a branch, i.e. the maximum number of nodes in a branch, minus one; so that a derivation consisting only of an initial sequent has height 0.

Example 2.8. We illustrate now with an example how derivations are obtained in the calculi $\mathcal{I}_{\mathcal{L}}$: we derive the characteristic axiom T of logic $\mathbb{VT}$ in some of the extensions

Initial sequents

$$\text{init}\ \frac{}{P, \Gamma \Rightarrow \Delta, P} \qquad L\bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta}$$

Propositional rules

$$L\&\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \& B, \Gamma \Rightarrow \Delta} \qquad R\&\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \& B} \qquad L\vee\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}$$

$$R\vee\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad L\supset\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} \qquad R\supset\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}$$

Conditional rules

$$L\preccurlyeq\ \frac{\Gamma \Rightarrow \Delta, [D, \Sigma \lhd A] \quad \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}{C \preccurlyeq D, \Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \qquad R\preccurlyeq\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A] \quad \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]} \qquad \text{jump}\ \frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}$$

Structural rules

$$\text{Ctr}_\text{L}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma, \Rightarrow \Delta} \qquad \text{Ctr}_\text{R}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{Ctr}_\text{B}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd A]}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \qquad \text{Ctr}_\text{i}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma, A, A \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma, A \lhd B]}$$

Rules for extensions

$$\text{N}\ \frac{\Gamma \Rightarrow \Delta, [\bot \lhd \top]}{\Gamma \Rightarrow \Delta} \qquad \text{T}\ \frac{\Gamma \Rightarrow \Delta, B \quad \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta}$$

$$\text{W}\ \frac{\Gamma \Rightarrow \Delta, \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \qquad \text{C}\ \frac{A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, B}{\Gamma, A \preccurlyeq B \Rightarrow \Delta} \qquad \text{A}\ \frac{A, \Gamma^{\preccurlyeq} \Rightarrow \Delta^{\preccurlyeq}, \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}$$

$\Gamma^{\preccurlyeq} \Rightarrow \Delta^{\preccurlyeq}$ is $\Gamma \Rightarrow \Delta$ restricted to $\preccurlyeq$-formulae and blocks.

$\mathcal{I}_{\mathbb{V}} := \{\text{init}, L\bot, L\&, R\&, L\vee, R\vee, L\supset, R\supset, L\preccurlyeq, R\preccurlyeq, \text{com}, \text{jump}, \text{Ctr}_\text{L}, \text{Ctr}_\text{R}, \text{Ctr}_\text{B}, \text{Ctr}_\text{i}\}$

$\mathcal{I}_{\mathbb{VN}} := \mathcal{I}_{\mathbb{V}} \cup \{\text{N}\}$ $\qquad$ $\mathcal{I}_{\mathbb{VW}} := \mathcal{I}_{\mathbb{V}} \cup \{\text{N}, \text{T}, \text{W}\}$ $\qquad$ $\mathcal{I}_{\mathbb{VA}} := \mathcal{I}_{\mathbb{V}} \cup \{\text{A}\}$

$\mathcal{I}_{\mathbb{VT}} := \mathcal{I}_{\mathbb{V}} \cup \{\text{N}, \text{T}\}$ $\qquad$ $\mathcal{I}_{\mathbb{VC}} := \mathcal{I}_{\mathbb{V}} \cup \{\text{N}, \text{T}, \text{W}, \text{C}\}$ $\qquad$ $\mathcal{I}_{\mathbb{VNA}} := \mathcal{I}_{\mathbb{V}} \cup \{\text{N}, \text{A}\}$

Figure 2.2: Sequent Calculi $\mathcal{I}_{\mathbb{V}}$ and extensions

of $\mathcal{I}_{\mathbb{V}}$. Axiom T is

$$(\bot \preccurlyeq \neg A) \supset A$$

and, making explicit the notation $\neg A \equiv A \supset \bot$, it is equivalent to

$$(\bot \preccurlyeq (A \supset \bot)) \supset A$$

We can derive axiom T in $\mathcal{I}_{\mathbb{VT}}$ as follows:

$$R\supset \dfrac{\mathsf{T}\,\dfrac{R\supset \dfrac{\text{init}\,\dfrac{}{A \Rightarrow A, \bot}}{\Rightarrow A, A \supset \bot} \qquad \text{jump}\,\dfrac{L\bot\,\dfrac{}{\bot \Rightarrow \bot}}{\Rightarrow A, [\bot \lhd \bot]}}{\bot \preccurlyeq (A \supset \bot) \Rightarrow A}}{\Rightarrow (\bot \preccurlyeq (A \supset \bot)) \supset A}$$

Notice that in this derivation we used the characteristic rule T of calculus $\mathcal{I}_{\mathbb{VT}}$, whose role is in fact making axiom T derivable in the calculus. Now, we show derivations of axiom T in $\mathcal{I}_{\mathbb{VW}}$ and $\mathcal{I}_{\mathbb{VC}}$ which do not make use of rule T. These are:

$$R\supset \dfrac{\mathsf{N}\,\dfrac{L\preccurlyeq \dfrac{\mathsf{W}\,\dfrac{R\supset \dfrac{\text{init}\,\dfrac{}{A \Rightarrow \bot, \bot, A}}{\Rightarrow A, \bot, \bot, A \supset \bot}}{\Rightarrow A, [(A \supset \bot), \bot \lhd \top]} \qquad \text{jump}\,\dfrac{L\bot\,\dfrac{}{\bot \Rightarrow \bot}}{\Rightarrow A, [\bot \lhd \bot]}}{\bot \preccurlyeq (A \supset \bot) \Rightarrow A, [\bot \lhd \top]}}{\bot \preccurlyeq (A \supset \bot) \Rightarrow A}}{\Rightarrow (\bot \preccurlyeq (A \supset \bot)) \supset A} \qquad R\supset \dfrac{\mathsf{C}\,\dfrac{L\bot\,\dfrac{}{\bot \Rightarrow A} \qquad R\supset \dfrac{\text{init}\,\dfrac{}{A \Rightarrow A, \bot}}{\Rightarrow A, A \supset \bot}}{\bot \preccurlyeq (A \supset \bot) \Rightarrow A}}{\Rightarrow (\bot \preccurlyeq (A \supset \bot)) \supset A}$$

Therefore axiom T is derivable, and the corresponding rule T could thus be omitted, in the rule sets of $\mathcal{I}_{\mathbb{VW}}$ and $\mathcal{I}_{\mathbb{VC}}$. We stress that the second derivation is a derivation in $\mathcal{I}_{\mathbb{VC}}$ and not in $\mathcal{I}_{\mathbb{VW}}$, since it makes use of rule C. The first derivation, instead, which is a derivation in $\mathcal{I}_{\mathbb{VW}}$, is actually a derivation also in $\mathcal{I}_{\mathbb{VC}}$, as the set of the rules of $\mathcal{I}_{\mathbb{VW}}$ is a subset of the set of rules of $\mathcal{I}_{\mathbb{VC}}$. Note that we use in all three derivations the fact that *generalized initial sequents* of form $A, \Gamma \Rightarrow \Delta, A$ are derivable in the calculi $\mathcal{I}_{\mathcal{L}}$: this fact will be proved in Lemma 2.11 below.

Remark 2.9. From the definition of the conditional $>$ in terms of $\preccurlyeq$, one can state its left and right rules as follows:

$$L{>}\,\dfrac{\bot \preccurlyeq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \,\&\, \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta} \qquad R{>}\,\dfrac{(A \,\&\, \neg B) \preccurlyeq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}$$

In fact, the sequent $A > B, \Gamma \Rightarrow \Delta$ is equivalent to $(\bot \preccurlyeq A) \vee \neg((A \,\&\, \neg B) \preccurlyeq A), \Gamma \Rightarrow \Delta$, which can be derived as follows:

$$L\vee\,\dfrac{\bot \preccurlyeq A, \Gamma \Rightarrow \Delta \qquad L\supset \dfrac{R\preccurlyeq \dfrac{\Gamma \Rightarrow \Delta, [A \,\&\, \neg B \lhd A]}{\Gamma \Rightarrow \Delta, (A \,\&\, \neg B) \preccurlyeq A} \qquad L\bot\,\dfrac{}{\bot, \Gamma \Rightarrow \Delta}}{\neg((A \,\&\, \neg B) \preccurlyeq A), \Gamma \Rightarrow \Delta}}{(\bot \preccurlyeq A) \vee \neg((A \,\&\, \neg B) \preccurlyeq A), \Gamma \Rightarrow \Delta}$$

Hence the sequent $A > B, \Gamma \Rightarrow \Delta$ can be derived from sequents $\bot \preccurlyeq A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, [A \,\&\, \neg B \lhd A]$ in any calculus $\mathcal{I}_{\mathcal{L}}$, and the left rule $L >$ is an abbreviation for the above derivation. Analogously, rule $R >$ is an abbreviation for the derivation:

$$\dfrac{\dfrac{\dfrac{(A \,\&\, \neg B) \preccurlyeq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{(A \,\&\, \neg B) \preccurlyeq A, \Gamma \Rightarrow \Delta, \bot \preccurlyeq A}\, R\preccurlyeq}{\Gamma \Rightarrow \Delta, \bot \preccurlyeq A, \neg((A \,\&\, \neg B) \preccurlyeq A)}\, R\supset}{\Gamma \Rightarrow \Delta, (\bot \preccurlyeq A) \vee \neg((A \,\&\, \neg B) \preccurlyeq A)}\, R\vee$$

The next theorem is a key result to tell what the relationship between the family of sequent calculi $\mathcal{I}_{\mathcal{L}}$ and Lewis' logics $\mathbb{V}$ and extensions is.

Theorem 2.10 (Soundness)**.** If $\mathcal{I}_{\mathcal{L}} \vdash \Gamma \Rightarrow \Delta$, then $\iota(\Gamma \Rightarrow \Delta)$ is a theorem of $\mathcal{L}$.

Proof. By induction on the height of the derivation of $\Gamma \Rightarrow \Delta$, analysing different cases for any rule of $\mathcal{I}_{\mathcal{L}}$. In particular it is possible to show that, for each rule of the calculus $\mathcal{I}_{\mathcal{L}}$, if there is a countermodel in the class of models for $\mathcal{L}$ for the formula interpretation ι of its conclusion, then there is also a countermodel for at least one of its premisses. The cases of the propositional rules and initial sequents are standard. For a detail of the other cases we address to [14], pp. 124-125. □

We show now some of the structural properties of the calculi $\mathcal{I}_{\mathcal{L}}$.

Lemma 2.11 (Derivability of generalized initial sequents)**.** Generalized initial sequents, i.e. sequents of the form $A, \Gamma \Rightarrow \Delta, A$, are derivable in $\mathcal{I}_{\mathcal{L}}$ for any formula A.

Proof. As in the propositional case, by induction on the weight of the formula A (cf. [29]). Notice that, in case A is a propositional formula, the potential blocks in Δ are not involved in the proof. Then it is enough to extend the proof of the propositional case of [29] with the case in which $A = B \preccurlyeq C$:

$$\dfrac{\dfrac{\dfrac{C \Rightarrow C, B}{\Gamma \Rightarrow \Delta, [C, B \lhd B]}\,\mathsf{jump} \qquad \dfrac{B \Rightarrow B}{\Gamma \Rightarrow \Delta, [B \lhd B]}\,\mathsf{jump}}{B \preccurlyeq C, \Gamma \Rightarrow \Delta, [B \lhd C]}\, L\preccurlyeq}{B \preccurlyeq C, \Gamma \Rightarrow \Delta, B \preccurlyeq C}\, R\preccurlyeq$$

We conclude observing that the premisses are derivable by inductive hypothesis. □

2.2.2 Cut elimination for $\mathcal{I}_{\mathbb{V}}$

We now show the main structural properties of the calculus $\mathcal{I}_{\mathbb{V}}$: the fundamental one is the cut elimination. As for contraction rules, there is a cut rule handling "cuts within blocks". Hence, the cut rules that we consider are:

$$\dfrac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\,\mathsf{cut}_1 \qquad \dfrac{\Gamma \Rightarrow \Delta, [\Omega \lhd A] \quad \Sigma \Rightarrow \Pi, [A, \Theta \lhd B]}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega, \Theta \lhd B]}\,\mathsf{cut}_2$$

We write $\mathcal{I}_{\mathcal{L}}\mathsf{cut}$ for calculi $\mathcal{I}_{\mathcal{L}}$ extended with the cut rules cut_1 and cut_2. Moreover, we write $\mathcal{I}_{\mathcal{L}}(\mathsf{cut})$ for a uniform notation for both the calculi $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{L}}\mathsf{cut}$. Finally, we denote derivability in n steps by $\vdash_n$: we write $\mathcal{I}_{\mathcal{L}}(\mathsf{cut}) \vdash_n \Gamma \Rightarrow \Delta$ if there exists a derivation of height less or equal than n in $\mathcal{I}_{\mathcal{L}}(\mathsf{cut})$, with endsequent $\Gamma \Rightarrow \Delta$.

Definition 2.12. The **complexity** of an application of cut_1 or cut_2 is the complexity of the cut formula, i.e. the number $|A|$ of symbols in the cut formula A. Given a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathcal{L}}\mathsf{cut}$, its **formula cut rank** $rk_{\mathsf{cut}_1}(\mathcal{D})$ is the maximal complexity of an application of cut_1 in it. Analogously, its **structural cut rank** $rk_{\mathsf{cut}_2}(\mathcal{D})$ is the maximal complexity of an application of cut_2 in it.

The idea for the cut elimination proof is eliminating the topmost applications of cut of maximal complexity by first permuting them into the left premiss, until we reach an occurrence of the cut formula which is principal, and then permuting them into the right one. To do this, we need as a preliminary result the height-preserving admissibility of the weakening rules:

$$\mathsf{Wk}\,\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \qquad \mathsf{Wk_i}\,\frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}{\Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A]}$$

where the rule Wk is the usual one, while $\mathsf{Wk_i}$ is the rule for weakening inside the blocks.

Lemma 2.13 (Weakening admissibility)**.** The weakening rules are height-preserving admissible in $\mathcal{I}_{\mathcal{L}}(\mathsf{cut})$, namely: If $\mathcal{I}_{\mathcal{L}}(\mathsf{cut}) \vdash_n \Gamma \Rightarrow \Delta$, then $\mathcal{I}_{\mathcal{L}}(\mathsf{cut}) \vdash_n \Gamma, \Sigma \Rightarrow \Delta, \Pi$; and if $\mathcal{I}_{\mathcal{L}}(\mathsf{cut}) \vdash_n \Gamma \Rightarrow \Delta, [\Sigma \lhd A]$, then $\mathcal{I}_{\mathcal{L}}(\mathsf{cut}) \vdash_n \Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A]$. Moreover, both the formula cut rank and the structural cut rank are preserved.

Proof. Simultaneously for both the weakening rules, by induction on the height of the derivation, distinguishing cases according to the last rule applied (cf. [14], p.126 to see the proof of some cases in detail). □

Now, we report the strategy that brings to the result of syntactic completeness via the cut elimination from [15]. Cut elimination is proved through three lemmas: the first one, Lemma 2.14 shows how to eliminate occurrences of cut_1; the two following lemmas are needed to eliminate occurrences of cut_2. Lemma 2.16 permutes the cut upwards on the left premiss until an occurrence of jump is reached. When such an occurrence is reached, Lemma 2.15 eliminates it, by permuting it upwards on the right premiss. More detailed proofs are contained in [15], while for a much more complete discussion on this topic we address to [14].

Lemma 2.14 (cut_1-reduction)**.** Suppose $\mathcal{I}_{\mathbb{V}}\mathsf{cut} \vdash \Gamma \Rightarrow \Delta, A^n$ and $\mathcal{I}_{\mathbb{V}}\mathsf{cut} \vdash A^m, \Sigma \Rightarrow \Pi$ by derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, with $rk_{\mathsf{cut}_1}(\mathcal{D}_1) < |A| > rk_{\mathsf{cut}_1}(\mathcal{D}_2)$ and $rk_{\mathsf{cut}_2}(\mathcal{D}_1) < |A| > rk_{\mathsf{cut}_2}(\mathcal{D}_2)$. There is a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathbb{V}}\mathsf{cut}$ of $\Gamma, \Sigma \Rightarrow \Delta, \Pi$, with $rk_{\mathsf{cut}_1}(\mathcal{D}) < |A| > rk_{\mathsf{cut}_2}(\mathcal{D})$.

Proof. By induction on the sum of the heights of $\mathcal{D}_1$ and $\mathcal{D}_2$. □

Lemma 2.15 (Shift-right)**.** Suppose that for $k_1, \ldots, k_n \geq 1$ we have $\mathcal{I}_{\mathbb{V}}\mathsf{cut}$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\Gamma \Rightarrow \Delta, [\Omega \lhd A]$ and $\Sigma \Rightarrow \Pi, [A^{k_1}, \Theta_1 \lhd B_1], \ldots, [A^{k_n}, \Theta_n \lhd B_n]$ respectively, with $rk_{\mathsf{cut}_1}(\mathcal{D}_1) \leq |A| \geq rk_{\mathsf{cut}_1}(\mathcal{D}_2)$ and $rk_{\mathsf{cut}_2}(\mathcal{D}_1) < |A| > rk_{\mathsf{cut}_2}(\mathcal{D}_2)$ such that the last applied rule in $\mathcal{D}_2$ is jump. There is a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathbb{V}}\mathsf{cut}$, with $rk_{\mathsf{cut}_1}(\mathcal{D}) \leq |A| > rk_{\mathsf{cut}_2}(\mathcal{D})$, of the sequent

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega, \Theta_1 \lhd B_1], \ldots, [\Omega, \Theta_n \lhd B_n]$$

Proof. By induction on the height of $\mathcal{D}_2$, distinguishing cases according to the last applied rule R in $\mathcal{D}_2$. □

Lemma 2.16 (cut_2-reduction)**.** Suppose we have $\mathcal{I}_{\mathbb{V}}$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_n \lhd A]$ and $\Sigma \Rightarrow \Pi, [A, \Theta \lhd B]$ respectively, with $rk_{cut_1}(\mathcal{D}_1) \leq |A| \geq rk_{cut_1}(\mathcal{D}_2)$ and $rk_{cut_2}(\mathcal{D}_1) < |A| > rk_{cut_2}(\mathcal{D}_2)$. There is a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathbb{V}}Cut$, with $rk_{cut_1}(\mathcal{D}) \leq |A| > rk_{cut_2}(\mathcal{D})$, of the sequent

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_n, \Theta \lhd B]$$

Proof. By induction on the height of $\mathcal{D}_1$, distinguishing cases according to the last applied rule R in $\mathcal{D}_1$. □

Theorem 2.17 (Cut elimination)**.** If $\mathcal{I}_{\mathbb{V}}\mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, then $\mathcal{I}_{\mathbb{V}} \vdash \Gamma \Rightarrow \Delta$. In particular, there is a procedure to eliminate cuts from a derivation in $\mathcal{I}_{\mathbb{V}}\mathsf{cut}$.

Proof. By lexicografic induction on the tuples $\langle rk_{\mathsf{cut}_1}(\mathcal{D}), \#_{\mathsf{cut}_2}(\mathcal{D}), \#_{\mathsf{cut}_1}(\mathcal{D})\rangle$, where $\#_{\mathsf{cut}_1}(\mathcal{D})$ and $\#_{\mathsf{cut}_2}(\mathcal{D})$ are respectively the number of applications of cut_1 and cut_2 in $\mathcal{D}$ with cut formula of complexity $max\{rk_{\mathsf{cut}_1}(\mathcal{D}), rk_{\mathsf{cut}_2}(\mathcal{D})\}$. In fact we eliminate the topmost applications of the rules cut_1 and cut_2 with maximal complexity by decreasing the inductive parameter through the lemmas of cut_1-reduction and cut_2-reduction respectively. □

A direct corollary of the cut admissibility is the syntactical proof of the completeness of $\mathcal{I}_{\mathbb{V}}$ with respect to logic $\mathbb{V}$:

Theorem 2.18 (Completeness)**.** If a formula F is valid in $\mathbb{V}$, then it is valid in $\mathcal{I}_{\mathbb{V}}$, i.e. there is a derivation of $\Rightarrow F$ in $\mathcal{I}_{\mathbb{V}}$.

Proof. By deriving the axioms of the Hilbert-style calculus for $\mathbb{V}$ in $\mathcal{I}_{\mathbb{V}}\mathsf{cut}$ and showing that $\mathcal{I}_{\mathbb{V}}\mathsf{cut}$ is closed with respect to the inference rules of the Hilbert-calculus for $\mathbb{V}$. By way of illustration, we report here some of the cases.

To prove the admissibility of the rule (CPR) we need to show that if its premiss $B \supset A$ is derivable, then also its conclusion $A \preccurlyeq B$ is. Notice that by $L\supset$ the sequent $B, B \supset A \Rightarrow A$ is derivable:

$$\dfrac{\dfrac{}{B \Rightarrow A, B}\ \mathsf{init} \qquad \dfrac{}{B, A \Rightarrow A}\ \mathsf{init}}{B \supset A, B \Rightarrow A}\ L\supset$$

Notice now that the sequent $B \Rightarrow A, B \supset A$ is derivable from the premiss $B \supset A$ of (CPR), derivable by hypothesis, by admissibility of weakening. Therefore, we conclude by the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, B \supset A \qquad B, B \supset A \Rightarrow A}{B, B \Rightarrow A, A}\ \mathsf{cut}_1}{B \Rightarrow A}\ \mathsf{Ctr_L}+\mathsf{Ctr_R}}{\Rightarrow [A \lhd B]}\ \mathsf{jump}}{\Rightarrow A \preccurlyeq B}\ R\preccurlyeq$$

The derivation of *modus ponens* involves the propositional rules and rule cut_1, and is standard.

$$
L{\preccurlyeq}^{\mathsf{i}}\ \frac{A \preccurlyeq B, \Gamma \Rightarrow \Delta, [B, \Sigma \lhd C] \quad A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]}{A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}
$$

$$
\mathsf{com}^{\mathsf{i}}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}
$$

$$
\mathsf{T}^{\mathsf{i}}\ \frac{A \preccurlyeq B, \Gamma \Rightarrow \Delta, B \quad A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{A \preccurlyeq B, \Gamma \Rightarrow \Delta} \qquad \mathsf{W}^{\mathsf{i}}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A], \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}
$$

$$
\mathsf{C}^{\mathsf{i}}\ \frac{A \preccurlyeq B, \Gamma \Rightarrow \Delta, B \quad A \preccurlyeq B, A, \Gamma \Rightarrow \Delta}{A \preccurlyeq B, \Gamma \Rightarrow \Delta} \qquad \mathsf{A}^{\mathsf{i}}\ \frac{A, \Gamma^{\preccurlyeq} \Rightarrow \Delta^{\preccurlyeq}, [\Sigma \lhd A], \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}
$$

Figure 2.3: Modified rules of Sequent Calculi $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ and extensions

Axiom (CPA) is derived as follows:

$$
R\vee\ \dfrac{R{\preccurlyeq}\ \dfrac{R{\preccurlyeq}\ \dfrac{\mathsf{com}\ \dfrac{\mathsf{jump}\ \dfrac{L\vee\ \dfrac{\mathsf{init}\ \dfrac{}{A \Rightarrow A, B} \quad \mathsf{init}\ \dfrac{}{B \Rightarrow A, B}}{A \vee B \Rightarrow A, B}}{\Rightarrow [A, B \lhd A \vee B]} \quad \mathsf{jump}\ \dfrac{L\vee\ \dfrac{\mathsf{init}\ \dfrac{}{A \Rightarrow A, B} \quad \mathsf{init}\ \dfrac{}{B \Rightarrow A, B}}{A \vee B \Rightarrow A, B}}{\Rightarrow [A, B \lhd A \vee B]}}{\Rightarrow [A \lhd A \vee B], [B \lhd A \vee B]}}{\Rightarrow [A \lhd A \vee B], B \preccurlyeq A \vee B}}{\Rightarrow A \preccurlyeq A \vee B, B \preccurlyeq A \vee B}}{\Rightarrow (A \preccurlyeq A \vee B) \vee (B \preccurlyeq A \vee B)}
$$

The derivations of the other axioms of $\mathbb{V}$ are similar to the one of (CPA). □

We stress the fact that this syntactical proof of completeness strongly leans on the cut elimination result 2.17 of $\mathcal{I}_{\mathbb{V}}$: the derivation proving the admissibility of rule (CPR) is a derivation in $\mathcal{I}_{\mathbb{V}}\mathsf{cut}$ and not in $\mathcal{I}_{\mathbb{V}}$, due to an application of rule cut_1. At that point the cut elimination 2.17 is what allows us to conclude.

2.2.3 Sequent calculi $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ and completeness of $\mathcal{I}_{\mathbb{VW}}$

To complete the discussion on internal calculi $\mathcal{I}_{\mathcal{L}}$, we mention in this section the version of cacluli $\mathcal{I}_{\mathcal{L}}$ with invertible rules and admissible contraction rules. These calculi are equivalent to $\mathcal{I}_{\mathcal{L}}$ but have better structural properties. Our main interest in calculi $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ lies in the fact that it is possible to prove their completeness not only with respect to $\mathbb{V}$, but also for $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ (which will play a key role in Chapter 4) and $\mathbb{VC}$. For a more detailed treatment of this topic refer to [15].

We report in Figure 2.3 the table of the rules of $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ that are modified from $\mathcal{I}_{\mathcal{L}}$ (see Figure 2.2). All the rules, with the exception of jump and A^{i}, are invertible. Weakening rules are height-preserving admissible in these calculi. Contraction rules are admissible as well, so they do not need to be included in the calculi.

We state the equivalence mentioned above between $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ as follows:

Theorem 2.19 (Equivalence)**.** Let A be a formula in $\mathcal{F}_{\preccurlyeq}$. The formula A is derivable in the calculus $\mathcal{I}_{\mathcal{L}}$ if and only if A is derivable in the calculus $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$.

This result can be proved by induction on the height of the derivation of A, and exploits the admissibility of weakening and contraction rules. Finally, we mention that the calculi $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ allow terminating proof-search, thus providing a decision procedure for the respective logics, and this gives as a consequence the semantic completeness for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$:

Theorem 2.20 (Semantic completeness)**.** If a sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathbb{V}$ (resp. $\mathbb{VN}, \mathbb{VT}, \mathbb{VW}, \mathbb{VC}$), then it is derivable in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ (resp. $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}, \mathcal{I}^{\mathsf{i}}_{\mathbb{VT}}, \mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}, \mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$).

This result, together with the equivalence with the calculi $\mathcal{I}_{\mathcal{L}}$, yields the completeness of the $\mathcal{I}_{\mathcal{L}}$-version of the calculi mentioned above. In particular the sequent calculus $\mathcal{I}_{\mathbb{VW}}$, which plays an important rule in Chapter 4, is thus sound and complete with respect to logic $\mathbb{VW}$.

We conclude this section remarking that another way for proving completeness of the calculi mentioned above is taking advantage of the non-standard sequent calculi $\mathcal{R}_{\mathcal{L}}$, presented in [21]. These calculi have infinite rules, but it is possible to show that $\mathcal{I}_{\mathcal{L}}$ can simulate derivations in $\mathcal{R}_{\mathcal{L}}$. This allows to prove also the completeness of the calculi $\mathcal{I}_{\mathcal{L}}$ with respect to logics $\mathbb{VA}$ and $\mathbb{VNA}$, that were left out of the treatment due to the non-invertibility of rule A^{i}. We refer to [15] for the details of these arguments.

Chapter 3

Adams' Logic of High Probability

According to Adams, the appropriate way to interpret (several classes of) conditionals should be probabilistic. In fact, besides pointing out that it is questionable whether conditionals with false antecedents can meaningfully be classified as either true or false, as other authors do as well, Adams' innovation in [1] is that of observing that most of the conditionals of everyday language should be understood in terms of assertability conditions, which naturally align with probability-based reasoning.

In fact, many conditionals seem to indicate a probabilistic connection between the antecedent and consequent, rather than a purely deductive relation. The conditional

If $a = b$, then $a + 1 = b + 1$ (from [11])

expresses indeed a strict logical relationship; but many kinds of conditionals used in the ordinary language, such as

If you boil this egg, it will become hard (from [20])

or, also,

If the light does not turn on, then the lamp must be broken (from [10])

actually imply that the consequents follow from the antecedents with a certain degree of probability: in most cases, a high probability.

Central to Adams' treatment of conditionals is the notion of conditional probability, very similar to the one given by Kolmogorov [19]. Suppose that to every proposition A is associated a probability $P(A)$. Given two propositional formulas A and B, the conditional probability of B *given* A is

$$P(B \mid A) = \frac{P(A \,\&\, B)}{P(A)}$$

provided that $P(A) > 0$. It can be shown that the probability $P(A \supset B)$ of a material implication is never less than the conditional probability $P(B \mid A)$.

Adams' central assumption is that for every propositional sentences A and B, the probability of the conditional $A > B$ equals the corresponding conditional probability:

$$P(A > B) = P(B \mid A) \tag{3.1}$$

provided that $P(A) > 0$.

We make a few comments on this approach. First, as pointed out by Lewis, the application of the probabilistic account to counterfactuals is problematic, since the antecedent of a counterfactual should have probability zero; similarly, Adams himself remarks ([1]) that this theory works well just for *assertable* conditionals, treating conditionals as static. Assuming to stick to to this class of conditionals, the second point is that Adams' theory doesn't cover embeddings of conditionals, but just conditionals of propositional formulas. This is a weakness of his logic compared to the ones of Adams and Lewis, and an overview of possible solutions is delineated in [12], sec. 5.4. Finally, stating that a conclusion may be inferred from given premises with a reasonable probability, is actually different from assigning a degree of probability to the corresponding sentences, and could be hard to do in the practice. For this reason, the theory of high probability proposed by Adams in [5], our main focus in this chapter, simplifies Adams' probabilistic interpretation of conditionals, accounting just for conditional sentences that are "normally" verified.

For this introduction we followed [12], to which we refer for an overview on the probabilistic interpretation of conditionals. For the complete description of Adams' probabilistic account we relate to his seminal works [1], [2] and [3].

In this chapter we present the logic of high probability **HPA** proposed by Adams in [5], namely an extension of classical propositional logic with a binary relation H, where the meaning of the formula $H(A, B)$ is that the ordinary language conditional with antecedent A and consequent B is highly probable.

3.1 Logic HPA

The language on which this logic is based has two levels. The first is the *basic-level*, or *zero-level* language, consisting of the terms built up from constants and variables of a propositional calculus. We will denote this level of the language with $\mathcal{L}_0^A$. The second level is the *first-level* language, in which the only primitive operator is the "high probability predicate" H. We will denote this level with $\mathcal{L}_1^A$. The two levels are linked since the arguments of the first-level predicate H are terms from the zero-level language $\mathcal{L}_0^A$. Formally, the languages introduced above are as follows:

Notation. Take a countably denumerable set of propositional variables **Prop**. Formulas in the zero-level language $\mathcal{L}_0^A$ are generated by the following grammar, for p in **Prop** and A, B in $\mathcal{L}_0^A$:

$$\mathcal{L}_0^A ::= f \mid p \mid A \wedge B \mid A \vee B \mid A \rightarrow B$$

Formulas in the first-level language $\mathcal{L}_1^A$ are generated by the following grammar, for A, B in $\mathcal{L}_0^A$ and φ, ψ in $\mathcal{L}_1^A$:

$$\mathcal{L}_1^A ::= H(A, B) \mid \bot \mid \varphi \,\&\, \psi \mid \varphi \vee \psi \mid \varphi \supset \psi$$

In both languages negation, true and biconditional can be defined as usual in terms of the already existing connectives. For A, B in $\mathcal{L}_0^A$, the negation formula $\sim\! A$ is an abbreviation for $A \rightarrow f$; t stands for $\sim f$; and $A \leftrightarrow B$ is an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$. Analogously, for A, B in $\mathcal{L}_1^A$, $\neg A$ stands for $A \supset \bot$; $\top$ stands for $\neg\bot$; and $A \supset\subset B$ is an abbreviation for $(A \supset B) \,\&\, (B \supset A)$. Moreover, for A in $\mathcal{L}_0^A$, we will write $H(A)$ as an abbreviation for $H(t, A)$.

Given the syntax of the two levels of the language, we describe here their usual interpretation. Regarding the zero-level language $\mathcal{L}_0^A$ and its main abbreviations, we have:

- the symbols for the constants t and f are interpreted respectively as tautology and contradiction at the zero-level;
- the unary operation symbol $\sim$ is interpreted as negation at the zero-level;
- the binary operation symbols $\wedge, \vee, \rightarrow$ and $\leftrightarrow$ are interpreted respectively as conjunction, disjunction, material implication and material biconditional at the zero-level.

The first-level language $\mathcal{L}_1^A$ and its abbreviations, on the other hand, are interpreted as follows:

- the application $H(A, B)$ of the binary predicate H to formulas A and B of the zero-level language means that the conditional of the ordinary language, i.e. the non-material conditional, with antecedent A and consequent B is highly probable;
- the constants $\top$ and $\bot$ stand respectively for tautology and contradiction in the first-level language;
- the unary operation $\neg$ stands for negation in the first-level language;
- the binary operations $\&, \vee, \supset$ and $\supset\subset$ stand respectively for conjunction, disjunction, material implication and material biconditional in the first-level language.

The abbreviation $H(A)$ for $H(t, A)$ reads as "A is highly probable", thinking of the non conditional formula A as equivalent to the corresponding conditional with tautological antecedent.

Notice that there is no connective for the (non-material) conditional $>$ in the zero-level language: this is because of the assumption that the probabilities of non-material conditionals are conditional probabilities, as discussed in detail by the same Adams in [3]. Hence, also high probabilities of non-material conditionals are conditional high probabilities, which in our language, enriched with the conditional operator $>$ at the zero-level, can be translated as:

$$H(A > B) \equiv H(A, B) \tag{3.2}$$

Hence the non-material conditional is implicitly contained in the high probability operator, as we will investigate more deeply in Section 3.3. In the following, before officially introducing the conditional in the language, in Section 3.3 again, we will often informally refer to it using the symbol "$>$". This symbol in fact is not part of the zero-level language, but it is useful to give an idea of the meaning of the axioms, given the usual interpretation "normally, B follows from A" for the formula $A > B$.

We call **HPA** the logic from Adams' paper [5]. It is axiomatized as follows:

H1. If $A, B \in \mathcal{L}_1^A$ are such that B is a tautological consequence of A, then $H(A, B)$ is a theorem;

H2. If $A, B \in \mathcal{L}_1^A$ are tautologically equivalent, then $H(A, C) \supset H(B, C)$ is a theorem;

H3. $H(A, C) \,\&\, H(B, C) \supset H(A \vee B, C)$

H4. $H(A, B) \,\&\, H(A, C) \supset H(A \wedge B, C)$

H5. $H(A, B) \,\&\, H(A \wedge B, C) \supset H(A, C)$

H6. $H(A, C) \supset H(A \wedge B, C) \vee H(A, {\sim}B)$

The first two axioms are actually axiom schemas stated in natural language in terms of tautological equivalence and tautological entailment in the original paper by Adams. The succeding four axioms H3 – H6 are presented in Adams' paper [5] as universal closures of formulas in $\mathcal{L}_1^A$.

Remark 3.1. Axioms H1 and H2 can be expressed through rules of inference as:

$$\text{H1}\,\frac{A \to B}{H(A, B)} \qquad \text{H2}\,\frac{A \leftrightarrow B}{H(A, C) \supset H(B, C)}$$

meaning that if $A \to B$ (resp. $A \leftrightarrow B$) is derivable in the axiomatic calculus, then also $H(A, B)$ (resp. $H(A, C) \supset H(B, C)$) is derivable. This formulation will be particularly useful in the following, especially in studying the relation between Adams' logic **HPA** and Lewis' family of logics for conditionals extending $\mathbb{V}$.

Before making some remarks about this axiomatic system, we give the following definitions:

Definition 3.2. A **probability function** at the zero-level language is a map

$$P \colon \mathcal{L}_0^A \longrightarrow [0, 1]$$

assigning a probability value in the unit interval to each formula of the zero-level language, and satisfying the usual axioms for probability functions.

A **truth function** is a probability function that has values only in $\{0, 1\}$, where 0 is interpreted as falsehood and 1 is interpreted as truth. We will usually denote a truth function with T.

Moreover, any probability function P can be extended to a **conditional probability function**

$$\begin{aligned} P \colon \mathcal{L}_0^A \times \mathcal{L}_0^A &\longrightarrow [0, 1] \\ \langle A, B \rangle &\longmapsto P(A, B) \end{aligned}$$

where

$$P(A, B) := \begin{cases} \frac{P(A \wedge B)}{P(A)} & \text{if } P(A) > 0 \\ 1 & \text{if } P(A) = 0 \end{cases}$$

Finally, the **improbability function** associated with P is

$$\begin{aligned} I \colon \mathcal{L}_0^A &\longrightarrow [0, 1] \\ A &\longmapsto I(A) := 1 - P(A) \\ I \colon \mathcal{L}_0^A \times \mathcal{L}_0^A &\longrightarrow [0, 1] \\ \langle A, B \rangle &\longmapsto I(A, B) := 1 - P(A, B) \end{aligned}$$

depending on whether P is a probability or a conditional probability function.

Remark 3.3. 1. We can translate axiom H3 in a “vague ordinary language” by stating that if both $A > C$ and $B > C$ are highly probable, then $(A \vee B) > C$ is highly probable, too. In fact, H3 vaguely expresses the following improbability inequality:

$$I(A > C) + I(B > C) \geq I((A \vee B) > C)$$

which means that the improbability of $(A \vee B) > C$ cannot be greater than the sum of the improbabilities of $A > C$ and $B > C$. The axiom H3 can be derived in fact from the improbability inequality above, although we do not report here a detailed proof of this fact. Intuitively, assuming that the improbabilities of the antecedents are arbitrarily low we get that the improbability of the conclusion is lower; this “vaguely implies” that it is possible for the conclusion $(A \vee B) > C$ to be arbitrarily high probable, provided that $A > C$ and $B > C$ have high enough probabilities.

2. Actually all of the axioms from H1 to H5 can be seen as expressions of improbability inequalities in which the improbability of the conclusion cannot exceed the sum of the improbabilities of the premisses, like for H3. This means, as in the case of H3 again, that we can ensure an arbitrarily high probability for the conclusion just by requiring the premisses to be sufficiently highly probable, and the axioms can be derived from the corresponding improbability inequalities as outlined for H3. There is an analogy between this relationship and the one between premisses and conclusions of a valid non-conditional inference: this fact makes this interpretation of axioms very natural.

3. Also H6 can be seen in terms of improbability inequalities, but with a slight difference: this is the only axiom, representing in fact the novelty of Adams’ paper in relation to previous works ([1], [3], [4] among the others), having a disjunction in the positive part of the formula and making the formula non-Harrop. Also the theory axiomatized this way becomes non-Harrop, and this makes it much harder to describe. The improbability inequality corresponding to H6 is

$$I(A > C) \geq I((A \wedge B) > C) \times I(A > {\sim}B)$$

4. All of the axioms from H1 to H6 are formulas of the following form (C), which gives us a way to treat together all of the six axioms:

$$(\mathsf{C}) \quad (H(A_1, B_1) \,\&\, \ldots \,\&\, H(A_n, B_n)) \supset (H(C_1, D_1) \vee \cdots \vee H(C_m, D_m))$$

where $n, m \in \mathbb{N}$ and all the arguments of H are terms of the zero-level language. Varying n and m, this schema gives us the structure of all of the six axioms: H1 for $n = 0$ and $m = 1$, H2 for $n = m = 1$, H3, H4 and H5 for $n = 2$ and $m = 1$, and H6 for $n = 1$ and $m = 2$ (with the convention that when $n = 0$ the antecedent is true and for $m = 0$ the consequent is false). Previous works mentioned above, and in particular [4], dealt only with “one-conclusion inferences” and a “one-conclusion” form of the schema (C), in which $m = 1$ and only n varies. The valid formulas of this kind are entailed by the only axioms H1-H5. Thanks to the axioms H6, our system of axioms is instead sufficient to entail all the valid formulas of form (C).

3.2 A characterization of theoremhood in HPA

In the original paper Adams looks at formulas of form (C) in detail, focusing in particular on their theoremhood or non-theoremhood according to the evaluation given by improbability functions. We will not focus on this kind of results, but rather we will have a quick look at just one of them. This result represents indeed a useful practical tool, giving a decision procedure to determine whether a formula of form (C) is a theorem or not.

A truth-function T defined as in 3.2 is defined over $\mathcal{L}_0^A$. We want to extend in a certain sense the domain of such functions to the set of all high probability formulas $H(A, B)$:

Definition 3.4. A high probability formula $H(A, B)$ is evaluated via a **truth function** T as follows:

- If $T(A) = T(B) = 1$ then $T(H(A, B)) = 1$, that is T verifies the high probability formula $H(A, B)$;
- If $T(A) = 1$ and $T(B) = 0$ then $T(H(A, B)) = 0$, that is T falsifies the high probability formula $H(A, B)$;
- Otherwise, namely if $T(A) = 0$, the high probability formula $H(A, B)$ is neither verified nor falsified.

Notice that the truth function T extended this way is in fact a partial function, being undefined on a subset of the high probability formulas. This is a way of having a grey area between the truth values 0 and 1, but avoiding the structure of a many-valued logic. Multivalent logical systems with finitely many truth values have been proved to be unsuitable to model probabilistic or modal theories of conditionals, as McGee proves in [25]. The fact that logic **HPA** with the truth-functions 3.4 is apparently three-valued is not incompatible with McGee's result, as explained in [6]. Also, we remark that the material implication $A \supset B$ associated to the high probability implication $H(A, B)$ is falsified by a truth function T if and only if the high probability implication is falsified by T, and verified otherwise. Therefore, to have a correspondence with the zero-level language interpretation, sometimes we will talk about a truth function T verifying the material implication corresponding to the high probability formula $H(A, B)$, rather than of T not falsifying $H(A, B)$; and of T falsifying $A \supset B$ instead of T falsifying $H(A, B)$.

We now extend the valuation function not to all $\mathcal{L}_1^A$, but to inferences that are formulas of the form (C); we see a (C) formula as an inference in which the set of the premisses *yields* the ordered set of conclusions, according to the following definition:

Definition 3.5. Let m, n be in $\mathbb{N}$. Given a (C) formula

$$H(A_1, B_1)\,\&\, \dots \,\&\, H(A_n, B_n) \supset H(C_1, D_1) \vee \cdots \vee H(C_m, D_m)$$

with premisses $H(A_i, B_i)$, for $i \in \{1, \dots, n\}$ and ordered conclusions $H(C_j, D_j)$ for $j \in \{1, \dots, m\}$, we say that the premisses **yield** the conclusions *iff* $m + n > 0$ and the following two conditions hold:

(1) any truth function T that verifies at least one premiss and falsifies none, verifies at least one conclusion;

(2) for every $j \in \{1, \dots, m\}$, any truth function T that falsifies one of $H(C_1, D_1), \dots, H(C_j, D_j)$ and verifies none of them, falsifies at least one premiss.

The Definition 3.5 of premisses yielding conclusions uses the convention that if $n = 0$, the empty premiss is $\top$, verified by every truth function T; while if $m = 0$ the conclusion is $\bot$, falsified by every truth function T. We remark that Definition 3.5 actually depends on the order of the conclusions in the fact that clause (2) considers, for any $j \in \{1, \ldots, m\}$, the truth functions T falsifying the first j high probability formulas; we are not interested in the subsequent ones, and for this reason one can notice that the order of the conclusions plays a role in the definition above. In the following examples formula 3.4 better illustrates the dependence of the definition of yielding on the order of the conclusions by showing how the definition applies in a specific notable case.

We can now state the result that gives a decision procedure for the logic:

Theorem 3.6. A formula of form (C) is a theorem if and only if a subset of its premisses yields an ordered subset of its conclusions.

Examples 3.7. We give some examples of application of Theorem 3.6.

1. Consider the formula
$$H(A, C) \supset H(A \wedge B, C) \tag{3.3}$$
called *strengthening of the antecedent*, that is valid for the material implication. The premiss does not yield the conclusion though, since the truth function T such that $T(A) = T(C) = 1$ and $T(B) = 0$ verifies the premiss but not the conclusion. Then no subset of the premisses yields an ordered subset of the conclusions, and by 3.6 this formula is not a theorem.

2. By adding an alternative conclusion to 3.3 we obtain the formula:
$$H(A, C) \supset (H(A \wedge B, C) \vee H(A, {\sim}B)) \tag{3.4}$$
which is axiom H6. In this case, $n = 1$ and $m = 2$, so $m+n = 3 > 0$. Furthermore:
   - Clause (1) of Definition 3.5 is satisfied: consider a truth function T that verifies the premiss $H(A, C)$, i.e. $T(A) = T(C) = 1$; now, either $T(B) = 1$ and the first conclusion $H(A \wedge B, C)$ is verified, or $T(B) = 0$ and the second conclusion $H(A, {\sim}B)$ is verified;
   - Clause (2) of 3.5 is satisfied: both for any truth function T falsifying the first conclusion $H(A \wedge B, C)$, and for any truth function T falsifying the second conclusion $H(A, {\sim}B)$ while not verifying the first one, $T(A) = T(B) = 1$ and $T(C) = 0$; then also the premiss $H(A, C)$ is falsified.

   We just proved that 3.4 with the subset of all premisses and the ordered subset of all the conclusions given in the order of 3.4 satisfies the Definition 3.5. Then for 3.6 the closure of 3.4 is a theorem.

   Notice that choosing the other ordered subset of two conclusions, namely the one in which $H(A, {\sim}B)$ comes before $H(A \wedge B, C)$, wouldn't satisfy Definition 3.5. This is because the truth function T for which $T(A) = T(B) = T(C) = 1$, falsifying just the first conclusion $H(A, {\sim}B)$ (we are not interested in the action of T on the second conclusion, since we are considering the case $j = 1$ of clause (2) of Definition 3.5), actually verifies the only premiss $H(A, C)$. We want to highlight that the theoremhood of a formula is not sensitive to the dependence on the order of the conclusions of Definition 3.5: in the formula 3.4 the premisses yield the conclusions, while in
$$H(A, C) \supset (H(A, {\sim}B) \vee H(A \wedge B, C))$$

they do not; however, according to Theorem 3.6, one of the two formulas is a theorem if and only if the other one is, since it asks just for *the existence* of a subset of the premisses yielding an ordered subset of the conclusions. Therefore, Theorem 3.6 overcomes the dependence on the order of the conclusions of Definition 3.5; this allows us to consider as essentially equivalent formulas that differ only in the order of the conclusions.

3. In the case of $n = 0$, the formula (C) has form:

$$H(C_1, D_1) \vee \cdots \vee H(C_m, D_m) \tag{3.5}$$

with $m > 0$, and by convention the premiss is $\top$, verified by any truth function T. If 3.5 is a theorem, then by Theorem 3.6 a subset of the premisses yields an ordered subset of the conclusions. Clause (1) of the Definition 3.5 says that for every truth function T that verifies a premiss and falsifies none (i.e. for every truth function), there exists a conclusion verified by such T; this is not fundamental for the characterization we are going to give, anyway. Let's consider now clause (2) of Definition 3.5. If the formula 3.5 is a theorem, then there exists an ordered subset of the conclusions of k elements such that for every $j \in \{1, \ldots, k\}$, any truth function T that falsifies at least one of the j conclusions and verifies none, falsifies at least one of the premisses. Let's consider the easy case $j = 1$, and let $H(C_i, D_i)$ the first element of the ordered set of the conclusions: for every truth function T that falsifies $H(C_i, D_i)$, T needs to falsify at least one of the premisses; but no truth function T can falsify the premiss $\top$, so no T can falsify $H(C_i, D_i)$. This is equivalent to state that every truth finction T verifies the material implication $C_i \supset D_i$.

   Hence, if formula 3.5 is a theorem, then there exists $i \in \{1, \ldots, m\}$ such that $C_i \supset D_i$ is a tautology. Since the converse trivially holds, the choice of the ordered set of the only $H(C_i, D_i)$ as the subset mentioned in Theorem 3.6 is in fact a characterization of the theoremhood of form 3.5.

4. Quite similarly to what was done in the previous example, if $m = 0$ and $n > 0$, the (C) formula is

$$\neg H(A_1, B_1) \vee \cdots \vee \neg H(A_n, B_n) \tag{3.6}$$

as an equivalent formulation of

$$(H(A_1, B_1) \mathbin{\&} \ldots \mathbin{\&} H(A_n, B_n)) \supset \bot$$

In this case, if 3.6 is a theorem, then a subset of the premisses satisfies clause (2) of Definition 3.5: for every truth function T that falsifies $\bot$ (i.e. for any truth function T), T falsifies at least one premiss. This way, also clause (1) is trivially satisfied, since no truth function T verifies at least a premiss while falsifies none, as a consequence of clause (2). This means that, if $m = 0$, if 3.6 is a theorem then no truth function T can falsify no premiss; or, equivalently, that no truth function T verifies all of the material implications $A_1 \supset B_1, \ldots, A_n \supset B_n$.

   Therefore, if 3.6 is a theorem then the material implications $A_1 \supset B_1, \ldots, A_n \supset B_n$ corresponding to the premisses are tautologically inconsistent. Again, the converse trivially holds choosing the set of all premisses as the subset mentioned in Theorem 3.6. Hence this last condition is a characterization for the theoremhood of 3.6.

5. If for any $j \in \{1, \dots, m\}$, $C_j = \top$, the formula (C) has form:

$$(H(A_1, B_1) \,\&\, \dots \,\&\, H(A_n, B_n)) \supset (H(D_1) \vee \cdots \vee H(D_m)) \tag{3.7}$$

Notice that, for any truth function T, for any $j \in \{1, \dots, m\}$, since $C_j = \top$, T verifies (or, respectively, falsifies) the conclusion $H(C_j, D_j)$ if and only if it verifies (resp. falsifies) D_j. Let $H(D_i)$ be the first element of the ordered subset of conclusions mentioned in Theorem 3.6.

Then the material implication having the conjunction of $A_1 \supset B_1, \dots, A_n \supset B_n$ as antecedent of the implication and D_i as conclusion is verified by any truth function T: if one tries to think of a truth function T verifying this last conjunction (i.e. falsifying no premiss) and not D_i, then T wouldn't satisfy clause (2) of 3.5, falsifying $H(D_i)$ but none of the premisses.

Now again, since the converse trivially holds choosing the set of all premisses and the ordered set with $H(D_i)$ as only element, respectively, as the subset of premisses and the ordered subset of conclusions mentioned in 3.6, we can state that 3.7 is a theorem if and only if $A_1 \supset B_1, \dots, A_n \supset B_n$ tautologically imply D_i for some $i \in \{1, \dots, m\}$.

The last three examples tell something about the link between high probability implications and their material counterparts in the basic-level language, in some particular cases: formula 3.5 above being a theorem means that one of the material conditionals must be highly probable no matter what; formula 3.6 tells us that several high probability conditionals are tautologically inconsistent if and only if their corresponding material counterparts are; and finally, formula 3.7 is the case in which all conclusions are actually inconditional. In this last case, we see that the inference to alternative conclusions is valid only if there is a specific one of the conclusions that is validly inferable: in fact, reasoning about factual propositions it is impossible to deduce that the disjunction of two or more conclusions must be high probable, without being able to say which one of the conclusions actually is.

3.3 The conditional operator > in probabilistic logic

Having displayed Adams' logic for high probability from [5], given a tool to determine the theoremhood of formulas in the logic, and shown some of the links of the high probability operator H introduced there with the material implication $\supset$, we are now ready to investigate the relationship between this logic and the non-material conditional $>$. More precisely, we are interested in what happens when we add it to the zero-level language, trying to answer the following questions:

1) What happens if we (*reasonably*) add $>$-formulas as a possible argument for high-probability formulas?

2) By translating the high probability formulas into $>$-formulas, do we find a correspondence with an already existing logic?

To answer the first question, we extend $\mathcal{L}_0^A$ with the non-material conditional $>$. We talk of *reasonably* adding $>$ to the language in the sense that we want to complement this extension with appropriate definitions and axioms handling the relationship between $>$ and the other operations. First, as we already claimed in remark 3.2 we define

$H(A > B)$ to be equivalent to $H(A, B)$, since we want high probabilities of non-material conditional to be conditional high probabilities:

$$H(A > B) \equiv H(A, B) \tag{3.8}$$

One can prove that adding this definition is *conservative*, in the sense that no new formulas of form (C), not including $>$, can be derived just by the addition of 3.8. We now add a further plausible assumption, under which we will prove an equivalence in high probability of $>$ and $\supset$, and then a sort of "trivialization" of the logic. So, we add as an assumption a qualitative version of the law of probability change by conditionalization, known also as "import-export law":

$$H(A \wedge B, C) \supset\subset H(A, B > C) \tag{3.9}$$

What happens under this assumption is a trivialization of the high probability operator H, that comes to be equivalent for a conditional formula and the corresponding material implication formula. This aligns with the triviality results proved by Lewis in [24], that underline the fact that the nesting of conditionals of this kind is problematic.

To prove the equivalence in high probability of the two implications we need the following lemma of distributivity of the high probability operator H:

Lemma 3.8. For any formulas A, B in $\mathcal{L}_0^A$

$$H(A, B) \supset (H(A) \supset H(B)) \tag{3.10}$$

is a theorem in **HPA**.

Proof. Formula 3.10 is classically equivalent to

$$(H(A) \,\&\, H(A, B)) \supset H(B)$$

which is an abbreviation for

$$(H(t, A) \,\&\, H(A, B)) \supset H(t, B)$$

This last formula is an instance of axiom H5, substituting in it t for A, A for B and B for C, and considering that $t \wedge A$ is classically equivalent to A. Therefore, formula 3.10 is a theorem. □

We can now state the "equivalence in probability" mentioned above. While the *if* direction holds in general, to prove the *only if* direction we found it necessary to assume axiom W: $(A > B) \supset (A \rightarrow B)$, although Adams does not specify this choice in his paper [5].

Theorem 3.9. The following equivalence of formulas is a theorem:

$$H(A, B) \supset\subset H(A \rightarrow B) \tag{3.11}$$

Proof. To prove the equivalence above, we prove the two implications separately:

$\subset$) The formula $({\sim}A \wedge A) \rightarrow B$ is a classical tautology. Then, by axiom H1,

$$H({\sim}A \wedge A, B) \tag{3.12}$$

is a theorem. Now, the formula

$$H(\sim A \wedge A, B) \supset H(\sim A, A > B) \tag{3.13}$$

is an instance of the *only if* direction of 3.9, and is therefore a theorem, too. By modus ponens, 3.12 and 3.13 produce

$$H(\neg A, A > B) \tag{3.14}$$

Analogously, the classical tautology $B \,\&\, A \rightarrow B$ makes $H(B \,\&\, A, B)$ a theorem by axiom H1. This last formula, with the *only if* instance of 3.9 $H(B \,\&\, A, B) \supset H(B, A > B)$ and by modus ponens, produces

$$H(B, A > B) \tag{3.15}$$

Consider now the instance of axiom H3

$$(H(\sim A, A > B) \,\&\, H(B, A > B)) \supset H(\sim A \vee B, A > B)$$

If we apply this last theorem to the conjunction of 3.14 and 3.15, by modus ponens we obtain the formula $H(\neg A \vee B, A > B)$, which is classically equivalent to

$$H(A \rightarrow B, A > B) \tag{3.16}$$

If we apply now the Lemma 3.10 to 3.16, we obtain as a consequence

$$H(A \rightarrow B) \supset H(A > B)$$

which is equivalent to

$$H(A \rightarrow B) \supset H(A, B)$$

by the Definition 3.8.

$\supset$) We prove this direction by assuming axiom W to be valid. Thus $(A > B) \supset (A \rightarrow B)$ holds. By the following derivation we conclude.

$$\dfrac{\dfrac{\dfrac{}{(A > B) \supset (A \rightarrow B)}\,\mathsf{W}}{H(A > B, A \rightarrow B)}\,\mathsf{H1}}{H(A > B) \supset H(A \rightarrow B)}\,3.10$$

□

Notice that the first formula $H(A, B)$ is equivalent to $H(A > B)$ by 3.8. This result yields then an equivalence of the material and the non-material implication when they are in the scope of high probability operators. Finally, we remark the following about the $\supset$)-direction of the proof above:

Remark 3.10. 1. We have proved that $H(A > B) \supset H(A \rightarrow B)$ with the strong assumption that axiom W is valid. It is not clear in [5] whether Adams himself proceeds this way or not;

2. The application of rule H1 in the proof of $\supset$) does not strictly follow the axiom H1, which deals about "classical tautologies". Anyway, extending the basic language $\mathcal{L}_0^A$ with the conditional operator $>$, it looks natural to extend the rule to the formulas arose from the new axioms introduced to handle the conditional.

To answer the second question at the beginning of this section, we introduce the non-material implication > into Adams' language in the following way. Consider the language $\mathcal{F}_>$, whose formulas are recursively generated by the grammar:

$$A, B ::= p \mid \bot \mid A \,\&\, B \mid A \vee B \mid A \supset B \mid A > B$$

with p in **Prop** and A, B formulas in $\mathcal{F}_>$. We translate then Adams' full language $\mathcal{L}_0^A \cup \mathcal{L}_1^A$ into $\mathcal{F}_>$ via the translation τ, that replaces every high probability formula with the corresponding >-formula, and every connective in $\mathcal{L}_0^A$ with its counterpart in $\mathcal{L}_1^A$. Formally, translation τ is the map

$$\tau \colon \mathcal{L}_0^A \cup \mathcal{L}_1^A \longrightarrow \mathcal{F}_> \tag{3.17}$$

inductively defined as follows:

$\tau(p) := p$

$\tau(f) := \bot$

$\tau(A \wedge B) := \tau(A) \,\&\, \tau(B)$

$\tau(A \vee B) := \tau(A) \vee \tau(B)$

$\tau(A \to B) := \tau(A) \supset \tau(B)$

$\tau(H(A, B)) := \tau(A) > \tau(B)$

$\tau(\bot) := \bot$

$\tau(\varphi \,\&\, \psi) := \tau(\varphi) \,\&\, \tau(\psi)$

$\tau(\varphi \vee \psi) := \tau(\varphi) \vee \tau(\psi)$

$\tau(\varphi \supset \psi) := \tau(\varphi) \supset \tau(\psi)$

for p in **Prop**, A, B in $\mathcal{L}_0^A$ and φ, ψ in $\mathcal{L}_1^A$. Notice that the translation τ is not surjective: in the language $\mathcal{F}_>$ nested applications of the conditional, i.e. conditionals having conditional formulas as arguments, are allowed; the image of τ, on the other hand, corresponds to the flat fragment of $\mathcal{F}_>$, namely the fragment in which the conditional operators > are not nested. This restriction is due to the clause that the high probability operator's arguments are in $\mathcal{L}_0^A$, and thus non-conditional.

Remark 3.11. Under the translation τ, formulas of form (C) get the form of counterfactual (CFC) formulas:

$$(\tau(A_1) > \tau(B_1) \,\&\, \dots \,\&\, \tau(A_n) > \tau(B_n)) \supset (\tau(C_1) > \tau(D_1) \vee \cdots \vee \tau(C_m) > \tau(D_m))$$

For a notational convenience, we will often write

$$\text{(CFC)} \quad (A_1 > B_1 \,\&\, \dots \,\&\, A_n > B_n) \supset (C_1 > D_1 \vee \cdots \vee C_m > D_m)$$

where for any i in $\{1, \dots, n\}$ and for any j in $\{1, \dots, m\}$ we omitted to make explicit the translations via τ of the formulas A_i, B_i, C_j, D_j of $\mathcal{L}_0^A$.

Notice that, given the standard interdefinitions between the conditional > and the comparative plausibility operator $\preccurlyeq$, the language $\mathcal{F}_>$ is equivalent to language $\mathcal{F}_\preccurlyeq$ introduced in Definition 1.13. We will discuss in the next chapter how the translation τ is used to compare Adams' logic **HPA** with the logics of Lewis' $\mathbb{V}$ family.

Chapter 4

Relationship between logics $\mathbb{V}$ and HPA

In this section we discuss the key results of this study: we figure out what is the relationship between Adams' logic for high probability **HPA**, that he presented in [5], and the better-known family of conditional logics that extend system $\mathbb{V}$, introduced by Lewis in [23]. In his paper [5] Adams claims an equivalence between a fragment of his logic **HPA** and its translation in Lewis' $\mathbb{VW}$, and gives a sketch of the proof of this fact. We refine the proof that Adams outlined, giving a complete and more detailed one, and proving in addiction a stronger completeness result, namely that not only $\mathbb{VW}$ is complete with respect to **HPA**, but in fact even Lewis' basic logic $\mathbb{V}$ is. To do this, we exploit at first the family of calculi $\mathcal{I}_{\mathcal{L}}$ from [15], which are built in a modular way to be sound and complete with respect to logic $\mathbb{V}$ and many of its extensions, among which $\mathbb{VW}$. These calculi are the tool that allows us to give an explicit proof of the completeness results. We report then an adaptation to our language of the proof sketched in [5] by Adams, strongly based on Adams' previous works, in order to prove also the other direction of the equivalence between the two fragments mentioned above.

4.1 Completeness of $\mathbb{V}$ wrt HPA

In this section we present the proof of the main result we are concerned with: completeness of the logic $\mathbb{V}$ with respect to Adams' logic **HPA**. As we mentioned in Section 2.2, the sequent calculus $\mathcal{I}_{\mathbb{V}}$ is sound and complete with respect to the logic $\mathbb{V}$; then we will directly prove the completeness of calculus $\mathcal{I}_{\mathbb{V}}$ in order to achieve the completeness of $\mathbb{V}$ as an equivalent condition. The completeness of $\mathbb{VW}$ sketched by Adams in [5] immediately follows from this fact.

We prove the following theorem by directly showing the derivability of the axioms and the admissibility of the rules of inference of the (Hilbert-style) axiomatic system **HPA** in the calculus $\mathcal{I}_{\mathbb{V}}$. To do this we have to make the language between the two logics uniform. We recall first from Definition 1.1 the conditional language $\mathcal{F}_{>}$ defined by the following grammar:

$$A, B ::= p \mid \bot \mid A \,\&\, B \mid A \vee B \mid A \supset B \mid A > B$$

with p in **Prop** and A, B in $\mathcal{F}_{>}$. The translation τ (defined in 3.17) is a map

$$\tau \colon \mathcal{L}_0^A \cup \mathcal{L}_1^A \longrightarrow \mathcal{F}_{>}$$

that actually replaces every logical symbol in $\mathcal{L}_0^A$ with its counterpart in $\mathcal{L}_1^A$ and every high-probability formula with the corresponding $>$-formula. The image of $\mathcal{L}_0^A \cup \mathcal{L}_1^A$ through τ thus is not the whole language $\mathcal{F}_{>}$, but its flat fragment, i.e. its subset in which conditionals are not nested. We remind now that the set of well-formed formulas used in Section 2.2 to describe the internal calculi $\mathcal{I}_{\mathcal{L}}$ for Lewis' logics is the language $\mathcal{F}_{\preccurlyeq}$ introduced in Definition 1.13, which is defined by:

$$A, B ::= p \mid \bot \mid A \,\&\, B \mid A \vee B \mid A \supset B \mid A \preccurlyeq B$$

with p in **Prop** and A, B in $\mathcal{F}_{\preccurlyeq}$. Notice that the language $\mathcal{F}_{\preccurlyeq}$ is suitable to cover the language $\mathcal{F}_{>}$, since $>$ is definable in terms of $\preccurlyeq$ (cf. Definition 1.1) and its addition to $\mathcal{F}_{\preccurlyeq}$ would thus be inessential; then $\mathcal{F}_{>}$ is a subset of $\mathcal{F}_{\preccurlyeq}$ enriched with the conditional $>$. This shows that translation τ links the languages and allows us to state the following completeness result:

Theorem 4.1 (Completeness)**.** If a formula F is valid in **HPA**, then its translation $\tau(F)$ is derivable in $\mathcal{I}_{\mathbb{V}}$.

Proof. We show that, in $\mathcal{I}_{\mathbb{V}}$, the translations via τ of the inference rules of the axiomatic system **HPA** are admissible, and that the translations via τ of the axioms of **HPA** are derivable.

H1:

$$\scriptstyle R> \displaystyle\frac{\scriptstyle L\preccurlyeq \displaystyle\frac{\scriptstyle \text{jump} \displaystyle\frac{\scriptstyle \text{init} \displaystyle\frac{}{A \Rightarrow A, \bot}}{\Rightarrow [A, \bot \lhd A]} \qquad \scriptstyle \text{jump} \displaystyle\frac{\scriptstyle L\&+R\supset \displaystyle\frac{\scriptstyle R\supset-inv. \displaystyle\frac{\scriptstyle \text{Wk} \displaystyle\frac{\Rightarrow A \supset B}{\Rightarrow \bot, A \supset B}}{A \Rightarrow \bot, B} \qquad \scriptstyle L\bot \displaystyle\frac{}{\bot, A \Rightarrow \bot}}{A \,\&\, \neg B \Rightarrow \bot}}{\Rightarrow [\bot \lhd A \,\&\, \neg B]}}{(A \,\&\, \neg B) \preccurlyeq A \Rightarrow [\bot \lhd A]}}{\Rightarrow A > B}$$

The derivations showing the admissibility of the rule H2 and the derivability of the axioms H3 – H6 are reported in Appendix A. □

We report now some comments about the proof above:

Remark 4.2. The derivations in $\mathcal{I}_{\mathbb{V}}$ of the proof of completeness of $\mathcal{I}_{\mathbb{V}}$ with respect to **HPA** can be transformed in derivations in $\mathcal{I}^{\mathrm{i}}_{\mathbb{V}}$ (since the two calculi are equivalent) by actually adding some formulas to the premises of certain rules - the ones that are modified from $\mathcal{I}_{\mathbb{V}}$. Then, thanks to the translation between derivations in $\mathcal{I}^{\mathrm{i}}_{\mathbb{V}}$ and in **G3V**, we can build the corresponding derivations in **G3V**. Hence, equivalent proofs of completeness can be achieved by making use of other sequent calculi for $\mathbb{V}$.

As we mentioned above, Theorem 4.1, together with soundness 2.10 of the calculus $\mathcal{I}_{\mathbb{V}}$ with respect to logic $\mathbb{V}$, immediately entail the following result:

Corollary 4.3. If a formula F is a theorem in **HPA**, then its translation $\tau(F)$ is derivable in $\mathbb{V}$.

Finally, also the following corollary immediately descends from Theorem 4.1:

Corollary 4.4. If a formula F is a theorem in **HPA**, then its translation $\tau(F)$ is derivable in $\mathbb{VW}$.

Proof. Notice that a derivation in $\mathcal{I}_{\mathbb{V}}$ is a derivation also in $\mathcal{I}_{\mathbb{VW}}$. Hence Theorem 4.1 proves that if a formula F is a theorem in **HPA**, then its translation $\tau(F)$ is derivable in $\mathcal{I}_{\mathbb{VW}}$. We conclude thanks to soundness of $\mathcal{I}_{\mathbb{VW}}$ with respect to $\mathbb{VW}$ 2.10. □

This result will be crucial to prove the equivalence of $\mathbb{VW}$ and **HPA** in the next section.

4.2 Equivalence of $\mathbb{VW}$ and HPA

In this section we prove as a main result the equivalence of the fragments of logics $\mathbb{VW}$ and **HPA**. To do this, we mimic the proof that Adams outlined in [5]: he actually proves the equivalence of six conditions among which are the desired theoremhood of a (C)-formula in **HPA** and the derivability of its translation in $\mathbb{VW}$. The completeness of this fragment of $\mathbb{VW}$ with respect to **HPA** is an immediate consequence of the completeness of $\mathbb{V}$, proved in the previous section. For the other direction of the equivalence, instead, it is essential to have logic $\mathbb{VW}$. To do this we proceed similarly to what Adams did in the original paper, presenting in fact a more detailed version of the proof of the equivalence of four of the six conditions mentioned above. We obtain as a by-product also the proof of Theorem 3.6.

We recall from Chapter 3 that a formula of form (C) in the logic **HPA** is:

$$\text{(C)}\quad (H(A_1, B_1)\,\&\, \dots \,\&\, H(A_n, B_n)) \supset (H(C_1, D_1) \vee \cdots \vee H(C_m, D_m))$$

where for any i in $\{1, \dots, n\}$ and for any j in $\{1, \dots, m\}$ A_i, B_i, C_j and D_j are formulas in the zero-level language $\mathcal{L}_0^A$. As mentioned in 3.11 the translation via τ of the (C) formula above is the following counterfactual formula:

$$\text{(CFC)}\quad (A_1 > B_1\,\&\, \dots \,\&\, A_n > B_n) \supset (C_1 > D_1 \vee \cdots \vee C_m > D_m)$$

where for any i in $\{1, \dots, n\}$ and for any j in $\{1, \dots, m\}$ we identified for simplicity the formulas A_i, B_i, C_j, D_j in $\mathcal{L}_0^A$ with their translations via τ.

Finally, we adapt from Chapter 3 the Definition of a truth function 3.4 to any >-formula stipulating that it is the same as its high probability counterpart: for any propositional formulas A, B and any truth function T

$$T(A > B) := T(H(A, B))$$

As an immediate consequence, this allows to extend the Definition of yielding 3.5 to sets of >-formulas, and thus to the conditional version of (C)-formulas: formulas of form (CFC).

We present now p-orderings, defined also in [4] and first introduced in [2]. The notion of p-ordering is involved in the four equivalent conditions of Theorem 4.8: the idea we pursue here, following Adams' strategy, is to make his language as similar as possible to Lewis' language, and including p-orderings of formulas of $\mathcal{L}_0^A$ in our language is what enables us to link Adams' logic **HPA** to Lewis' $\mathbb{V}$ and its extensions.

Definition 4.5. A probability ordering (or ***p*-ordering**) $\preccurlyeq$ is any weak ordering in $\mathcal{L}_0^A$, i.e. a transitive (for any A, B, C in $\mathcal{L}_0^A$, if $A \preccurlyeq B$ and $B \preccurlyeq C$, then $A \preccurlyeq C$), strongly connected (for any A, B in $\mathcal{L}_0^A$, $A \preccurlyeq B$ or $B \preccurlyeq A$) relation in $\mathcal{L}_0^A$, that satisfies also the following conditions. For any formulas A, B in $\mathcal{L}_0^A$:

(1) if A tautologically implies B, then $B \preccurlyeq A$;

(2) $A \preccurlyeq (A \vee B)$ or $B \preccurlyeq (A \vee B)$;

(3) $t \prec f$;

where for any A, B in $\mathcal{L}_0^A$, $A \prec B$ stands for *not* $B \preccurlyeq A$.

This ordering satisfies all the properties of the comparative plausibility defined in [23]. Notice that if we enrich $\mathcal{L}_0^A$ with the p-ordering relational symbol $\preccurlyeq$ (and eventually translate the other connectives of the language via τ in order to have a uniform notation), what we obtain is the fragment of language $\mathcal{F}_{\preccurlyeq}$ from section 1.2 in which $\preccurlyeq$ can only be the main connective. Now we need this language to fit the first-level language of **HPA**: given the formulas A, B in $\mathcal{L}_0^A$ we say that the conditional $A > B$ **holds** in the ordering $\preccurlyeq$ *iff* either $f \preccurlyeq A$ or $(A \,\&\, B) \prec (A \,\&\, \neg B)$. This coincides with the definition of the conditional in terms of the comparative plausibility operator:

Remark 4.6 (Comparative plausibility and p-orderings)**.** Definition 4.5 above coincides with the one of comparative plausibility given by Lewis in [23]; this brings the probabilistic and conditional logics we are studying closer. We remark that the definition of p-ordering given by Adams is the converse of the one above, and for that definition Adams himself highlights in [5] that *normal possibility orderings*, namely comparative plausibility orderings satisfying the normality condition N, are the converse of his p-orderings. We decided to switch the p-ordering symbol so that it effectively coincides with the comparative plausibility operator. This also allows us to have a semantic interpretation for $\preccurlyeq$ and for $>$ too, since its definition in terms of p-orderings is the same as the one by Lewis.

We internalize now the p-ordering defined above in the structure of our language: assume that the zero-level language is generated from a non-zero, finite number k of propositional variables $p_1, \dots, p_k$. The atoms are then arranged, for any p-ordering $\preccurlyeq$, on levels according to their probability as done in [4]. Since k is finite, suppose that there are s levels in total. Atoms at level 1 are the ones equivalent to $\top$ in the ordering; we suppose that there is at least an atom at this level. An atom p_i is at level 2 if and only if $\top \prec p_i$ and for no atom p_j, $\top \prec p_j \prec p_i$. The idea is that atoms at level 1 are the most probable (or plausible) ones, atoms at level 2 are less probable than just atoms at level 1 with no space between levels. In the end, at every level there are the atoms that are less probable than and still the nearest to the ones at the previous levels, until we meet the less probable atoms at level s. For any $r \in \{1, \dots, s\}$ we call $k(r)$ the number of atoms at level r; then $k = k(1) + \cdots + k(s)$.

Consider now a formula C of form (C) and its translation via τ, C', of form (CFC). C' can be seen also as an inference in which the premiss set is $X = \{A_1 > B_1, \dots, A_n > B_n\}$ and the conclusion set is $Y = \{C_1 > D_1, \dots, C_m > D_m\}$.

Definition 4.7. Given a formula C of form (C) and C', X, Y as above, a p-ordering is said to be **counterexample** to the inference C *iff* all members of X hold in it, while none of the members of Y does.

We have described now all the necessary elements to state the equivalence of the four conditions mentioned above, that allows to prove not only the equivalence between logics $\mathbb{VW}$ and **HPA**, but also Theorem 3.6. The proof strongly relies on the notions presented above and hence on Adams' previous works, as well as on Lewis' sphere semantics. We report here our proof of the implication from the condition **H** to the condition **VW**, and a formalization of Adams' proof of the implication from **VW** to **C**. For the proof of the implications "**C** *implies* **D**" and "**D** *implies* **H**", not involving Lewis' logic $\mathbb{VW}$, we relate to the proof given by Adams in the appendix of [5], that is based on notions introduced in various previous works, and in particular similar proofs from [4].

Theorem 4.8. The following four conditions are equivalent:

C. There is no p-ordering counterexample to C';

D. A subset of X yields a subset of Y;

H. C is a theorem in **HPA**;

VW. C' is valid in $\mathbb{VW}$.

Proof. **H** *implies* **VW**: Follows from Corollary 4.4, since the translation via τ of the (C)-formula C is the (CFC)-formula C'.

VW *implies* **C**: We report the proof proposed by Adams, adapting the notation to the present work.

We prove that if **C** is false, then also **VW** is. Assume then that there is a p-ordering $\preccurlyeq$ counterexample to C. To prove that **VW** does not hold, we build a weak-centered sphere model $\mathcal{M} = \langle W, S, [\![\ ]\!]\rangle$ in which the formula C' is not valid. We can suppose without loss of generality that the zero-level language is built upon a finite number of atoms, since we can consider just the atoms involved in C: the ones not involved are negligible. Let these atoms be called $p_1, \ldots, p_k$ and arrange them in levels of probability as explained above. Let the positive number s in $\mathbb{N}$ be the number of levels. Finally, let $\mathcal{S}_r$ be the set of the atoms of level at most r.

We can interpret the atoms as possible worlds:

$$W \coloneqq \{p_1, \ldots, p_k\}$$

The sphere system mentioned above is defined in the following way: for any $i \in \{1, \ldots, k\}$, the sphere system corresponding to the world p_i at level r is the collection of nested sets

$$S(p_i) \coloneqq \{\mathcal{S}_j | j \in \{r, \ldots, s\}\}$$

Notice that for any i, the sphere system $S(p_i))$ is weakly centered on the world p_i. Focusing on p_1, starting from the hypothesis that $\preccurlyeq$ is a counterexample to C, it is routine to show that, according to Lewis' definition of truth at a possible world, all of the formulas $A_1 > B_1, \ldots, A_n > B_n$ are true at p_1, while none of $C_1 > D_1, \ldots, C_m > D_m$ is. Therefore the counterfactual formula C' corresponding to C is not valid in $\mathbb{VW}$. □

The two additional conditions included by Adams in the Appendix of [5] and proved equivalent to the four conditions of Theorem 4.8 prove results about the validity of (C)-formulas through the evaluation of improbability functions, given in the same paper and left out of this work. We refer to the original paper for the details of these results and a sketch of the proof of the equivalence of these two more conditions to the four of Theorem 4.8. We observe that in the proof of "**C** *implies* **D**" the clause (3) of the Definition 4.5, representing the normality of the p-ordering, is essential.

We conclude this section with an important remark about Adams' proof of "**VW** *implies* **C**", that we have adapted to our notation and explicited as much as possible above.

Remark 4.9. In his proof of Theorem 4.8, in [5], Adams does not make explicit how the weakened sphere model that he mentions is defined: in fact, he does not specify what is the assignment of the evaluation $[\![\ ]\!]$ on the atoms, and to us this is a factor of considerable relevance. Also, we noticed that it is not trivial to establish what is the evaluation function to which Adams relates.

We came back to the previous works [4] and [23], on which Adams relies throughout all the proof. Our idea is to exploit *truth assignments* as defined at p.154 of [4], and build a sphere model in which the worlds are truth assignments, or sets of propositional variables. This would imitate the construction of the proof of "*Deriving (D) from (C)*" of [4], p.173. Moreover, p-orderings as we formulated them are equivalent to Lewis' comparative possibility systems ([23], sec. 2.5), as Adams himself claims. Lewis shows a way of building a sphere model starting from a comparative possibility system, and shows that his construction preserves the satisfiability of formulas.

Combining these two notions, our idea is to consider the finite set W of truth assignments over the propositional variables, and to build a comparative possibility system on W. Finally, taking advantage of Lewis' construction, build a sphere model corresponding to the p-ordering counterexample for the formula C: since Lewis' construction preserves the satisfiability of formulas, the sphere model thus designed would be a countermodel for the formula C'.

We stress that the proof we outlined here is not the construction of Adams' proof; nevertheless, in our view this is a clearer and more explicit attempt at proving the result in exam.

Having provided an explicit proof of one of the directions involving the condition **VW**, namely the completeness of $\mathbb{VW}$ with respect to **HPA**, and outlined a sketch of the proof of the other direction of our main interest ("**VW** *implies* **C**"), we leave the details of this proof to future work.

Conclusions

In this thesis we have explored two approaches for conditional logics: Lewis' possible worlds account and the family of conditional logics extending $\mathbb{V}$ on the one hand, and Adams' probabilistic account together with his logic **HPA** for high probability on the other hand. We have examined both semantic and syntactic aspects of these logics and focused on already existing proof systems for Lewis' logics, to finally investigate the interconnections between the extensions of $\mathbb{V}$ and **HPA**.

Through our study, we have formalized and clarified Adams' theory of high probability logic **HPA** by giving explicit proofs of the main results claimed by Adams.

Additionally, we have demonstrated the completeness of Lewis' logic $\mathbb{V}$ with respect to **HPA**, showing that every theorem in **HPA** can be translated into a derivable formula in $\mathbb{V}$. Furthermore, we have highlighted the equivalence of a significant fragment of $\mathbb{VW}$ and **HPA**, revisiting and elaborating on Adams' original sketch of proof: one direction is entailed by the previous completeness result; for the other way round we have proposed an alternative proof strategy than the one outlined by Adams, lacking relevant details.

Despite the significant progress made in this work, several open questions remain. Future research could extend the rigorous formulation of Adams' theory, in particular completing a formalization of the steps of the proof of soundness of the logic $\mathbb{VW}$ with respect to the fragment of formulas of form (C) of **HPA**: in this regard, an extensive account of the proof strategy we have proposed in Remark 4.9 needs to be worked out in detail.

Additionally, since we proved completeness of $\mathbb{V}$, and some kind of soundness of $\mathbb{VW}$ with respect to **HPA** has been established by Adams himself, we remark the non-symmetric connection between the family of logics of $\mathbb{V}$ and **HPA**. An interesting developement of this study would be determining if a weakened version of Theorem 4.8 is provable, in order to establish a connection between **HPA** and some of Levis' logics that are between $\mathbb{V}$ and $\mathbb{VW}$.

Finally, in the proof-theoretic framework it would be interesting to strengthen the construction of translations between the various sequent calculi, like the maps between derivations defined in [16] and the ones proposed in Remark 2.6. Such maps allow to move from one calculus to another, making it possible to take advantage of the one that better fits a certain context, especially in explicit uses of the calculus like the proof of 4.1.

In conclusion, this thesis contributes to the formal study of conditional logics by bridging the gap between the well-established logics of Lewis and Adams' probabilistic approaches. By determining a clearer connection between Lewis' and Adams' frameworks, we have laid the groundwork for further advancements in the logic of conditionals and high probability reasoning.

Appendix A

Appendix: Derivations of the axioms

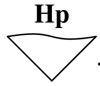
Proof of Theorem 4.1: We report here the derivations that show the admissibility of the rule H2 and the derivability of the axioms H3 – H6.

In the derivations we adopt the following notation: given rules R_1, R_2, R of the sequent calculus $\mathcal{I}_{\mathbb{V}}$ we denote by $R_1 + R_2$ the ordered application of the rule R_1 followed by the rule R_2 bottom-up, displayed in a single inference step; analogously, by R^* we denote multiple applications of the rule R. For an invertible rule R we write $R - inv.$ for its inversion. Finally, (generalized) instances of the rule init are considered leaves; we mark such instances, as well as the other leaves of the derivations, with the symbol

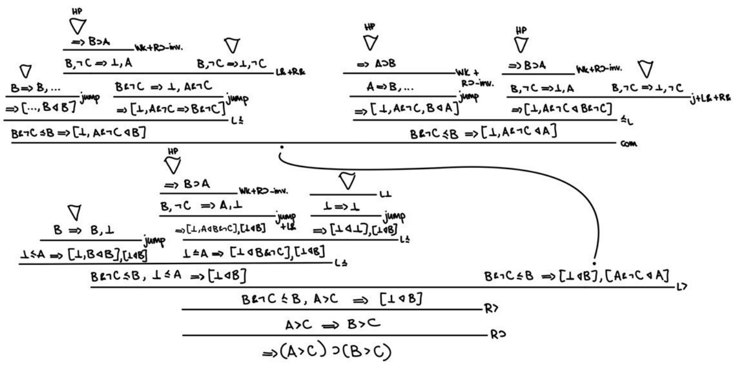
.

In the derivation of H2, the leaves in which the premiss of the rule H2 is reported are marked with

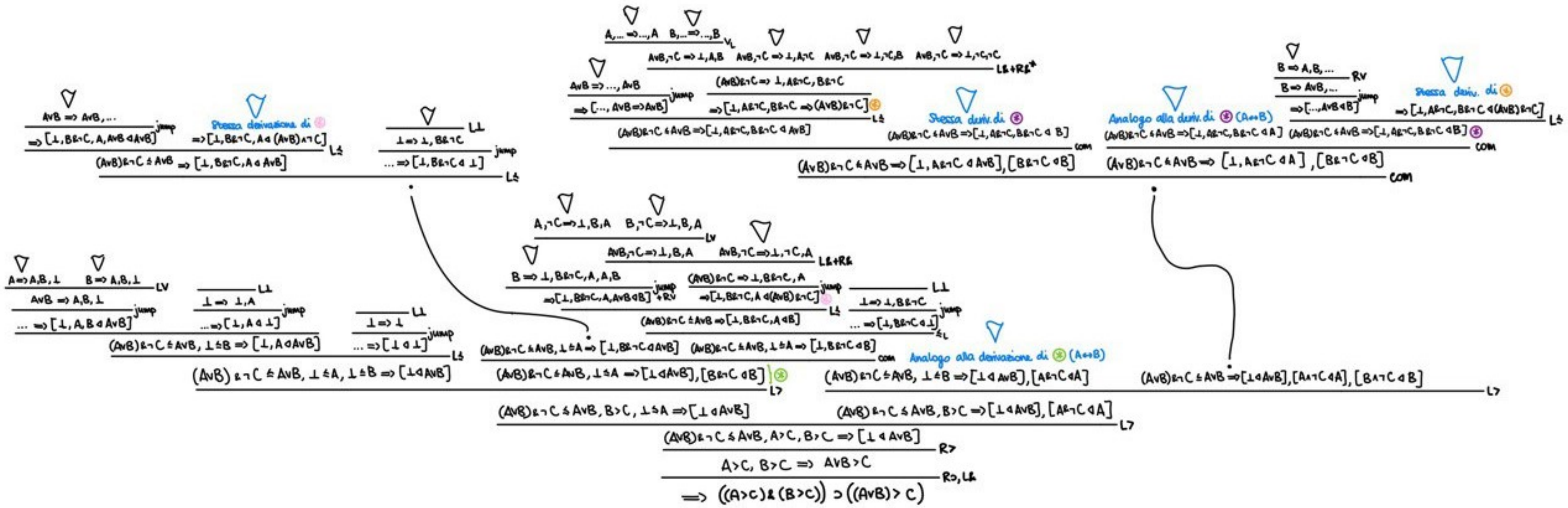
Hp

.

H2:

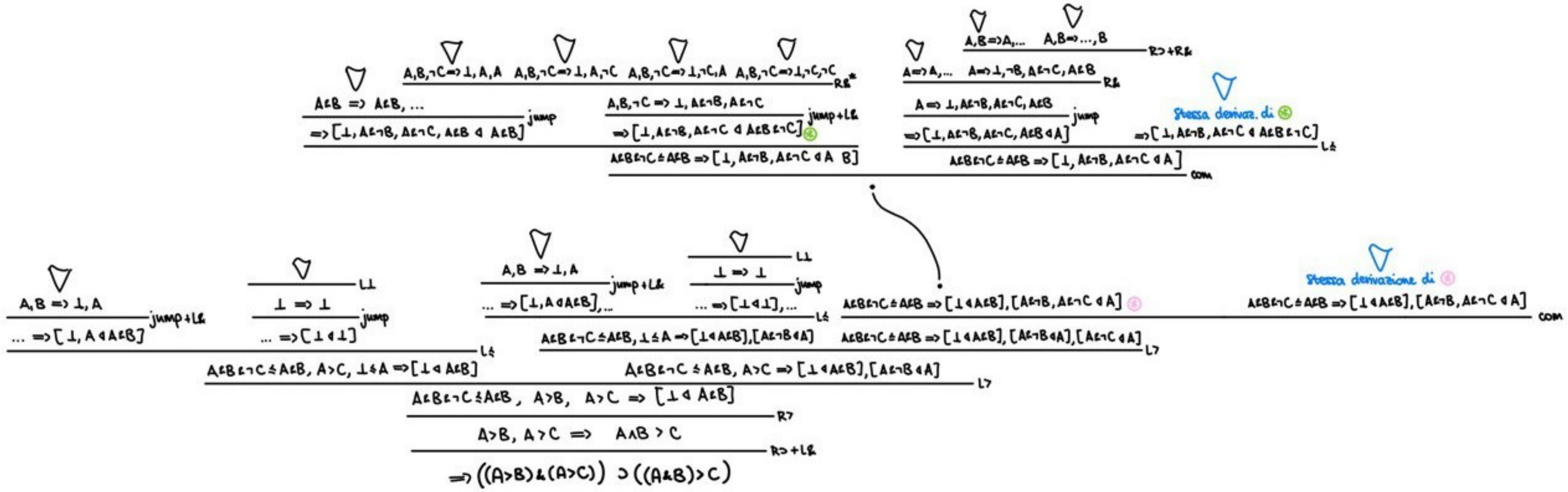


H3:

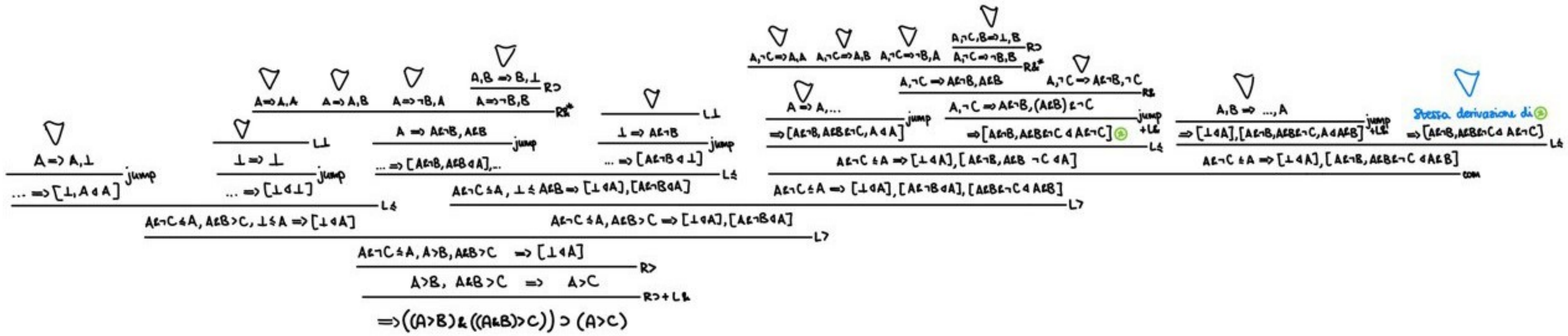


A>C, B>C ⟹ AvB>C
⟹ ((A>C)&(B>C)) ⊃ ((AvB)>C)
R>
R⊃, L&
Stessa derivazione di
Stessa deriv. di
Analogo alla deriv. di
Analogo alla derivazione di
jump
com
L>

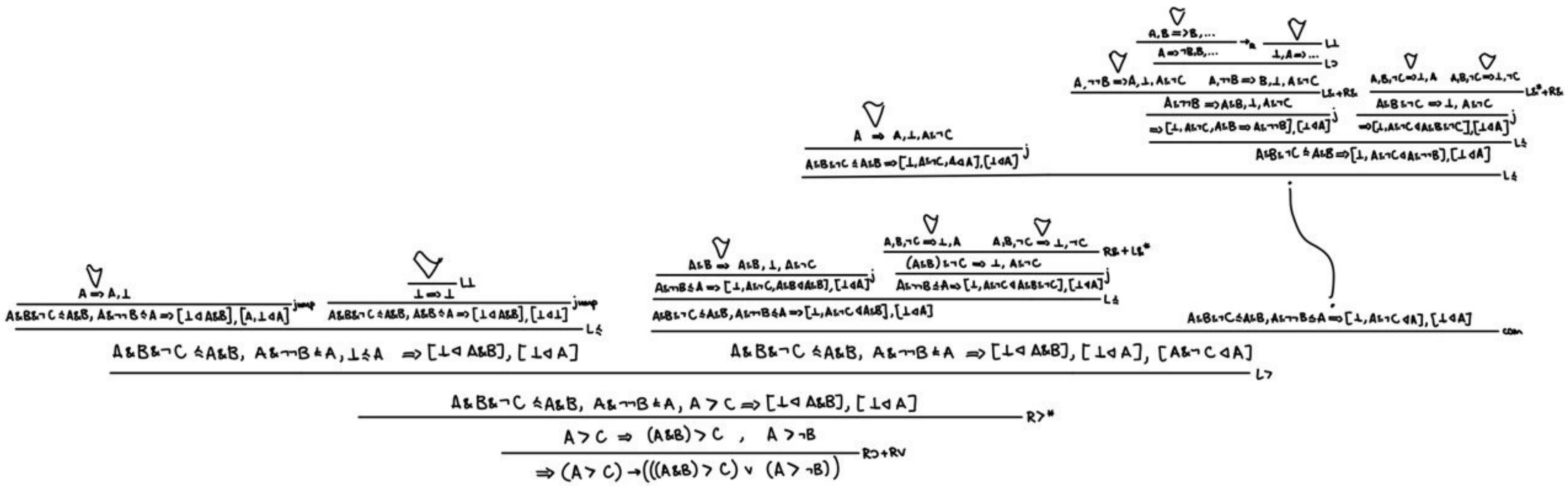
H4:



H5:



H6:



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