Scuola di Scienze Matematiche, Fisiche e Naturali

Dipartimento di Matematica

Tesi di Laurea Magistrale

Fibrations with comprehensions and their completions

CANDIDATO:
Andrea Giusto

RELATORE:
Prof. Jacopo Emmenegger

CORRELATORE:
Prof. Francesco Dagnino

ANNO ACCADEMICO 2023–24

Contents

Introduction **1**

1 Preliminary concepts **3**
1.1 2-categories in a nutshell 3
1.2 Fibrations: definitions and basic properties 11
1.3 Fibred products and terminal objects 20
1.4 Fibred product completion 22

2 Lawvere comprehensions **25**
2.1 Fibrations with Lawvere comprehension 25
2.2 The free fibration with Lawvere comprehension 29

3 Jacobs comprehension **38**
3.1 Comprehension categories: definition and first properties 38
3.2 From Lawvere to Jacobs 40
3.3 The free comprehension category over a fibration 44
3.3.1 Towards the bi-adjunction 48

Conclusions **56**

Introduction

The notion of comprehension is central in Mathematics and in Logic. Informally, it allows us to add hypothesis or structure to the objects we are studying. In fact, in set theory the Axiom Schema of Comprehension can be expressed in the following way: given a well-formed formula ϕ with one free variable and given a set X there exist a set whose elements are exactly the elements of X that satisfy ϕ. Formally it is expressed by the following formula:

$$\forall_X \exists_Y \forall_z (z \in Y \iff (z \in X \wedge \phi(z)))$$

If one thinks of a set X as a model of a theory, then comprehension gives the possibility to consider a model of the theory extended with ϕ, namely the subset $S \subseteq X$ obtained by comprehension. Studying the relations between the comprehensions of different formulas provides information about the relation between formulas themselves. In particular, one is interested into studying logical consequences.

Analogously to the set-theoretical case, one can consider comprehension from a type-theoretical perspective. In this scenario adding an hypothesis corresponds to extend a context: given a judgement of type $\Gamma \vdash \sigma\colon \mathrm{Type}$ the context extension rule yields the extended context $\Gamma, x\colon \sigma$. This is substantially a proof-relevant version of the previous case: we can study not only logical consequences, but also their proofs.

This thesis analyzes two different kinds of comprehension structures, namely Lawvere comprehension and Jacobs comprehension, through the lens of fibered category theory. A fibered category, also known as Grothendieck fibration, consists of a family of categories indexed by a category, packaged into a (strict) functor $p\colon \mathcal{E} \to \mathcal{B}$. From the point of view of logic, one can see the base category $\mathcal{B}$ of p as the objects they want to study together with their transformations. Each fiber $\mathcal{E}_X = p^{-1}(X)$ over an object X of $\mathcal{B}$ consists of the properties of X (the predicates) together with the order induced by logical consequences. The crucial property of fibrations, with respect to functors, is that each transformation in $\mathcal{B}$ induces a so-called reindexing functor between the fibers which, from the point of view of logic, performs a substitution into each predicate.

Lawvere's notion of comprehension arises in the study of hyperdoctrines, which he studied in his work [11]. These correspond to faithful fibrations with additional properties, so one can generalize it to non-faithful fibrations. They represent a suitable framework in which one can interpret logical theories thanks to the observation, also due to Lawvere, that logical operators can be interpreted as suitable adjoint functors. Theories in first order logic give rise to faithful fibrations: arrows in the total category represent the consequence relation determined by the theory.

On the other hand, Jacobs' notion of comprehension arise to give a framework in which one can model dependent type theories. These are particularly useful both in Theoretical Computer Science and in Mathematics: for example several interactive theorem provers (including Coq and Lean) and some functional programming languages (such as Agda and Idris) are based on them.

It is well known that one can model simple type theories, i.e. type theories in which types do not depend on variables of any kind, inside cartesian closed categories. Dependent type theories, instead, allow types to depend on term variables: for example, one can form the type $\mathrm{vec}(n)$ of vectors of length n, where $n\colon \mathbb{N}$. In [8] Jacobs examines different ways to interpret simple, dependent and polymorphic type theories inside specific kinds of fibered categories. In particular, he defines comprehension categories specifically to model dependent type theories. These fibrations are usually far from being faithful. This reflects the type-theoretic concepts of "propositions-as-types" and "proofs-as-terms": in general, there are different possible proofs for the same proposition, as there are different possible terms of a given type.

Our aim is first of all to describe these different kinds of comprehension from a 2-categorical perspective. Later on, we get to the main results of the thesis by describing how to freely add both types of comprehensions. Here it appears crucial the 2-categorical environment: the two constructions that we provide are described by two 2-functors that give rise to two bi-adjunctions which are (in general) not strict. These universal constructions are important for two different reasons: first of all, they enlighten the algebraic nature of both kinds of comprehension. Secondly, they provide a standard way to build models, for example of logical theories or of dependent type theories. These completions are part of our original contribution.

In the first chapter we recall some basic notions of 2-category theory and fibred category theory. We often make use of the facts and constructions given in this chapter, that may be considered as a container for the prerequisites one needs on 2-categories and fibered categories in order to understand the main results.

Then, in the second chapter we investigate Lawvere comprehension. First we give a characterization of those fibrations with Lawvere comprehension that are faithful. Afterwards we get to the main result of the chapter, namely the completion for Lawvere comprehension. In particular, we build the free fibration with Lawvere comprehension and finite fibred products starting from a fibration with finite fibred products, and prove its universality.

Finally, the third chapter is focused on comprehension categories and their relation to fibrations with Lawvere comprehension. In fact there we show that Lawvere comprehension implies Jacobs comprehension, and we characterize those comprehension categories that are fibrations with Lawvere comprehension. This characterization is the other part of our original contribution. In the end we also build the free comprehension category over an arbitrary fibration, and prove its universality.

Chapter 1

Preliminary concepts

In this chapter we set up the stage for the subsequent developments by recalling some necessary preliminary notions and fixing the notations used throughout this thesis. More in detail, in Section 1.1 we recall some definitions about 2-category theory. Section 1.2 introduces the main character of this thesis, namely, fibrations (Definition 1.2.3). Then, in Section 1.3 we define fibrations with finite fibred products (Definition 1.3.1) and we provide some examples. Finally, in Section 1.4 we give the completion for finite fibred product of a fibration (Theorem 1.4.5). We refer the reader to [2, 9, 10, 13, 14] for the definitions and results appearing in this chapter.

Notation. Given a category $\mathcal{C}$, we refer to the class of its object as $|\mathcal{C}|$, and to the class of morphisms between objects A and B with $\mathcal{C}(A, B)$. We denote by $\mathbf{1}$ the terminal category, i.e. the category with a single object and only the identity on it. Also, we write $\mathcal{C} \times \mathcal{D}$ for the product of the categories $\mathcal{C}$ and $\mathcal{D}$, whose objects and arrows are pairs of object and arrows, respectively, of $\mathcal{C}$ and $\mathcal{D}$.

1.1 2-categories in a nutshell

In this section we briefly recall basic notions from 2-dimensional category theory we will use throughout this thesis. We are particularly interested in the definition of bi-adjunctions (see Definition 1.1.17), that will allow us to speak with a 2-categorical approach about free constructions. We refer the reader to [2] for a detailed introduction to the subject.

The notion of 2-category arises to describe a category in which the collection of morphisms between a fixed pair of objects forms a category itself. This abstracts the essential structure of the "category of categories", which gives rise to the paradigmatic example of a 2-category. Indeed, roughly, fixing two categories $\mathcal{C}$ and $\mathcal{D}$, the morphisms between them are organized in a category in which objects are functors and arrows are natural transformations.

Definition 1.1.1. A 2-category $\mathbf{C}$ consists of:

- a class $|\mathbf{C}|$, whose elements are said 0-cells;
- for each $A, B \in |\mathbf{C}|$, a category $\mathbf{C}(A, B)$, whose objects are called 1-cells (or arrows) and whose arrows are called 2-cells;

- for each $A, B, C \in |\mathbf{C}|$, a functor

$$c_{ABC}: \mathbf{C}(A,B) \times \mathbf{C}(B,C) \to \mathbf{C}(A,C);$$

- for each $A \in |\mathbf{C}|$, a functor $u_A: \mathbf{1} \to \mathbf{C}(A,A)$;

These data have to satisfy the following conditions:

- **Associativity:** given $A, B, C, D \in |\mathbf{C}|$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}(A,B) \times \mathbf{C}(B,C) \times \mathbf{C}(C,D) \arrow[r, "\mathrm{Id}_{\mathbf{C}(A,B)} \times c_{BCD}"] \arrow[d, "c_{ABC} \times \mathrm{Id}_{\mathbf{C}(C,D)}"'] & \mathbf{C}(A,B) \times \mathbf{C}(B.D) \arrow[d, "c_{ABD}"] \\
\mathbf{C}(A,C) \times \mathbf{C}(C,D) \arrow[r, "c_{ACD}"'] & \mathbf{C}(A,D)
\end{tikzcd}$$

- **Identity:** given $A, B \in |\mathbf{C}|$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{1} \times \mathbf{C}(A,B) \arrow[d, "u_A \times \mathrm{Id}_{\mathbf{C}(A,B)}"'] & \mathbf{C}(A,B) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\mathrm{Id}_{\mathbf{C}(A,B)}"'] & \mathbf{C}(A,B) \times \mathbf{1} \arrow[d, "\mathrm{Id}_{\mathbf{C}(A,B)} \times u_B"] \\
\mathbf{C}(A,A) \times \mathbf{C}(A,B) \arrow[r, "c_{AAB}"] & \mathbf{C}(A,B) & \mathbf{C}(A,B) \times \mathbf{C}(B,B) \arrow[l, "c_{ABB}"']
\end{tikzcd}$$

Remark 1.1.2. In a 2-category $\mathbf{C}$ there are at least two different ways to compose 2-cells. Given 1-cells f, g, h from A to B and 2-cells $\alpha: f \Rightarrow g$ and $\beta: g \Rightarrow h$ we can define the composition $\beta \circ \alpha: f \Rightarrow h$ as their composition inside the category $\mathbf{C}(A,B)$. This is called vertical composition.

Instead, given 1-cells f, g from A to B and f', g' from B to C and 2-cells $\alpha: f \Rightarrow g$, $\beta: f' \Rightarrow g'$, we can define the composition $\beta * \alpha: f' \circ f \Rightarrow g' \circ g$ as the action of the composition functor c_{ABC} on them. This is called horizontal composition.

Horizontal and vertical compositions satisfy an interchange law following by functoriality of the composition functor c_{ABC}: given the diagram

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f_1"] \arrow[r, "f_2" description] \arrow[r, bend right=50, "f_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha_1"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha_2"] & B \arrow[r, bend left=50, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=50, "g_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta_1"] \arrow[r, phantom, shift right=3, "\Downarrow\beta_2"] & C
\end{tikzcd}$$

one has $(\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1) = (\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1)$.

Furthermore, there is a way to "compose" 1-cells and 2-cells. Given 1-cells $f: A \to B$ and $g, h: B \to C$ and a 2-cell $\alpha: g \Rightarrow h$, we define the 2-cell $\alpha f: g \circ f \Rightarrow h \circ f$ as the horizontal composition $\alpha * \mathrm{i}_f$, where with i_f we denote the identity 2-cell on the 1-cell f. Analogously, given 1-cells $g, h: A \to B$ and $f: B \to C$ and a 2-cell $\alpha: g \Rightarrow h$, we define the 2-cell $f\alpha: g \circ f \Rightarrow h \circ f$ as the horizontal composition $\mathrm{i}_f * \alpha$. These operations are called whiskering of α with f.

Example 1.1.3. Any standard category can be regarded as a particular 2-category. Indeed, given a category $\mathcal{C}$, one can define the so-called **2-discrete** 2-category $\mathbf{Disc}(\mathcal{C})$ over it in the following way: 0-cells are the objects of $\mathcal{C}$, 1-cells are its arrows and the 2-cells are only the identities. Note that in a 2-discrete 2-category all compositions involving 2-cells become trivial.

Example 1.1.4. The paradigmatic example of a 2-category is **Cat**, where 0-cells are the categories, 1-cells are functors and the 2-cells are the natural transformations. The vertical composition of natural transformations is as follows: given functors F, G, H from $\mathcal{C}$ to $\mathcal{D}$ and two natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ we define the composition $\beta \circ \alpha\colon F \Rightarrow H$ as the family $\{\beta_c \circ \alpha_c\}_{c\in|\mathcal{C}|}$.

Instead, the horizontal composition of natural transformations is as follows: given functors F, G from $\mathcal{C}$ to $\mathcal{D}$ and F', G' from $\mathcal{D}$ to $\mathcal{E}$ and natural transformations $\alpha\colon F \Rightarrow G$, $\beta\colon F' \Rightarrow G'$, we define the composition $\beta * \alpha\colon F' \circ F \Rightarrow G' \circ G$ as the family $\{\beta_{Gc} \circ F'\alpha_c\}_{c\in|\mathcal{C}|}$, or equivalently (by naturality of β) as the family $\{G'\alpha_c \circ \beta_{Fc}\}_{c\in|\mathcal{C}|}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=40, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=40, "H"'] \arrow[rr, phantom, bend left=20, "\alpha\Downarrow"] \arrow[rr, phantom, bend right=20, "\beta\Downarrow"] & & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

$$\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[rr, bend left, "\beta_c\circ\alpha_c"] & Gc \arrow[r, "\beta_c"] & Hc
\end{tikzcd}
\qquad
\begin{tikzcd}
F'Fc \arrow[r, "F'\alpha_c"] \arrow[d, "\beta_{Fc}"'] & F'Gc \arrow[d, "\beta_{Gc}"] \\
G'Fc \arrow[r, "G'\alpha_c"'] & G'Gc
\end{tikzcd}$$

The composition functor act as the composition on 1-cells and as horizontal composition on 2-cells, while the composition between 2-cells internal to $\mathbf{Cat}(\mathcal{A}, \mathcal{B})$ is the vertical composition.

Definition 1.1.5. Let $F \dashv G$ and $F' \dashv G'$ be adjunctions, with $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{C}' \to \mathcal{D}'$. Consider also two functors $A\colon \mathcal{C} \to \mathcal{C}'$ and $B\colon \mathcal{D} \to \mathcal{D}'$ together with a 2-cell $\omega\colon F' \circ A \Rightarrow B \circ F$. The **mate** of ω is the 2-cell $\omega^{\#}\colon A \circ G \Rightarrow G' \circ B$ defined as follows: $\omega^{\#} := (G'B\epsilon) \circ (G'\omega G) \circ (\eta' AG)$, where ϵ is the counit of $F \vdash G$ and η' is the unit of $F' \vdash G'$.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "G"] \arrow[dr, "\mathrm{Id}_{\mathcal{D}}"'] & \mathcal{C} \arrow[r, "A"] \arrow[d, "F"] \arrow[dl, phantom, "\overset{\epsilon}{\Leftarrow}"] & \mathcal{C}' \arrow[d, "F'"] \arrow[dr, "\mathrm{Id}_{\mathcal{C}'}"] \arrow[dl, phantom, "\omega"] & \\
& \mathcal{D} \arrow[r, "B"'] & \mathcal{D}' \arrow[r, "G'"'] \arrow[ur, phantom, "\eta'\Downarrow"] & \mathcal{C}'
\end{tikzcd}$$

Proposition 1.1.6. *Let $F \dashv G$ and $F' \dashv G'$ be adjunctions, with $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{C}' \to \mathcal{D}'$. Consider also two functors $A\colon \mathcal{C} \to \mathcal{C}'$ and $B\colon \mathcal{D} \to \mathcal{D}'$ together with a natural transformation $\omega\colon F' \circ A \Rightarrow B \circ F$ and its mate $\omega^{\#}$. Then the following square commutes:*

$$\begin{tikzcd}
F'AG \arrow[r, Rightarrow, "\omega G"] \arrow[d, Rightarrow, "F'\omega^{\#}"'] & BFG \arrow[d, Rightarrow, "B\epsilon"] \\
F'G'B \arrow[r, Rightarrow, "\epsilon' B"'] & B
\end{tikzcd}$$

Proof. See [10]. □

Proposition 1.1.7. *Let $F \dashv G$, $F' \dashv G'$ and $F'' \dashv G''$ be adjunctions, with $F\colon \mathcal{C} \to \mathcal{D}$, $F'\colon \mathcal{C}' \to \mathcal{D}'$ and $F''\colon \mathcal{C}'' \to \mathcal{D}''$. Consider also four functors*

$A\colon \mathcal{C} \to \mathcal{C}'$, $B\colon \mathcal{D} \to \mathcal{D}'$, $C\colon \mathcal{C}' \to \mathcal{C}''$ and $D\colon \mathcal{D}' \to \mathcal{D}''$, together with two natural transformations $\omega\colon F' \circ A \Rightarrow B \circ F$ and $\sigma\colon F'' \circ C \Rightarrow D \circ F'$. Then $(D\omega \circ \sigma A)^{\#} = (\sigma^{\#} B) \circ (C\omega^{\#})$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "A"] \arrow[d, "F"'] & \mathcal{C}' \arrow[r, "C"] \arrow[d, "F'"] \arrow[dl, phantom, "\omega"] & \mathcal{C}'' \arrow[d, "F''"] \arrow[dl, phantom, "\sigma"] \\
\mathcal{D} \arrow[r, "B"'] & \mathcal{D}' \arrow[r, "D"'] & \mathcal{D}''
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "C\circ A"] \arrow[d, "F"'] & & \mathcal{C}'' \arrow[d, "F''"] \\
\mathcal{D} \arrow[rr, "D\circ B"'] \arrow[r, phantom, "D\omega\circ\sigma A"] & & \mathcal{D}''
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "A"] \arrow[dr, phantom, "\omega^{\#}"] & \mathcal{C}' \arrow[r, "C"] \arrow[dr, phantom, "\sigma^{\#}"] & \mathcal{C}'' \\
\mathcal{D} \arrow[r, "B"'] \arrow[u, "G"] & \mathcal{D}' \arrow[r, "D"'] \arrow[u, "G'"'] & \mathcal{D}'' \arrow[u, "G''"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "C\circ A"] & & \mathcal{C}'' \\
\mathcal{D} \arrow[rr, "D\circ B"'] \arrow[u, "G"] \arrow[urr, phantom, "(\sigma^{\#}B)\circ(C\omega^{\#})"] & & \mathcal{D}'' \arrow[u, "G''"']
\end{tikzcd}$$

Proof. See [3]. □

Proposition 1.1.8. *Let $F \dashv G$ and $F' \dashv G'$ be adjunctions, with $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{C}' \to \mathcal{D}'$. Consider also functors $A, C\colon \mathcal{C} \to \mathcal{C}'$ and $B, D\colon \mathcal{D} \to \mathcal{D}'$ together with natural transformations $\omega\colon F' \circ A \Rightarrow B \circ F$ and $\rho\colon F' \circ C \Rightarrow D \circ F$. Finally, take also natural transformations $\alpha\colon A \Rightarrow C$ and $\beta\colon B \Rightarrow D$. Then the left-hand square below commutes if and only if the right-hand square commutes.*

$$\begin{tikzcd}
F'A \arrow[r, Rightarrow, "\omega"] \arrow[d, Rightarrow, "F'\alpha"'] & BF \arrow[d, Rightarrow, "\beta F"] \\
F'C \arrow[r, Rightarrow, "\rho"'] & DF
\end{tikzcd}
\qquad
\begin{tikzcd}
AG \arrow[r, Rightarrow, "\omega^{\#}"] \arrow[d, Rightarrow, "\alpha G"'] & G'B \arrow[d, Rightarrow, "G'\beta"] \\
CG \arrow[r, Rightarrow, "\rho^{\#}"'] & G'D
\end{tikzcd}$$

Proof. Suppose that the left-hand square commutes. By definition of mate, the right-hand square above is the following composition

$$\begin{tikzcd}
AG \arrow[r, Rightarrow, "\eta' AG"] \arrow[d, Rightarrow, "\alpha G"'] & G'F'AG \arrow[r, Rightarrow, "G'\omega G"] \arrow[d, Rightarrow, "G'F'\alpha G"'] & G'BFG \arrow[r, Rightarrow, "G'B\epsilon"] \arrow[d, Rightarrow, "G'\beta FG"] & G'B \arrow[d, Rightarrow, "G'\beta"] \\
CG \arrow[r, Rightarrow, "\eta' CG"'] & G'F'CG \arrow[r, Rightarrow, "G'\rho G"'] & G'DFG \arrow[r, Rightarrow, "G'D\epsilon"'] & G'D
\end{tikzcd}$$

The two lateral squares commute since they are whiskerings of naturality squares. The central one commutes since it is a whiskering of a commutative one.

The converse follows from the fact that $(\omega^{\#})^{\#} = \omega$. □

Example 1.1.9. Analogously to the one dimensional case, given a 2-category $\mathbf{C}$, one can define the 2-category $\mathbf{C}^2$ in which 0-cells are 1-cells of $\mathbf{C}$, 1-cells are pairs of 1-cells in $\mathbf{C}$ such that the appropriate square commutes, and the 2-cells are pairs of 2-cells satisfying some coherence condition. More precisely, given the diagram below, one requires that $g \circ h = k \circ f$ (also $g \circ h' = k' \circ f$) and

$\beta f = g\alpha$.

$$\begin{tikzcd}
A \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\alpha\Downarrow"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, bend left, "k"] \arrow[r, bend right, "k'"'] \arrow[r, phantom, "\beta\Downarrow"] & D
\end{tikzcd}$$

As an instance we recover the 2-category of functors $\mathbf{Cat}^2$. A 1-cell F is a pair of functors $F = (\overline{F}, \widehat{F})$, where the first component is between the domains of the 0-cells and the latter is between the codomains. A 2-cell $\alpha: F \Rightarrow G$ is a pair of natural transformations $\alpha = (\overline{\alpha}, \widehat{\alpha})$ where $\overline{\alpha}: \overline{F} \Rightarrow \overline{G}$ and $\widehat{\alpha}: \widehat{F} \Rightarrow \widehat{G}$.

Definition 1.1.10. Let $\mathbf{C}$ be a 2-category, and consider a pair of 1-cells $f: X \to Y$ and $g: Z \to Y$. A 2-pullback is an object $X \underset{f,g}{\times} Z$ such that there is an isomorphism of categories $(\mathbf{C}(S,X) \times \mathbf{C}(S,Z))_{f,g} \to \mathbf{C}(S, X \underset{f,g}{\times} Z)$, where $(\mathbf{C}(S,X) \times \mathbf{C}(S,Z))_{f,g}$ indicates the subcategory of $\mathbf{C}(S,X) \times \mathbf{C}(S,Z)$ such that the composition on the right in the following diagram factorizes through the diagonal.

$$\begin{tikzcd}
\mathbf{C}(S,X) \times \mathbf{C}(S,Z) \arrow[r, "\cong"] & (\mathbf{C}(S,X) \times \mathbf{1}) \times (\mathbf{C}(S,Z) \times \mathbf{1}) \arrow[d, "(\mathrm{Id}_{\mathbf{C}(S,X)} \times f) \times (\mathrm{Id}_{\mathbf{C}(S,Z)} \times g)"] \\
 & (\mathbf{C}(S,X) \times \mathbf{C}(X,Y)) \times (\mathbf{C}(S,Z) \times \mathbf{C}(Z,Y)) \arrow[d, "c_{SXY} \times c_{SZY}"] \\
\mathbf{C}(S,Y) \arrow[r, "\Delta"'] & \mathbf{C}(S,Y) \times \mathbf{C}(S,Y)
\end{tikzcd}$$

A 2-pullback satisfies an analogue version of the universal property of a pullback in a 1-category.

Remark 1.1.11. Consider a pullback in $\mathcal{C}at$. Then it is a 2-pullback in $\mathbf{Cat}$. This implies that we can use the universal property of the pullback on natural transformations as well.

Since in a 2-category there are two types of arrows, morphisms between 2-categories have an additional degree of freedom as compared to functors between ordinary categories: the idea is that one may use 2-cells to compare 1-cells. So for example there are morphisms that preserve composition and identity only up to isomorphism (i.e. up to an invertible 2-cell), or even that preserve them up to an arbitrary 2-cell (lax functors and colax functors, depending on the direction of the 2-cell). Here we are interested in the strict and in the up-to-iso type of morphism, called respectively 2-functors and pseudo-functors.

Definition 1.1.12. Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. A pseudo-functor $\boldsymbol{F}: \mathbf{C} \to \mathbf{D}$ consists of:

- for each $A \in |\mathbf{C}|$, an object $FA \in |\mathbf{D}|$;
- for each $A, B \in |\mathbf{C}|$, a functor

$$F_{AB}: \mathbf{C}(A,B) \to \mathbf{D}(FA, FB);$$

- for each $A, B, C \in |\mathbf{C}|$, a natural isomorphism $\gamma_{ABC}\colon c_{FA,FB,FC} \circ (F_{AB} \times F_{BC}) \Rightarrow F_{AC} \circ c_{ABC}$ as in the following diagram:

$$\begin{tikzcd}
\mathbf{C}(A,B) \times \mathbf{C}(B,C) \arrow[r, "c_{ABC}"] \arrow[d, "F_{AB}\times F_{BC}"'] \arrow[dr, phantom, "\gamma_{ABC}\Rightarrow"] & \mathbf{C}(A,C) \arrow[d, "F_{AC}"] \\
\mathbf{D}(FA,FB) \times \mathbf{D}(FB,FC) \arrow[r, "c_{FA,FB,FC}"'] & \mathbf{D}(FA,FC)
\end{tikzcd}$$

- for each $A \in |\mathbf{C}|$, a natural isomorphism $\delta_A\colon u_{FA} \Rightarrow F_{AA} \circ u_A$ as in the following diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "u_A"] \arrow[dr, "u_{FA}"'] & \mathbf{C}(A,A) \arrow[d, "F_{AA}"] \arrow[dl, phantom, "\overset{\delta_A}{\Rightarrow}"] \\
& \mathbf{D}(FA,FA)
\end{tikzcd}$$

These data have to satisfy the following conditions:

- **Composition coherence:** given 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

the following diagram commutes:

$$\begin{tikzcd}
Fh \circ Fg \circ Ff \arrow[r, Rightarrow, "\mathrm{i}_{Fh} * \gamma_{f,g}"] \arrow[d, Rightarrow, "\gamma_{g,h} * \mathrm{i}_{Ff}"'] & Fh \circ F(g \circ f) \arrow[d, Rightarrow, "\gamma_{g\circ f,h}"] \\
F(h \circ g) \circ Ff \arrow[r, Rightarrow, "\gamma_{f,h\circ g}"'] & F(h \circ g \circ f)
\end{tikzcd}$$

- **Identity coherence:** given $f\colon A \to B$ a 1-cell, the following diagrams commute:

$$\begin{tikzcd}
Ff \circ \mathrm{id}_{FA} \arrow[r, Rightarrow, "\mathrm{i}_{Ff} * \delta_A"] \arrow[dr, Rightarrow, "\mathrm{i}_{Ff}"'] & Ff \circ F\mathrm{id}_A \arrow[d, Rightarrow, "\gamma_{\mathrm{id}_A,f}"] \\
& F(f \circ \mathrm{id}_A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{FB} \circ Ff \arrow[r, Rightarrow, "\delta_B * \mathrm{i}_{Ff}"] \arrow[dr, Rightarrow, "\mathrm{i}_{Ff}"'] & F\mathrm{id}_B \circ Ff \arrow[d, Rightarrow, "\gamma_{f,\mathrm{id}_B}"] \\
& F(\mathrm{id}_B \circ f)
\end{tikzcd}$$

where with i_{Ff} we denote the identity of the object Ff in the category $\mathbf{C}(A, B)$.

Intuitively, a pseudo-functor is an assignation that preserves the compositions and the identities of 1-cells and 2-cells up to isomorphism. We call 2-functor a pseudo-functor that preserves these compositions (and the identities) strictly, i.e., where the coherence isomorphisms are actually identities.

Definition 1.1.13. A 2-functor is a pseudo-functor such that γ and δ are identities.

It is easy to see that the coherence diagrams commute automatically whenever the natural transformations γ and δ are identities. Hence, we can give the following simpler definition of a 2-functor.

Definition 1.1.14 (Alternative)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. A 2-functor $\boldsymbol{F}\colon \mathbf{C} \to \mathbf{D}$ consists of:

- for each $A \in |\mathbf{C}|$, an object $FB \in |\mathbf{D}|$;
- for each $A, B \in |\mathbf{C}|$, a functor

$$F_{AB}\colon \mathbf{C}(A,B) \to \mathbf{D}(FA,FB);$$

These data have to satisfy the following conditions:

- **Composition coherence:** given $A, B, C \in |\mathbf{C}|$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}(A,B) \times \mathbf{C}(B,C) \arrow[r, "c_{ABC}"] \arrow[d, "F_{AB} \times F_{BC}"'] & \mathbf{C}(A,C) \arrow[d, "F_{AC}"] \\
\mathbf{D}(FA,FB) \times \mathbf{D}(FB,FC) \arrow[r, "c_{FA,FB,FC}"'] & \mathbf{D}(FA,FC)
\end{tikzcd}$$

- **Identity coherence:** given $A \in |\mathbf{C}|$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "u_A"] \arrow[dr, "u_{FA}"'] & \mathbf{C}(A,A) \arrow[d, "F_{AA}"] \\
 & \mathbf{D}(FA,FA)
\end{tikzcd}$$

Adjunctions are widely used in 1-dimensional category theory since they allow us to relate different functors. So we want to extend the notion of adjunction to a 2-categorical approach: this can be done using the notion of bi-adjunction. Roughly, a bi-adjunction satisfies up to isomorphism the conditions required in a classical adjunction, as we see in Proposition 1.1.20.

Definition 1.1.15. Let $\mathbf{C}$ be a 2-category. A sub-2-category of it is a 2-category $\mathbf{D}$ such every 0-cell (1-cell, 2-cell) of $\mathbf{D}$ is a 0-cell (1-cell, 2-cell) of $\mathbf{C}$. A sub-2-category is said 2-full if for any 1-cells f, g in $\mathbf{D}(A,B)$ the 2-cells between them in $\mathbf{C}$ lie all in $\mathbf{D}$.

Definition 1.1.16. Let $\mathbf{C}$, $\mathbf{D}$ be 2-categories and $\boldsymbol{F}, \boldsymbol{G}\colon \mathbf{C} \to \mathbf{D}$ 2-functors. A pseudo-natural transformation $\boldsymbol{\alpha}\colon \boldsymbol{F} \Rightarrow \boldsymbol{G}$ is a family of 1-cells $\alpha_c\colon Fc \to Gc$ indexed by 0-cells of $\mathbf{C}$ together with a family of invertible 2-cells $\alpha_f\colon \alpha_d \circ Ff \Rightarrow Gf \circ \alpha_c$ indexed by 1-cells of $\mathbf{C}$ such that for any 2-cell $\beta\colon f \Rightarrow g$ in $\mathbf{C}$ the equality $G\beta * \alpha_f = \alpha_g * F\beta$ holds.

$$\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow \alpha_f"] \\
Fd \arrow[r, "\alpha_d"'] & Gd
\end{tikzcd}$$

Definition 1.1.17. Two 2-functors $\boldsymbol{R}\colon \mathbf{C} \to \mathbf{D}$ and $\boldsymbol{L}\colon \mathbf{D} \to \mathbf{C}$ are **bi-adjoint** if for each $X \in |\mathbf{D}|$ and $Y \in |\mathbf{C}|$ there is a pseudo-natural equivalence of categories $\boldsymbol{\phi}_{XY}\colon \mathbf{C}(LX, Y) \Rightarrow \mathbf{D}(X, RY)$. In this case we denote the bi-adjunction as $\boldsymbol{L} \dashv \boldsymbol{R}$.

Bi-adjoint functors are relevant to our purpose since they let us speak about free constructions, even if they are not free in the usual sense (i.e. arising from a strict adjunction). In particular we will provide some completions that arise from non-strict bi-adjunctions.

Definition 1.1.18. Let $\boldsymbol{\eta}, \boldsymbol{\epsilon}\colon \boldsymbol{F} \Rightarrow \boldsymbol{G}$ be pseudo-natural transformations, with $\boldsymbol{F}, \boldsymbol{G}\colon \mathbf{C} \to \mathbf{D}$ 2-functors. A **modification** (see [9]) $\alpha\colon \boldsymbol{\eta} \Rightarrow \boldsymbol{\epsilon}$ is a family of 2-cells $\alpha_X\colon \eta_X \Rightarrow \epsilon_X$ indexed by 0-cells $\mathbf{C}$ such that for every 1-cell $f\colon X \to Y$ the following diagram commutes:

$$\begin{tikzcd}
(Gf) \circ \eta_X \arrow[r, Rightarrow, "(Gf)\alpha_X"] \arrow[d, Rightarrow, "\eta_f"'] & (Gf) \circ \epsilon_X \arrow[d, Rightarrow, "\epsilon_f"] \\
\eta_Y \circ (Ff) \arrow[r, Rightarrow, "\alpha_Y Ff"'] & \epsilon_Y \circ (Ff)
\end{tikzcd}$$

Remark 1.1.19. As for natural transformations, one can give a notion of "vertical" composition between modifications. It is easy to see that a modification is invertible w.r.t. this composition if and only if every component is an invertible 2-cell.

Proposition 1.1.20. *Let $\boldsymbol{R}\colon \boldsymbol{C} \to \boldsymbol{D}$ and $\boldsymbol{L}\colon \boldsymbol{D} \to \boldsymbol{C}$ be pseudo-functors. Then there is a bi-adjunction $\boldsymbol{L} \dashv \boldsymbol{R}$ if and only if there are two pseudo-natural transformations $\boldsymbol{\eta}\colon \mathbf{Id}_{\boldsymbol{D}} \Rightarrow \boldsymbol{RL}$ and $\boldsymbol{\epsilon}\colon \boldsymbol{LR} \Rightarrow \mathbf{Id}_{\boldsymbol{C}}$ (called respectively* ***unit*** *and* ***counit*** *of the bi-adjunction) and invertible modifications α, β as in the following two diagrams (i.e. triangular identities):*

$$\begin{tikzcd}
R \arrow[r, Rightarrow, "\eta R"] \arrow[dr, Rightarrow, "\mathbf{1}_R"'] & RLR \arrow[d, Rightarrow, "R\epsilon"] \arrow[dl, phantom, "\Leftarrow\alpha"] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, Rightarrow, "L\eta"] \arrow[dr, Rightarrow, "\mathbf{1}_L"'] & LRL \arrow[d, Rightarrow, "\epsilon L"] \arrow[dl, phantom, "\Leftarrow\beta"] \\
& L
\end{tikzcd}$$

Proof. First suppose that there is a biadjunction $\boldsymbol{L} \dashv \boldsymbol{R}$, and let $\boldsymbol{\phi}_{X,Y}\colon \mathbf{C}(LX, Y) \Rightarrow \mathbf{D}(X, RY)$, $\boldsymbol{\psi}_{X,Y}\colon \mathbf{D}(X, RY) \Rightarrow \mathbf{C}(LX, Y)$ the pseudo-natural equivalence of categories, with $\tilde{\beta}\colon \phi_{XY}\psi_{XY} \Rightarrow \mathrm{Id}_{\mathbf{D}(X,RY)}$, $\tilde{\alpha}\colon \psi_{X,Y}\phi_{X,Y} \Rightarrow \mathrm{Id}_{\mathbf{D}(LX,Y)}$ invertible and, for $f\colon X' \to X$ and $g\colon Y \to Y'$, the following pseudo-naturality square:

$$\begin{tikzcd}
\mathbf{C}(LX, Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "g\circ - \circ Lf"'] & \mathbf{D}(X, RY) \arrow[d, "Rg\circ - \circ f"] \\
\mathbf{C}(LX', Y') \arrow[r, "\phi_{X',Y'}"'] \arrow[ur, phantom, "\Uparrow\gamma_{f,g}"] & \mathbf{D}(X', RY')
\end{tikzcd}$$

with $\gamma_{f,g}$ a natural iso. Then one can define $\eta_X := \phi_{X,LX}(\mathrm{id}_{LX})$ and $\epsilon_Y :=$

$\psi_{RY,Y}(\mathrm{id}_{RY})$. Given $X \in |\mathbf{D}|$, consider the naturality diagram

$$\begin{tikzcd}
\mathbf{C}(LRLX, LX) \arrow[r, "\phi_{RLX,LX}"] \arrow[d, "-\circ L\eta_X"'] & \mathbf{D}(RLX, RLX) \arrow[d, "-\circ \eta_X"] \\
\mathbf{C}(LX, LX) \arrow[r, "\phi_{RLX,LX}"'] \arrow[ur, phantom, "\Rightarrow \gamma_{\eta_X,\mathrm{id}_{LX}}"] & \mathbf{D}(X, RLX)
\end{tikzcd}$$

Looking at the component indexed by ϵ_{LX} of $\gamma_{\eta_X,\mathrm{id}_{LX}}$ and using also the equivalence, one gets the 2-cells

$$\phi_{X,LX}(\epsilon_{LX} \circ L\eta_X) \xRightarrow{(\gamma_{\eta_X,\mathrm{id}_{LX}})_{\epsilon_{LX}}} \phi_{RLX,LX}(\epsilon_{LX}) \circ \eta_X \xRightarrow{\tilde{\beta}_{\mathrm{id}_{RLX}} * \mathrm{id}_{\eta_X}} \eta_X$$

Applying $\psi_{X,LX}$ at this diagram, precomposing it with $\tilde{\alpha}^{-1}$ and postcomposing it with $\tilde{\alpha}$, one gets

$$\epsilon_{LX} \circ L\eta_X \xRightarrow{\tilde{\alpha}_{\mathrm{id}_{LX}} \circ \psi_{X,LX}((\tilde{\beta}_{\mathrm{id}_{RLX}} * \mathrm{id}_{\eta_X}) \circ (\gamma_{\eta_X,\mathrm{id}_{LX}})_{\epsilon_{LX}}) \circ \tilde{\alpha}^{-1}_{\epsilon_{LX} \circ L\eta_X}} \mathrm{id}_{LX}$$

Since $\tilde{\alpha}$, $\tilde{\beta}$ and $\gamma_{\eta_X,\mathrm{id}_{LX}}$ are invertible and by functoriality we have that this is an invertible 2-cell, namely α. In an analogous way one can define β. They satisfy our request by construction.

Conversely, given η and ϵ, one defines $\phi_{X,Y}(f)$ as $Rf \circ \eta_X$ and $\psi_{X,Y}(g)$ as $\epsilon_Y \circ Lg$. It is then straightforward to verify the pseudo-naturality of ϕ and ψ together with the equivalence that they satisfy. □

We will use this proposition to prove our principal results, namely Theorem 2.2.14 and Theorem 3.3.28. Sometimes it happens that some of natural transformations involved above are identities, meaning that some of the diagrams commute strictly. This is just a particular case of a bi-adjunction, a sort of middle term between it and a strict adjunction (i.e. a 2-adjunction).

1.2 Fibrations: definitions and basic properties

In many areas of mathematics, we often end up in studying families of structures indexed by a category $\mathcal{C}$. These are typically described by contravariant functors from $\mathcal{C}$ into the category of structures we are interested in. The paradigmatic example is given by presheaves, that is, functors of shape $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, which are ubiquitous, for instance, in algebraic geometry. A natural question is what happens if we go one dimension higher, considering families of categories indexed by a category $\mathcal{C}$, namely presheaves of categories. In order to handle the two dimensional nature of categories, one soon realizes that the right way of representing these families is by pseudo-functors on (the opposite of) a 2-discrete 2-category, that is, of shape $\mathbf{Disc}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}$. These are rather complicated objects, but here it is where fibrations come into play. Fibrations were introduced by Grothendieck [5] to generalize problems arising in algebraic geometry, and were developed further by Bénabou [1]. They are functors $p \colon \mathcal{E} \to \mathcal{B}$ between (1-)categories, which however can encode the same information as a pseudo-functor of shape $\mathbf{Disc}(\mathcal{B})^{\mathrm{op}} \to \mathbf{Cat}$. Let us illustrate this fact by the following example, which describes the family of categories of modules indexed by the category of commutative rings.

Example 1.2.1. Let us consider the category $\mathcal{R}ing$ whose objects are commutative rings with unit, and whose morphisms are ring homomorphisms preserving units. Then we define the pseudo-functor[1] $\boldsymbol{Mod}\colon \mathbf{Disc}(\mathcal{R}ing)^{\mathrm{op}} \to \mathbf{Cat}$, which assigns to any ring A the category $\mathcal{M}od(A)$ of modules over A and to any ring homomorphism $f\colon A \to B$ a functor $f^*\colon \mathcal{M}od(B) \to \mathcal{M}od(A)$, given by restriction of scalars along f. More precisely, the objects of $\mathcal{M}od(A)$ are pairs $M = (|M|, \alpha_M)$, where $|M|$ is an abelian group and $\alpha_M\colon A \times |M| \to |M|$ is a group homomorphism compatible with the multiplication in A, in the following sense: for any a, b in A and m in $|M|$ one has $\alpha_M(a, \alpha_M(b, m)) = \alpha_M(ab, m)$. Its arrows $g\colon M \to N$ are group homomorphisms $|g|\colon |M| \to |N|$ such that $|g| \circ \alpha_M = \alpha_N \circ (\mathrm{id}_A \times |g|)$. For a ring homomorphism $f\colon A \to B$ we define the functor $f^*\colon \mathcal{M}od(B) \to \mathcal{M}od(A)$ as follows: if $h\colon M \to N$ is a B-module homomorphism, with $M = (|M|, \alpha_M)$ and $N = (|N|, \alpha_N)$, then we set $f^*M := (|M|, \alpha_M \circ (f \times \mathrm{id}_{|M|}))$ and $|f^*h| := |h|$.

$$
\begin{tikzcd}
f^*M \arrow[d, "f^*g"] & M \arrow[d, "g"] & A \times |M| \arrow[r, "\mathrm{id}_A \times |h|"] \arrow[d, "f \times \mathrm{id}_{|M|}"'] & A \times |N| \arrow[d, "f \times \mathrm{id}_{|N|}"] \\
f^*N & N & B \times |M| \arrow[r, "\mathrm{id}_B \times |h|"] \arrow[d, "\alpha_M"'] & B \times |N| \arrow[d, "\alpha_N"] \\
A \arrow[r, "f"] & B & {|M|} \arrow[r, "|h|"'] & {|N|}
\end{tikzcd}
$$

We can rearrange all the data given by the pseudo-functor $\boldsymbol{Mod}$ in just one category $\mathcal{M}od$, together with a functor $U\colon \mathcal{M}od \to \mathcal{R}ing$. The objects of $\mathcal{M}od$ are pairs (A, M) of a ring A and M an object in $\mathcal{M}od(A)$, and morphisms $(A, M) \to (B, N)$ are pairs (f, g) of a ring homomorphism $f\colon A \to B$ and a morphism $g\colon M \to f^*N$ in $\mathcal{M}od(A)$. The composition of morphisms $(f, g) \circ (h, s)$ is given by $(f \circ h, h^*(g) \circ s)$, and identities are the pairs of identities.

$$(A, M) \xrightarrow{(h,s)} (B, N) \xrightarrow{(f,g)} (C, K)$$

$$M \xrightarrow{\quad s \quad} h^*(N) \xrightarrow{h^*(g)} h^*(f^*(K))$$

$$A \xrightarrow{\quad h \quad} B \xrightarrow{\quad f \quad} C$$

Finally the functor $U\colon \mathcal{M}od \to \mathcal{R}ing$ is the first projection.

We can see that morphisms whose second component is invertible play a special role: given objects (A, M), (B, N) and (C, K) in $\mathcal{M}od$ together with two morphisms $(f, s)\colon (A, M) \to (B, N)$ and $(g, r)\colon (C, K) \to (B, N)$ with r invertible and a ring homomorphism $h\colon A \to C$ such that $g \circ h = f$, then there

[1] Note that this is actually a 2-functor.

exist a unique $t: M \to h^*(K)$ such that $(f, s) = (g, r) \circ (h, t)$.

$$
\begin{tikzcd}
(A,M) \arrow[d, dashed, "(h,t)"'] \arrow[dr, "(f,s)"] & \\
(C,K) \arrow[r, "(g,r)"'] & (B,N)
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[rr, "r"] & & g^*(N) \\
M \arrow[r, "s"] \arrow[rr, bend right, "t"'] & f^*(N) = h^*(g^*(N)) \arrow[r, "h^*(r^{-1})"] & h^*(K)
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, "f"] & \\
C \arrow[r, "g"'] & B
\end{tikzcd}
$$

This unique element is given by the composition $h^*(r^{-1}) \circ s$. Furthermore, the converse is also true: if we assume that there exist a unique morphism as above, then r is invertible in $\mathcal{M}od(C)$. So we have shown that arrows whose second component is invertible encode the information given by the action of $\boldsymbol{Mod}$ on the first component.

The special arrows described at the end of the previous example provide the key structure making the functor $U: \mathcal{M}od \to \mathcal{R}ing$ a fibration. This observation leads us to the following definitions.

Definition 1.2.2. Let $p: \mathcal{E} \to \mathcal{B}$ be a functor. An arrow $g: A \to B$ in $\mathcal{E}$ is **cartesian** if for every $g': A' \to B$ and $v: pA' \to pA$ such that $p(g') = p(g) \circ v$ there exist unique $h: A' \to A$ such that $g' = g \circ h$ and $p(h) = v$.

$$
\begin{tikzcd}
A' \arrow[d, dashed, "h"'] \arrow[dr, "g'"] & \\
A \arrow[r, "g"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
pA' \arrow[d, "v"'] \arrow[dr, "pg'"] & \\
pA \arrow[r, "pg"'] & pB
\end{tikzcd}
$$

Definition 1.2.3. A **fibration** is a functor $p: \mathcal{E} \to \mathcal{B}$ such that for every arrow $f: X \to Y$ in $\mathcal{B}$ and object B in $\mathcal{E}$ over Y there exist a cartesian arrow $g: A \to B$ over f. The arrow g is called **cartesian lifting** of f at B. The category $\mathcal{B}$ is called basis of the fibration, and $\mathcal{E}$ is the total category.

We say that an object A (arrow g) of $\mathcal{E}$ is **over** an object X (arrow f) of $\mathcal{B}$ if $pA = X$ $(p(g) = f)$. We also say that an arrow f is vertical if it is over the identity.

Example 1.2.4. The functor $U: \mathcal{M}od \to \mathcal{R}ing$ defined in Example 1.2.1 is a fibration. A cartesian lifting of a ring homomorphism $f: A \to B$ at a B-module M is given by restriction of scalars of M, that is the pair $(f, \mathrm{id}_{f^*M}): (A, f^*M) \to (B, M)$. More in general cartesian arrows are exactly the morphisms whose second component is invertible.

Example 1.2.5. Consider a category $\mathcal{B}$ and the unique functor $\mathbf{!}_{\mathcal{B}}\colon \mathcal{B} \to \mathbf{1}$. This is trivially a fibration: it is easy to see that identities are cartesian (over the only arrow in $\mathbf{1}$), which means that cartesian arrows are exactly the isomorphisms of $\mathcal{B}$.

Example 1.2.6. One elementary example of a fibration is the domain functor: given any category $\mathcal{C}$, the functor $\mathrm{dom}\colon \mathcal{C}^2 \to \mathcal{C}$ is a fibration. Cartesian arrows are pairs whose codomain is invertible. A cartesian lifting of $f\colon A \to B$ at $g\colon B \to C$ is then $(f, \mathrm{id}_C)\colon g \circ f \to g$.

Example 1.2.7. Instead, the codomain functor $\mathrm{cod}\colon \mathcal{C}^2 \to \mathcal{C}$ is a fibration if and only if $\mathcal{C}$ has pullbacks, in fact an arrow in $\mathcal{C}^2$ is cartesian if and only if it is a pullback in $\mathcal{C}$. Hence a cartesian lifting of $f\colon A \to B$ at $g\colon C \to B$ is their pullback square.

Example 1.2.8. Consider a category $\mathcal{C}$. The category $\mathrm{Fam}(C)$ has pairs $(I, \{X_i\}_{i\in I})$ as objects, where I is a set and $\{X_i\}_{i\in I}$ is a set-indexed family of objects in $\mathcal{C}$. Arrows $f\colon (I, \{X_i\}_{i\in I}) \to (J, \{Y_j\}_{j\in J})$ are pairs $(|f|, \{f_i\}_{i\in I})$ where $|f|\colon I \to J$ is a function and $\{f_i\}_{i\in I}$ is a family of arrows $f_i\colon X_i \to Y_{|f|(i)}$ in $\mathcal{C}$. Then the first projection $\mathrm{Fam}_C\colon \mathrm{Fam}(C) \to \mathcal{S}et$ is a fibration. A cartesian lifting of $f\colon I \to J$ at $(J, \{Y_j\}_{j\in J})$ is given by $(f, \{\mathrm{id}_{Y_{f(i)}}\}_{i\in I})\colon (I, \{Y_{f(i)}\}_{i\in I}) \to (J, \{Y_j\}_{j\in J})$.

Example 1.2.9. A particularly relevant example is given by a term model fibration. Consider a calculus in a dependent type theory. Objects in the base category $\mathcal{B}$ are contexts Γ, and arrows $\Gamma \to \Delta$, where $\Delta = y_1\colon \tau_1, ..., y_n\colon \tau_n$, is a n-tuple of terms $(M_1, ..., M_n)$ satisfying $\Gamma \vdash M_i\colon \tau_i[M_1/y_1, ..., M_{i-1}/y_{i-1}]$. These terms are to be interpreted as substitutions, and their composition is then the composition of substitutions. The objects of the total category $\mathcal{E}$ are type judgements of the form $\Gamma \vdash \sigma\colon \mathrm{Type}$. The arrows $(\Gamma \vdash \sigma\colon \mathrm{Type}) \to (\Delta \vdash \tau\colon \mathrm{Type})$ are pairs $(\vec{M}, N)$ with $\vec{M}\colon \Gamma \to \Delta$ arrow in $\mathcal{B}$ and N a term satisfying $\Gamma, x\colon \sigma \vdash N\colon \tau[\vec{M}/\vec{y}]$. Then the projection on the first component is a fibration. A cartesian lifting of an arrow $\vec{M}$ at a type judgement $\Delta \vdash \tau\colon \mathrm{Type}$ is $(\vec{M}, x)$ with $\Gamma, x\colon \tau[\vec{M}/\vec{y}] \vdash x\colon \tau[\vec{M}/\vec{y}]$.

Let us notice that we are able to perform context extension: given σ over Γ, one can consider the extended context $\Gamma, x\colon \sigma$. There is also a canonical projection $\chi\sigma\colon \Gamma, x\colon \sigma \to \Gamma$ given by the n-tuple of variables. Moreover, given an arrow $(\Gamma \vdash \sigma\colon \mathrm{Type}) \to (\Delta \vdash \tau\colon \mathrm{Type})$, one can consider the following square:

$$\begin{tikzcd}
\Gamma, x\colon \sigma \arrow[r, "\chi\sigma"] \arrow[d, "(\vec{M}, N)"'] & \Gamma \arrow[d, "\vec{M}"] \\
\Delta, y\colon \tau \arrow[r, "\chi\tau"'] & \Delta
\end{tikzcd}$$

It is not hard to see that this is a pullback in $\mathcal{B}$.

Example 1.2.10. Consider a theory T in a first order logic. This gives rise to a faithful fibration. Objects in the base $\mathcal{B}$ are contexts Γ, and arrows $\Gamma \to \Delta$, where $\Delta = y_1\colon \tau_1, ..., y_n\colon \tau_n$, is a n-tuple of terms $(M_1, ..., M_n)$ such that $M_i\colon \tau_i$. Again, terms are to be interpreted as substitutions: the only difference with the base category of Example 1.2.9 is that here we do not allow dependent types.

The total category $\mathcal{E}$ is the category of well-formed formulas of the theory: objects are pairs (Γ, A) where the first component is a context and the second a formula in that context; morphisms $(\Gamma, A) \to (\Delta, B)$ are arrows $\vec{M}\colon \Gamma \to \Delta$ in $\mathcal{B}$ such that $\Gamma; A \vdash B[\vec{M}/\vec{y}]$. Then the projection on the first component is a faithful fibration. By faithfulness, a cartesian lifting of $\vec{M}\colon \Gamma \to \Delta$ at (Δ, A) is uniquely determined by its domain: we pick $(\Gamma, A[\vec{M}/\vec{y}])$, so that the condition imposed on arrows is trivially satisfied (by the rule of assumption).

Cartesian arrows have some nice closure properties.

Lemma 1.2.11. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration, and $f\colon A \to B$, $g\colon B \to C$ be cartesian arrows. Then their composition $g \circ f$ is cartesian.*

Lemma 1.2.12. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration, and $f\colon A \to B$, $g\colon B \to C$ be arrows in $\mathcal{E}$ such that g and $g \circ f$ are cartesian. Then f is also cartesian.*

Proof. See [14]. □

Definition 1.2.13. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration, and X in $\mathcal{B}$. The **fiber** over X is the subcategory of $\mathcal{E}$ consisting of all the objects over X and all vertical arrows (clearly over id_X).

Definition 1.2.14. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. A **cleavage** of p is a choice of a cartesian lifting for every $f\colon X \to Y$ and B over Y, and such cartesian lifting is denoted by $f^B\colon f^*B \to B$. A **cloven** fibration is a fibration equipped with a cleavage.

Remark 1.2.15. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. Using the Axiom of Choice, for every arrow $f\colon X \to Y$ in $\mathcal{B}$ and object B over Y, we can select a cartesian lifting of f at B. Hence, when the Axiom of Choice is assumed, every fibration is cloven.

Remark 1.2.16. Cartesian liftings are unique up to vertical iso, in the following sense: given an arrow $f\colon X \to Y$ in the base category, an object A over Y and $g\colon B \to A$, $h\colon C \to A$ cartesian liftings of f at A, the unique vertical arrows $u\colon B \to C$ and $v\colon C \to B$ given by cartesianity of, respectively, h and g, are each other inverses. So given a cleavage one can characterize cartesian arrows up to vertical iso.

Definition 1.2.17. Consider a fibration endowed with a cleavage, and fix $f\colon X \to Y$ an arrow in the base. The **reindexing functor** along f is a functor f^* from the fiber over Y to the fiber over X that sends an object A into f^*A, and its action on arrows is uniquely determined by the universal property of cartesian arrows.

$$\begin{tikzcd}
f^*A \arrow[r, "f^A"] \arrow[d, dashed, "f^*g"'] & A \arrow[d, "g"] \\
f^*B \arrow[r, "f^B"'] & B
\end{tikzcd}$$

Remark 1.2.18. Given a pair of composable arrows in $\mathcal{B}$, the composition of the reindexing functors is not in general the reindexing functor of the composition of the arrows; however, there is an isomorphism between these two.

Explicitly one have the isomorphism $\gamma_{h,f}$ depicted below:

$$\begin{tikzcd}
h^*f^*A \arrow[r, "h^{f^*A}"] \arrow[d, dashed, "\gamma_{h,f}"] \arrow[dd, dashed, bend right, "\mathrm{id}_{h^*f^*A}"'] & f^*A \arrow[r, "f^A"] & A \arrow[d, "\mathrm{id}_A"] \\
(f\circ h)^*A \arrow[rr, "(f\circ h)^A"'] \arrow[d, dashed, "\gamma^{-1}_{h,f}"] & & A \arrow[d, "\mathrm{id}_A"] \\
h^*f^*A \arrow[r, "h^{f^*A}"'] & f^*A \arrow[r, "f^A"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "\mathrm{id}_Z"] \arrow[dd, bend right, "\mathrm{id}_Z"'] & X \arrow[r, "f"] & Y \arrow[d, "\mathrm{id}_Y"] \\
Z \arrow[rr, "f\circ h"'] \arrow[d, "\mathrm{id}_Z"] & & Y \arrow[d, "\mathrm{id}_Y"] \\
Z \arrow[r, "h"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

It is an easy exercise to check that the morphisms defined in the diagram by cartesianity are each other inverses.

Furthermore, the reindexing functor of the identity is not in general the identity, but there is an isomorphism δ between them determined by cartesianity, as in the diagram below:

$$\begin{tikzcd}
A \arrow[d, dashed, "\delta_A"'] \arrow[dr, "\mathrm{id}_A"] & \\
(\mathrm{id}_X)^*A \arrow[r, "\mathrm{id}_X^A"'] & A
\end{tikzcd}$$

$$X \xrightarrow{\mathrm{id}_X} X$$

Using the concept of fiber one can think of assign a presheaf with value in $\mathcal{C}at$ to a fibration, by mapping, in fact, objects in their fibers and arrows into the reindexing functor. Unfortunately, we just showed that the reindexing functor along the identity is not forced to be the identity, and in general the reindexing functors are not closed under composition. The non-coherence of the composition of reindexing functors forces us to consider pseudo-functors instead of strict functors when speaking of fibrations. In fact, we are about to see that fibrations and presheaves to $\mathcal{C}at$ are essentially the same: we are going to assign a (pseudo-)presheaf to any fibration, and then to assign a fibration to any (pseudo-)presheaf in such a way that the two constructions are mutually essentially inverses.

Definition 1.2.19. Given a fibration $p\colon \mathcal{E} \to \mathcal{B}$ equipped with a cleavage, one can define a pseudo-functor $\boldsymbol{F}\colon \mathbf{Disc}(\mathcal{B}^{\mathrm{op}}) \to \mathbf{Cat}$ associated to it:

- for X in $|\mathbf{Disc}(\mathcal{B}^{\mathrm{op}})|$, FX is the fiber over X;
- for X, Y in $|\mathbf{Disc}(\mathcal{B}^{\mathrm{op}})|$, the functor $F_{XY}\colon \mathbf{Disc}(\mathcal{B}^{\mathrm{op}})(X,Y) \to \mathbf{Cat}(FX, FY)$ sends an arrow $f\colon Y \to X$ to the reindexing functor f^* along the fibers;

- for X, Y, Z in $|\mathbf{Disc}(\mathcal{B}^{\mathrm{op}})|$, the natural isomorphism $\gamma_{XYZ}\colon c_{FX,FY,FZ} \circ (F_{XY} \times F_{YZ}) \Rightarrow F_{XZ} \circ c_{XYZ}$ has $\gamma_{h,f}$ as the component indexed by the object (h, f);
- for X in $|\mathbf{Disc}(\mathcal{B}^{\mathrm{op}})|$, the natural isomorphism δ_X has δ_A as the component indexed by the object A.

Proposition 1.2.20. *The assignation in Definition 1.2.19 satisfy the composition and the identity coherence. In particular, $\boldsymbol{F}$ is a pseudo-functor.*

Definition 1.2.21. Given a pseudo-functor $F\colon \mathbf{Disc}(\mathcal{B}^{\mathrm{op}}) \to \mathbf{Cat}$, one can define a corresponding fibration $\int F\colon \mathcal{E} \to \mathcal{B}$ via the Grothendieck construction. First one defines the total category $\mathcal{E}$: its objects are pairs (X, A) with $X \in |\mathcal{B}|$ and $A \in |FX|$; its arrows $f\colon (X, A) \to (Y, B)$ are pairs (g, h) such that $g\colon X \to Y$ is an arrow in $\mathcal{B}$ and $h\colon A \to F(g)(B)$ is an arrow in FX. The composition $(g, h) \circ (k, r)$ is given by $(g \circ k, F(g)(h) \circ r)$, and identities are given by the pair of identities.

Proposition 1.2.22. *With the notation introduced in Definition 1.2.21, the first projection $\int F\colon \mathcal{E} \to \mathcal{B}$ is a fibration. Furthermore one recovers a cleavage by defining the cartesian lifting of f at B as the arrow $(f, \mathrm{id}_{Ff(B)})$.*

Notice that Example 1.2.1 is just an instance of this construction, applied to the 2-functor $\boldsymbol{Mod}\colon \mathbf{Disc}(\mathcal{Ring}^{\mathrm{op}}) \to \mathbf{Cat}$ induced by the functor Mod.

Remark 1.2.23. By Remark 1.2.16 cartesian arrows with fixed codomain and over a fixed arrow are unique up to vertical iso. Hence the cartesian arrows of $\int F$ are the morphisms whose second component is an isomorphism.

Remark 1.2.24. A presheaf $F\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{Set}$ gives rise to a 2-functor $G\colon \mathbf{Disc}(\mathcal{B}^{\mathrm{op}}) \to \mathbf{Cat}$, by regarding $\mathcal{Set}$ as a subcategory of $\mathcal{Cat}$. The ensuing fibration $\int G$ obtained via Grothendieck construction is then a faithful fibration. Conversely, if $p\colon \mathcal{E} \to \mathcal{B}$ is a faithful fibration, the corresponding fiber pseudo-presheaf can be restricted to a presheaf of sets.

Example 1.2.25. Consider the presheaf $\mathcal{R}\colon \mathcal{Set}^{\mathrm{op}} \to \mathcal{Cat}$ where $\mathcal{R}(X)$ is the preordered set of functions $s\colon X \to \mathcal{P}(\mathbb{N})$ (regarded as a category) with the order relation given by: $s \le t$ if and only if there exist a partial recursive function $\phi\colon \mathbb{N} \to \mathbb{N}$ such that for all $x \in X$ and $h \in s(x)$ the function ϕ is defined on h and $\phi(h) \in t(x)$. Given an arrow $f\colon X \to Y$, $\mathcal{R}(f)$ is given by precomposition with f. This presheaf is a presheaf for Kleene realizability, a particular case of a more general construction given in [6]. We can regard R as a 2-functor $\boldsymbol{\mathcal{R}}\colon \mathbf{Disc}(\mathcal{Set}^{\mathrm{op}}) \to \mathbf{Cat}$, and then we can apply the Grothendieck construction to it to get a faithful fibration $\mathcal{K}\colon \mathcal{Real} \to \mathcal{Set}$. Since object are pairs (X, a) of a set X and a function $a\colon X \to \mathcal{P}(\mathbb{N})$, we can identify the objects in $\mathcal{Real}$ with functions towards $\mathcal{P}(\mathbb{N})$. Then an arrow $f\colon a \to b$, where $a\colon X \to \mathcal{P}(\mathbb{N})$ and $b\colon Y \to \mathcal{P}(\mathbb{N})$, is given by a function $|f|\colon X \to Y$ such that $a \le b \circ |f|$ in $\mathcal{R}(X)$.

$$
\begin{tikzcd}
X \arrow[r, "|f|"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow\ \subseteq"] & Y \arrow[d, "b"] \\
\mathcal{P}(\mathbb{N}) & \mathcal{P}(\mathbb{N}) \arrow[l, "\mathcal{P}(\phi)=\phi^{-1}"]
\end{tikzcd}
\qquad\qquad
\mathbb{N} \xrightarrow[\phi]{} \mathbb{N}
$$

Let us notice that if one considers in the fibers only functions $a\colon X \to \mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$, then applying the Grothendieck construction he recovers the category of assemblies as the total category.

It is easy to see that these constructions are essentially mutually inverse, so there is a correspondence between pseudo-functors into **Cat** and fibrations. This correspondence extends to a bi-equivalence of 2-categories. We start defining the 2-category of fibrations.

Remark 1.2.26. A fibration is a 0-cell in the 2-category $\mathbf{Cat}^2$ (see Example 1.1.4).

Definition 1.2.27. Let $p\colon \mathcal{E} \to \mathcal{B}$ and $q\colon \mathcal{E}' \to \mathcal{B}'$ be fibrations. A **fibration morphism** $F\colon p \to q$ consists of a pair of functors $(\overline{F}, \widehat{F})$ such that the square

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\overline{F}"] \arrow[d, "p"'] & \mathcal{E}' \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "\widehat{F}"] & \mathcal{B}' \end{tikzcd}$$

commutes and $\overline{F}$ preserves cartesian arrows.

Remark 1.2.28. Let $F\colon p \to q$ be a fibration morphism, and fix a cleavage of $p\colon \mathcal{B} \to \mathcal{E}$ and one of q. Then, given an arrow $f\colon X \to Y$ in $\mathcal{B}$ and an object A in the fiber over Y, one can define the vertical arrow $\beta_{f,A}\colon \overline{F}(f^*A) \to (\widehat{F}f)^*\overline{F}A$ by cartesianity, since by Definition 1.2.27 we have $q\overline{F}f^A = \widehat{F}f = q(\widehat{F}f)^{\overline{F}A}$.

$$\begin{tikzcd} \overline{F}(f^*A) \arrow[d, dashed, "\beta_{f,A}"'] \arrow[dr, "\overline{F}f^A"] & \\ (\widehat{F}f)^*\overline{F}A \arrow[r, "(\widehat{F}f)^{\overline{F}A}"'] & \overline{F}A \\ \widehat{F}X \arrow[r, "\widehat{F}f"] & \widehat{F}Y \end{tikzcd}$$

Remark 1.2.29. A fibration morphism F is a 1-cell in $\mathbf{Cat}^2$ such that $\overline{F}$ preserves cartesian arrows. Moreover it is straightforward that the composition in $\mathbf{Cat}^2$ of fibration morphisms it is still a fibration morphism, and that the identity also is a fibration morphism.

Definition 1.2.30. The 2-category **Fib** is the 2-full sub-2-category of $\mathbf{Cat}^2$ on fibrations and fibration morphisms, i.e. it has fibrations as 0-cells, fibration morphisms as 1-cells and it is full on 2-cells.

Remark 1.2.31. The two constructions examined in Proposition 1.2.20 and Proposition 1.2.22 extend to a bi-equivalence of categories between **Fib** and the 2-category of pseudo-functors from a 2-discrete 2-category to **Cat**.

Fibrations have nice closure properties. For example they are closed under composition and under pullbacks in $\mathcal{C}at$.

Definition 1.2.32. Let $F\colon \mathcal{C} \to \mathcal{B}$ and $p\colon \mathcal{E} \to \mathcal{B}$ be functors. Then the pullback in $\mathcal{C}at$ of these functors is shown in the following diagram:

$$\begin{tikzcd}
\mathcal{C} \underset{p,F}{\times} \mathcal{E} \arrow[r] \arrow[d, "F^\star p"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}$$

We say that the fibration $F^\star p$ is obtained from p by **change of base along** F.

Lemma 1.2.33. *An arrow $f = (f_1, f_2)$ in $\mathcal{C} \underset{p,F}{\times} \mathcal{E}$ is cartesian (w.r.t. $F^\star p$) iff f_2 is cartesian.*

Proof. Let us first assume that f_2 is cartesian. Fix $(f_1, f_2)\colon (X, A) \to (Y, B)$. Then let $(g_1, g_2)\colon (Z, C) \to (Y, B)$ and $h_1\colon Z \to X$ such that $g_1 = f_1 \circ h_1$.

$$\begin{tikzcd}
(Z, C) \arrow[dr, bend left, "{(g_1,g_2)}"] & \\
(X, A) \arrow[r, "{(f_1,f_2)}"] & (Y, B)
\end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[d, "h_1"'] \arrow[dr, bend left, "g_1"] & \\
X \arrow[r, "f_1"] & Y
\end{tikzcd}$$

Clearly an arrow that makes the diagram above commute and being over the diagram below needs to have h_1 as first component.

By cartesianity of f_2, there exist a unique $h_2\colon C \to A$ such that $g_2 = f_2 \circ h_2$ and $ph_2 = Fh_1$. Then the arrow $(h_1, h_2)\colon (Z, C) \to (X, A)$ satisfies the existence required for the cartesianity of (f_1, f_2). The uniqueness is a consequence of the uniqueness of h_2.

Conversely, let us assume the cartesianity of (f_1, f_2). By cartesianity there exists a unique $(\mathrm{id}_X, h)\colon (X, (pf_1)^*B) \to (X, A)$ over the identity on X. Furthermore, using the first implication one has that also $(f_1, (pf_1)^B)$ is cartesian, so by cartesianity there exists a unique $(\mathrm{id}_X, u)\colon (X, A) \to (X, (pf_1)^*B)$. Furthermore, these are each other inverses by universality of cartesian arrows. Thus u and h are each other inverses; the cartesianity of f_2 follows straightforwardly.

$$\begin{tikzcd}
(X, (pf_1)^*B) \arrow[d, dashed, "{(\mathrm{id}_X,h)}"'] \arrow[dr, bend left, "{(f_1,(pf_1)^B)}"] & \\
(X, A) \arrow[u, dashed, bend right, "{(\mathrm{id}_X,u)}"'] \arrow[r, "{(f_1,f_2)}"'] & (Y, B)
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}_X"'] \arrow[dr, bend left, "f_1"] & \\
X \arrow[u, bend right, "\mathrm{id}_X"'] \arrow[r, "f_1"'] & Y
\end{tikzcd}$$

□

Proposition 1.2.34. *Let $F\colon \mathcal{C} \to \mathcal{B}$ be a functor and $p\colon \mathcal{E} \to \mathcal{B}$ a fibration. Then $F^\star p$ is a fibration, i.e. the change of base along F preserve fibrations. Furthermore, the square is a fibration morphism $F^\star p \to p$.*

Proof. It follows easily by Lemma 1.2.33. Notice that a cartesian lifting of $f\colon X \to Y$ at (Y, B) is given by $(f, (Ff)^B)$. □

1.3 Fibred products and terminal objects

Fibrations with finite fibred products are very important since they allow us to define the completions we are going to introduce in Section 2.2 and in Section 3.3: the former applies to fibrations with finite fibred products, while the latter relies on them in its very definition.

Definition 1.3.1. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. We say that p has finite fibred products if, given a finite product $\prod A_i$ in the fiber over Y, a cartesian lifting $g\colon B \to \prod A_i$ of $f\colon X \to Y$ and a family $g_i\colon B_i \to A_i$ of cartesian arrows over f, one has that the family of vertical arrows $u_i\colon B \to B_i$ determined by cartesianity is a product diagram in the fiber over X.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, dashed, "u_i"'] & \prod A_i \arrow[d] \\
B_i \arrow[r, "g_i"'] & A_i \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

We can give an equivalent definition using a universal property of fibred products.

Definition 1.3.2. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. We say that p has fibred terminal objects, or fibred 0-ary products, if for any X in the base there exist an object T_X with the following universal property: for any arrow $f\colon Y \to X$ and object A over Y there exist a unique arrow $u\colon A \dashrightarrow T_X$ over f.

Example 1.3.3. Consider the fibration cod of Example 1.2.7. A terminal object in the fiber over X is given by id_X.

Remark 1.3.4. It is easy to see that p has fibred terminal objects if and only if it has a right adjoint right inverse functor T^p. The objects in the image of the functor are the fibred terminal objects, and the unique arrows given by the universal property are the transposes of the arrow in the base category.

Remark 1.3.5. Given $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with fibred terminal objects functor T, one has that T is full and faithful. This is true since T is a right adjoint and a right inverse.

Definition 1.3.6. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. We say that p has fibred binary products if for any X in the base and A, B over X there exist an object $A \wedge B$ over X and two vertical arrows $\mathrm{pr}_A\colon A \wedge B \to A$ and $\mathrm{pr}_B\colon A \wedge B \to B$ with the following universal property: for any arrow $f: Y \to X$ and any pair of arrows $g: D \to A$ and $h: D \to B$ over f, there exists a unique arrow $u: D \to A \wedge B$ over f such that the following diagram commutes.

$$\begin{tikzcd} & D \arrow[dl, "g"'] \arrow[d, dashed, "u"] \arrow[dr, "h"] & \\ A & A \wedge B \arrow[l, "\mathrm{pr}_A"] \arrow[r, "\mathrm{pr}_B"'] & B \end{tikzcd}$$

Clearly one can generalize this to arbitrary finite products.

Definition 1.3.7. The category **FPFib** is the 2-full sub-2-category of **Fib** whose 0-cells are fibrations with finite fibred products and 1-cells are all the morphisms in **Fib** that preserve fibred products, i.e. that map fibred finite product diagrams in finite fibred product diagrams.

Remark 1.3.8. Since they are determined by a universal property, fibred products are unique up to a unique isomorphism. In particular, this implies that properties such as commutativity and associativity hold, up to a unique iso.

Remark 1.3.9. Let $F: p \to q$ be a morphism in **FPFib**, with $p: \mathcal{E} \to \mathcal{B}$, and A, B be objects in $\mathcal{E}$ on the same fiber. Then we get the arrow $\gamma_{A,B}: \overline{F}(A \wedge B) \to (\overline{F}A \wedge \overline{F}B)$ defined by the universal property of fibred products. It is easy to see that $\gamma_{A,B}$ is an isomorphism, since F preserves fibred finite products. So a morphism in **FPFib** preserves a choice of fibred products up to a unique iso.

Example 1.3.10. Consider the fibration in Example 1.2.10 given by a theory. A terminal object over a context Γ is just the pair $(\Gamma, \top)$. A fibred binary product of (Γ, A) and (Γ, B) is given by $(\Gamma, A \wedge B)$. So this fibration has finite fibred products.

Example 1.3.11. Consider the fibration $\mathcal{Mod} \to \mathcal{Ring}$ of Example 1.2.4. This has finite fibred products: a fibred terminal object over a ring R is given by $(R, 0)$, and a binary fibred products of (R, M) and (R, N) is given by $(R, M \oplus N)$.

Example 1.3.12. Let $\mathcal{B}$ be a category with finite products $\times$. We define the category $\mathrm{s}(\mathcal{B})$: its objects are pairs (I, X) of objects of $\mathcal{B}$. A morphism $(I, X) \to (J, Y)$ is a pair of arrows (u, f) in $\mathcal{B}$ such that $u: I \to J$ and $f: I \times X \to Y$. The composition of morphisms $(v, g) \circ (u, f)$ is given by $(v \circ u, g \circ \langle u \circ \pi_I, f\rangle)$.

$$I \times X \xrightarrow{\langle u \circ \pi_I, f\rangle} J \times Y \xrightarrow{\quad g \quad} Z$$

The identity on (I, X) is given by (id_I, π_X). Then we define the functor $\mathrm{s}_{\mathcal{B}}: \mathrm{s}(\mathcal{B}) \to \mathcal{B}$ as the first projection on both objects and arrows. This is called the **simple fibration** (see [8]) on $\mathcal{B}$. It is immediate to see that this is a fibration with finite fibred products: a fibred product of (I, X) and (I, Y) is given by $(I, X \times Y)$, and the fibred terminal object consists of the pair (I, T) where T is the terminal object of $\mathcal{B}$.

Example 1.3.13. Consider the fibration $\mathcal{K}: \mathcal{Real} \to \mathcal{Set}$ of Example 1.2.25. A fibred terminal object over X is given by a maximum in $\mathcal{R}(X)$. It is not hard to see that the function $c_{\mathbb{N}}: X \to \mathcal{P}(\mathbb{N})$ whose value is constantly $\mathbb{N}$ is such object. Moreover, given $s, t \in \mathcal{R}(X)$, one can consider a bijective recursive function $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ whose inverse f^{-1} is recursive. A fibred product of s and t is given by the function $s \wedge t\colon X \to \mathcal{P}(\mathbb{N})$ defined by $(s \wedge t)(x) := \{n \in \mathbb{N} | f^{-1}(n) \in (s(x) \times t(x))\}$. Let $r\colon X \to \mathcal{P}(\mathbb{N})$ be such that $r \leq s$ and $r \leq t$ together with ϕ and ψ satisfying the $\leq$ condition for s and t respectively. Then we can define a partial recursive function γ by setting $\gamma := f \circ \langle \phi, \psi \rangle$. It is straightforward to verify that $r \leq s \wedge t$ using γ. Moreover, we have that $s \wedge t \leq s$ (and analogously $s \wedge t \leq t$): it can be shown using the partial recursive function $\pi_1 \circ f^{-1}$ (respectively, $\pi_2 \circ f^{-1}$).

$$
\begin{tikzcd}
\mathbb{N} \arrow[rr, "f^{-1}"] \arrow[dr, swap, "\pi_1 \circ f^{-1}"] & & \mathbb{N} \times \mathbb{N} \arrow[ll, bend right, swap, "f"] \arrow[rr, bend left, "f"] \arrow[dl, swap, "\pi_1"] \arrow[dr, "\pi_2"] & & \mathbb{N} \arrow[ll, swap, "f^{-1}"] \arrow[dl, "\pi_2 \circ f^{-1}"] \\
& \mathbb{N} & & \mathbb{N} & \\
& & \mathbb{N} \arrow[ul, "\phi"] \arrow[ur, swap, "\psi"] \arrow[uu, "\langle \phi, \psi \rangle"] & &
\end{tikzcd}
$$

So $\mathcal{K}$ has finite fibred products.

Proposition 1.3.14. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with finite fibred products, and $F\colon \mathcal{C} \to \mathcal{B}$ be a functor. Then the pullback $F^\star p$ is again a fibration with finite fibred products. Furthermore, the induced 1-cell in **Fib** extends to a 1-cell in **FPFib**.*

Proof. We saw in Proposition 1.2.34 that it is a fibration. A fibred terminal object over X is given by $(X, \mathrm{T}FX)$ and a fibred product of (X, A) and (X, B) is given by $(X, A \wedge B)$. Moreover, it is straightforward that the projection on the second component preserves finite fibred products. $\square$

1.4 Fibred product completion

In this section we give the completion for finite fibred product for a fibration. We will use it to define the completion for comprehension in the sense of Jacobs in Section 3.3.

Notation. Given a natural number n, we denote with $\underline{n}$ the set $\{i \in \mathbb{N} | 1 \leq i \leq n\}$.

Definition 1.4.1. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. We define a category $\mathcal{FFP}(p)$ by setting as objects pairs $(X, \vec{A})$ with $X \in \mathcal{B}$ and $\vec{A}$ a finite list of objects in $\mathcal{E}$ over X, and as morphisms $f\colon (X, \vec{A}) \to (Y, \vec{B})$ triples $(|f|, \underline{f}, \widetilde{f})$ with: $|f|\colon X \to Y$ an arrow in $\mathcal{B}$, $\underline{f}\colon \underline{m} \to \underline{n}$ a function, where m is the length of $\vec{B}$ and n is the length of $\vec{A}$, and $\widetilde{f} = \{f_i\}_{i \in \underline{m}}$ a family of arrows in $\mathcal{E}$ such that $f_i\colon A_{\underline{f}(i)} \to B_i$.

The composition $g \circ f$ is given by setting $|(g \circ f)| := |g| \circ |f|$, then $\underline{(g \circ f)} := \underline{f} \circ \underline{g}$ and finally $\widetilde{(g \circ f)} := \{g_i \circ f_{\underline{g}(i)}\}_i$. The identity is given by setting $|\mathrm{id}_{(X,\vec{A})}| := \mathrm{id}_X$, then $\underline{\mathrm{id}_{(X,\vec{A})}} := \mathrm{id}_{\underline{n}}$ and finally $\widetilde{\mathrm{id}_{(X,\vec{A})}} := \{\mathrm{id}_{A_i}\}_{i \in \underline{n}}$.

We now want to show that this construction yields free finite fibred products for an arbitrary fibration.

Proposition 1.4.2. *The first projection $\mathit{ffp}(p)\colon \mathcal{FFP}(p) \to \mathcal{B}$ is an object in **FPFib**.*

Proof. It is obviously a functor. Let $f\colon X \to Y$ be an arrow in $\mathcal{B}$ and $(Y, \vec{B})$ an object in $\mathcal{FFP}(p)$ over Y. For each $i \in \underline{m}$, let f_i be a cartesian lifting of f that ends in B_i with respect to p. Then a cartesian lifting g of f is given by:

$|g| = f$;

$\underline{g} = \mathrm{id}_{\underline{m}}$;

$\widetilde{g} = \{f_i\}_{i \in \underline{m}}$.

Furthermore, it has finite fibred products: the terminal object in a fiber is given by the empty list, and binary products in a fiber are given by the concatenation of the lists. It is straightforward to verify that these satisfy the universal property required.

□

This construction can be extended to a 2-functor $\mathbf{ffp}\colon \mathbf{Fib} \to \mathbf{FPFib}$ which is left bi-adjoint to the inclusion 2-functor. The action on 1-arrows is described as follows: given a morphism (1-cell) F in **Fib** and given $f\colon (X, \vec{A}) \to (X', \vec{B})$ in $\mathcal{FFP}(p)$ one has $\widehat{\mathbf{ffp}(F)} := \widehat{F}$ and $\overline{\mathbf{ffp}(F)}(f)\colon (\widehat{F}X, \overline{F}\vec{A}) \to (\widehat{F}X', \overline{F}\vec{B})$ defined by $\overline{\mathbf{ffp}(F)}(f) := (\widehat{F}|f|, \underline{f}, \{F(\widetilde{f})_i\}_{i \in \underline{n}})$.

Now, for a 2-cell α one defines $\mathbf{ffp}(\alpha)$ in the following way: $\widehat{\mathbf{ffp}(\alpha)} := \widehat{\alpha}$ and $\overline{\mathbf{ffp}(\alpha)}_{(X,\vec{A})} := (\widehat{\alpha}_x, \mathrm{id}_{\underline{n}}, \{\overline{\alpha}_{A_i}\}_{i \in \underline{n}})$. It is easy to verify that $\mathbf{ffp}$ is a 2-functor, so that preserves identities and composition of both 1-cells and 2-cells. We are going to show that this construction is the completion for finite fibred products of a fibration.

Let us now define two pseudo-natural transformations that will be respectively the unit and the counit of the free-forgetful adjunction corresponding to the finite fibred products construction.

Definition 1.4.3. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. We define the fibration morphism $\boldsymbol{\eta}_p\colon p \to \mathit{ffp}(p)$ as the square

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\overline{\boldsymbol{\eta}_p}"] \arrow[d, "p"'] & \mathcal{FFP}(p) \arrow[d, "\mathit{ffp}(p)"] \\ \mathcal{B} \arrow[r, "\widehat{\boldsymbol{\eta}_p}"'] & \mathcal{B} \end{tikzcd}$$

where $\widehat{\boldsymbol{\eta}_p} := \mathrm{Id}_{\mathcal{B}}$, and $\overline{\boldsymbol{\eta}_p}$ sends an object A to (pA, A) (and it acts obviously on arrows). $\boldsymbol{\eta}$ is the family of $\boldsymbol{\eta}_p$ as p varies in **Fib**.

Definition 1.4.4. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with finite fibred products, together with a choice of fibred product functor P. We define the fibration morphism $\boldsymbol{\epsilon}_p\colon \mathit{ffp}(p) \to p$ as the square

$$\begin{tikzcd} \mathcal{FFP}(p) \arrow[r, "\overline{\boldsymbol{\epsilon}_p}"] \arrow[d, "\mathit{ffp}(p)"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "\widehat{\boldsymbol{\epsilon}_p}"'] & \mathcal{B} \end{tikzcd}$$

where $\widehat{\boldsymbol{\epsilon}_p} := \mathrm{Id}_{\mathcal{B}}$ and $\overline{\boldsymbol{\epsilon}_p} := P$. $\boldsymbol{\epsilon}$ is family of $\boldsymbol{\epsilon}_p$ as p varies in **FPFib**.

Theorem 1.4.5. *$\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$ can be extended to pseudo-natural transformations that are respectively unit and counit for a bi-adjunction. Hence* **ffp** *is left-biadjoint to the inclusion 2-functor $\boldsymbol{FPFib} \hookrightarrow \boldsymbol{Fib}$. Moreover the unit is strictly natural.*

Proof. By Proposition 1.1.20 we have to check triangular identities and pseudo-naturality of $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$. It is an easy exercise to check the pseudo-naturality (also, that $\boldsymbol{\eta}$ is strictly natural) and the triangular identities (that are strict as well). $\square$

Chapter 2

Lawvere comprehensions

Lawvere's notion of comprehension arises to generalize the comprehension schema from axiomatic set theory (see [11]). In particular this can be expressed by a comprehension structure on the fibration associated to the powerset doctrine. Consider the contravariant powerset functor $\mathcal{P}: \mathcal{Set}^{\mathrm{op}} \to \mathcal{Set}$. Applying the Grothendieck construction to the pseudo-functor $\boldsymbol{\mathcal{P}}: \mathbf{Disc}(\mathcal{Set}^{\mathrm{op}}) \to \mathbf{Cat}$ yields a fibration $\mathrm{Pr}: \mathcal{Pred} \to \mathcal{Set}$. The total category is the category of predicates: objects are pairs (X, I) where $I \subseteq X$ is a subset of X. Thanks to the axiom schema of comprehension (also known as separation) a predicate $\phi(x)$, with x a free variable for ϕ, corresponds to a subset $I \subseteq X$ of X. This enables us to turn the predicate (X, I) into the set I together with the inclusion $I \hookrightarrow X$. This action is clearly functorial, by definition of the Grothendieck construction. Furthermore, one can also assign a predicate to a set X by considering the formula $\top$, thus obtaining (X, X) as its comprehension. Also this is functorial. It is not hard to see that these functors are adjoint. Lawvere comprehension is a generalization in fibred category theory of this structure.

In this chapter our aim is to define fibrations with Lawvere comprehension, providing some relevant examples. A particular case are faithful fibrations with Lawvere comprehension, for which we will give a characterization in terms of some properties involving monicity and epicity of a specific family of arrows in Theorem 2.1.10. Afterwards we will then look at the 2-dimensional level by defining the 2-category of fibrations with Lawvere comprehension (Definition 2.1.12). In the end we will present the main result of the chapter, namely the free fibration with Lawvere comprehensions (Theorem 2.2.14).

2.1 Fibrations with Lawvere comprehension

In this section we present the definition of fibrations with Lawvere comprehension and the characterization of faithfulness.

Definition 2.1.1 ([11] and [4, Def. 5])**.** A fibration with Lawvere comprehension is a fibration $p: \mathcal{E} \to \mathcal{B}$ together with two functors $\mathrm{T}^p: \mathcal{B} \to \mathcal{E}$ and $\mathrm{C}^p: \mathcal{E} \to \mathcal{B}$ such that T^p is a terminal object functor (or equivalently T^p is right adjoint and right inverse to p) and C^p is right adjoint to T^p.

$$\begin{tikzcd} \mathcal{E} \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\mathrm{C}^p"] \\ \mathcal{B} \arrow[u, "\dashv \mathrm{T}^p \dashv" description] \end{tikzcd}$$

For simplicity we will omit the fibration index whenever it will be clear from the context.

Example 2.1.2. Consider the fibration cod of Example 1.2.7. It has Lawvere comprehension: the terminal object functor sends an object in the base to the identity, as described in Example 1.3.3, while the comprehension functor is dom. It is easy to see that the adjunctions $\mathrm{cod} \vdash \mathrm{T}^p \vdash \mathrm{dom}$ hold.

Example 2.1.3. Consider the fibration $\mathcal{P}: \mathcal{S} \to \mathcal{S}et$ obtained by applying the Grothendieck construction to the powerset functor $\mathcal{P}: \mathcal{S}et^{\mathrm{op}} \to \mathcal{C}at$ (where we consider the powerset of a set partially ordered by the inclusion relation $\subseteq$, and we regard it as a category). Objects in $\mathcal{S}$ are pairs of a set X and a subset $S \subseteq X$, and an arrow $(X,S) \to (Y,T)$ is a function $f: X \to Y$ such that $S \subseteq f^{-1}(T)$. First of all, this fibration has finite fibred products, given by the intersection. Furthermore, it has Lawvere comprehension: the comprehension functor $\mathrm{C}^{\mathcal{P}}$ is obtained by taking the second component of the pair, and on an arrow $f: (X,S) \to (Y,T)$ it gives the restriction and corestriction of f to, respectively, S and T.

Proposition 2.1.4. *Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere comprehension, and X an object in $\mathcal{B}$. Then the arrow $p\epsilon_{\mathrm{T}X}: \mathrm{CT}X \to X$ is invertible, with inverse given by η_X.*

Proof. Since T is full and faithful (see Remark 1.3.5), η is an iso (see [13]). Using triangular identities can be shown that its inverse is given by $p\epsilon_{\mathrm{T}}$. ☐

Proposition 2.1.5. *Let $p: \mathcal{E} \to \mathcal{B}$ be a faithful fibration with Lawvere comprehension, and A in $\mathcal{E}$ over X. Then the arrow $p\epsilon_A: \mathrm{C}A \to pA$ is a mono.*

Proof. Given $f, g: X \to \mathrm{C}A$ in $\mathcal{B}$ such that $p\epsilon_A \circ f = p\epsilon_A \circ g$, we consider the transposes $f^\#, g^\#: \mathrm{T}X \to A$. They are equal to the composition, respectively, $\epsilon_A \circ \mathrm{T}f$ and $\epsilon_A \circ \mathrm{T}g$ by the definition of transposes through counit. These arrows are over $p\epsilon_A \circ f = p\epsilon_A \circ g$, and by faithfulness they have to be equal. This proves that $p\epsilon_A$ is a mono. ☐

Proposition 2.1.6. *Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere comprehension, and A in $\mathcal{E}$. If p is faithful, then ϵ is cartesian.*

Proof. Consider a cartesian lifting $g: A^* \to A$ of $p\epsilon_A$ at A. Then we get a vertical arrow $h: A^* \to \mathrm{TC}A$ by the universal property of the terminal object. We also get a vertical arrow $u: \mathrm{TC}A \to A^*$ by cartesianity. They are each other's inverses: $h \circ u = \mathrm{id}_{\mathrm{TC}A}$ by unicity of vertical arrows into the terminal, and $u \circ h = \mathrm{id}_{A^*}$ by cartesianity again.

$$\begin{tikzcd}
\mathrm{TC}A \arrow[d, dashed, bend right, "u"'] \arrow[dr, "\epsilon_A"] & \\
A^* \arrow[u, dashed, bend right, "h"'] \arrow[r, "g"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{C}A \arrow[r, "p\epsilon_A"'] & pA
\end{tikzcd}$$

□

Definition 2.1.7. Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration. An arrow $f: A \to B$ in $\mathcal{E}$ is called **locally epic** if for any $g, h: B \to C$ such that $pg = ph$ and $g \circ f = h \circ f$ one has $h = g$.

Definition 2.1.8. Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere comprehension. We say that a functor $F: \mathcal{E} \to \mathcal{B}$ is **locally faithful** if, for any $f, g: A \to B$ in $\mathcal{E}$ such that $pf = pg$ and $Ff = Fg$, one has $f = g$.

It is immediate to see that if p is faithful, then every component of the counit is locally epic, and also that the comprehension functor C is locally faithful. This fact mirrors the equivalence between these concepts.

Proposition 2.1.9. *Let p be a fibration with Lawvere comprehension. Then every component of the counit is locally epic if and only if the comprehension functor* C *is locally faithful.*

These facts help us to give a characterization of faithful fibrations with Lawvere comprehension.

Theorem 2.1.10. *Let p be a fibration with Lawvere comprehension. Then p is faithful if and only if the natural transformation $p\epsilon$ is a mono and ϵ is locally epic.*

Proof. We already showed that one implication holds (in Proposition 2.1.5 and right after Definition 2.1.8). For the converse, let $f, g: A \to B$ be in $\mathcal{E}$ and suppose that $pf = pg$. Consider the naturality squares of f and g and their image under p:

$$\begin{tikzcd}
\mathrm{TC}A \arrow[r, "\epsilon_A"] \arrow[d, bend right, "\mathrm{TC}f"'] \arrow[d, bend left, "\mathrm{TC}g"] & A \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \\
\mathrm{TC}B \arrow[r, "\epsilon_B"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{C}A \arrow[r, "p\epsilon_A"] \arrow[d, bend right, "\mathrm{C}f"'] \arrow[d, bend left, "\mathrm{C}g"] & X \arrow[d, "h"] \\
\mathrm{C}B \arrow[r, "p\epsilon_B"'] & Y
\end{tikzcd}$$

By commutativity of the squares below, we have $p\epsilon_B \circ \mathrm{C}f = h \circ p\epsilon_A = p\epsilon_B \circ \mathrm{C}g$. Since $p\epsilon_B$ is mono, we have $\mathrm{C}f = \mathrm{C}g$. Then their transposes are equal, so $\epsilon_B \circ \mathrm{TC}f = \epsilon_B \circ \mathrm{TC}g$. This implies that in the upper squares the diagonal is the same, so $f \circ \epsilon_A = g \circ \epsilon_A$. Hence $f = g$, since ϵ_A is locally epic, proving the faithfulness of p. □

Now we are ready to study the 2-dimensional level by introducing the 2-category of fibrations with Lawvere comprehension. We also introduce the 2-category of fibrations with Lawvere comprehension and finite fibred products, which will reveal to be essential to us.

Definition 2.1.11. Given $p: \mathcal{E} \to \mathcal{B}$, $q: \mathcal{E}' \to \mathcal{B}'$ fibrations with Lawvere comprehension, a morphism of fibrations with Lawvere comprehensions (LC-morphism for short) from p to q is a fibration morphism $F: p \to q$ such that the natural isomorphism $\theta: \overline{F} \circ \mathrm{T}^p \Rightarrow \mathrm{T}^q \circ \widehat{F}$ determined as the mate of $\mathrm{i}_{\widehat{F} \circ p}$ is invertible and its inverse's mate $(\theta^{-1})^{\#}: \widehat{F} \circ \mathrm{C}^p \Rightarrow \mathrm{C}^q \circ \overline{F}$ is again invertible.

Furthermore, given LC-morphisms $F: p \to q$ and $G: q \to s$ together with the natural isomorphisms $\theta: \overline{F} \circ \mathrm{T}^p \Rightarrow \mathrm{T}^q \circ \widehat{F}$ and $\sigma: \overline{G} \circ \mathrm{T}^q \Rightarrow \mathrm{T}^s \circ \widehat{G}$, their composition is given by $G \circ F$ together with the natural iso $(\sigma\widehat{F}) \circ (\overline{G}\theta)$.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\overline{F}"] & \mathcal{E}' \arrow[r, "\overline{G}"] & \mathcal{E}'' \\
\mathcal{B} \arrow[u, "\mathrm{T}^p"] \arrow[r, "\widehat{F}"'] \arrow[ur, phantom, "\theta \Searrow"] & \mathcal{B}' \arrow[u, "\mathrm{T}^q"'] \arrow[r, "\widehat{G}"'] \arrow[ur, phantom, "\sigma \Searrow"] & \mathcal{B}'' \arrow[u, "\mathrm{T}^s"']
\end{tikzcd}$$

A LC-morphism is simply a fibration morphism that preserves fibred terminal and comprehensions up to iso. They are closed under composition since the mate of the inverse of $(\sigma\widehat{F}) \circ (\overline{G}\theta)$ is again invertible, as a consequence of Proposition 1.1.7.

Definition 2.1.12. We define the 2-category **LComp** by declaring:

0-cells: fibrations with Lawvere comprehension;

1 and 2-cells: given p, q fibrations with Lawvere comprehension, $\mathbf{LComp}(p, q)$ is the full subcategory of $\mathbf{Fib}(p, q)$ on LC-morphisms.

Remark 2.1.13. Let $\phi: F \to G$ be a 2-cell in **LComp**. Then $\sigma \circ \overline{\phi}\mathrm{T}^p = \mathrm{T}^q\widehat{\phi} \circ \theta$ and $(\sigma^{-1})^{\#} \circ \widehat{\phi}\mathrm{C}^p = \mathrm{C}^q\overline{\phi} \circ (\theta^{-1})^{\#}$ as a consequence of Proposition 1.1.8. Diagrammatically, the following squares of natural transformations commute:

$$\begin{tikzcd}
\overline{F}\mathrm{T}^p \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\overline{\phi}\mathrm{T}^p"'] & \mathrm{T}^q\widehat{F} \arrow[d, Rightarrow, "\mathrm{T}^q\widehat{\phi}"] \\
\overline{G}\mathrm{T}^p \arrow[r, Rightarrow, "\sigma"'] & \mathrm{T}^q\widehat{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{C}^q\overline{F} \arrow[d, Rightarrow, "\mathrm{C}^q\overline{\phi}"'] & \widehat{F}\mathrm{C}^p \arrow[l, Rightarrow, "(\theta^{-1})^{\#}"'] \arrow[d, Rightarrow, "\widehat{\phi}\mathrm{C}^p"] \\
\mathrm{C}^q\overline{G} & \widehat{G}\mathrm{C}^p \arrow[l, Rightarrow, "(\sigma^{-1})^{\#}"]
\end{tikzcd}$$

Definition 2.1.14. The 2-category **FPLComp** is the 2-full sub-2-category of **LComp** on fibrations with finite fibred products and the morphisms that preserve them.

Remark 2.1.15. Since every fibration with Lawvere comprehension has fibred terminal objects, requiring fibred finite products is equivalent to requiring just binary fibred products.

Proposition 2.1.16. *Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere comprehension and binary fibred products, and A, B be objects in $\mathcal{E}$ over X. Then the arrow $p\epsilon_{A \wedge B}$ is the diagonal of the pullback of $p\epsilon_A$ and $p\epsilon_B$.*

Proof. First of all, $p\epsilon_{A\wedge B} = p\epsilon_A \circ \mathrm{C}(\mathrm{pr}_A) = p\epsilon_B \circ \mathrm{C}(\mathrm{pr}_B)$ by functoriality of p and naturality of ϵ applied to the diagram

$$\begin{tikzcd}
\mathrm{TC}B \arrow[r, "\epsilon_B"] & B \\
\mathrm{TC}(A\wedge B) \arrow[u, "\mathrm{TCpr}_B"] \arrow[r, "\epsilon_{A\wedge B}"] \arrow[d, "\mathrm{TCpr}_A"'] & A\wedge B \arrow[u, "\mathrm{pr}_B"'] \arrow[d, "\mathrm{pr}_A"] \\
\mathrm{TC}A \arrow[r, "\epsilon_A"'] & A
\end{tikzcd}$$

Then, using the properties of the adjunction $\mathrm{T} \dashv \mathrm{C}$, one gets the universal property of the pullback: given two arrows $f\colon Y \to \mathrm{C}A$ and $g\colon Y \to \mathrm{C}B$ such that $p\epsilon_A \circ f = p\epsilon_B \circ g$, we take the transposes of f and g and the unique arrow $u\colon \mathrm{T}Y \to A \wedge B$ that they induce on the fibred product. Then the transpose of u is the unique arrow that makes the pullback diagram to commute.

$$\begin{tikzcd}
\mathrm{T}Y \arrow[r, "\mathrm{T}g"] \arrow[dr, "\mathrm{T}f"'] \arrow[drr, dashed, "u"] & \mathrm{TC}B \arrow[r, "\epsilon_B"] & B \\
 & \mathrm{TC}A \arrow[dr, "\epsilon_A"'] & A\wedge B \arrow[u] \arrow[d] \\
 & & A
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[drr, bend left, "g"] \arrow[dr, dashed, "u^{\#}"] \arrow[ddr, bend right, "f"'] & & \\
 & \mathrm{C}(A\wedge B) \arrow[r, "\mathrm{C}(\mathrm{pr}_B)"'] \arrow[d, "\mathrm{C}(\mathrm{pr}_A)"] & \mathrm{C}B \arrow[d, "p\epsilon B"] \\
 & \mathrm{C}A \arrow[r, "p\epsilon A"'] & X
\end{tikzcd}$$

□

2.2 The free fibration with Lawvere comprehension

Consider the following commutative square of 2-functors:

$$\begin{tikzcd}
\mathbf{FPLComp} \arrow[r, "\boldsymbol{U}_{FLC}"] \arrow[d, "\boldsymbol{U}_{FPC}"'] & \mathbf{FPFib} \arrow[d, "\boldsymbol{U}_{FP}"] \\
\mathbf{LComp} \arrow[r, "\boldsymbol{U}_{LC}"'] & \mathbf{Fib}
\end{tikzcd}$$

where the functors involved forget either the comprehension or the finite fibred product structure. In this section we describe a left biadjoint of $\boldsymbol{U}_{FLC}$ (Theorem 2.2.14), providing the free fibration with Lawvere comprehension with fibred products over a fibration with fibred products. Unfortunately, we need finite fibred products to create the completion, unlike in Chapter 3 for comprehension categories. Finding the completion for Lawvere comprehension without assuming finite fibred products may be a starting point for further research.

Definition 2.2.1. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with finite fibred products. The category $\mathcal{E}^p$ has objects given by pairs (A, B) of objects in the same fiber, and arrows $(A, B) \to (C, D)$ given by pairs (f, g) of arrows in $\mathcal{E}$ making the following diagram commute:

$$\begin{tikzcd}
A \wedge B \arrow[r, "g"] \arrow[d, "\mathrm{pr}_A"'] & C \wedge D \arrow[d, "\mathrm{pr}_C"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

The functor $\hat{p}\colon \mathcal{E}^p \to \mathcal{E}$ is the first projection on both objects and arrows.

Remark 2.2.2. There is a functor $I\colon \mathcal{E}^p \to \mathcal{E}^2$ which sends a pair (A, B) to the first fibred projection $\mathrm{pr}_A\colon A \wedge B \to A$ and an arrow $(f, g)\colon (A, B) \to (C, D)$ to the corresponding arrow $(f, g)\colon \mathrm{pr}_A \to \mathrm{pr}_C$. This functor is clearly full and faithful, by definition of arrows in $\mathcal{E}^p$. The fibration $\hat{p}$ is then equal to the composition $\mathrm{cod} \circ I$. Notice that I is not necessarily a subcategory since it is not necessarily injective on objects.

Lemma 2.2.3. *The functor $\hat{p}$ is a fibration.*

Proof. A cartesian lifting of $f\colon A \to B$ at (B, C) is determined by the following diagram:

$$\begin{tikzcd}
(pf)^*C \arrow[r, "(pf)^C"] & C \\
A \wedge (pf)^*C \arrow[u] \arrow[r, dashed] \arrow[d] & B \wedge C \arrow[u] \arrow[d] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

where the dashed arrow is given by the universal property of the fibred product and cartesian liftings are taken w.r.t. p. This is clearly a lifting of f. To prove cartesianity we consider an arrow $g\colon (D, E) \to (B, C)$ and an arrow $s\colon D \to A$ such that $\widehat{g} = f \circ s$. Then we consider the diagram

$$\begin{tikzcd}
 & (pf)^*C \arrow[r, "(pf)^C"] & C \\
D \wedge E \arrow[ur, dashed, "u"] \arrow[r, dashed, "h"'] \arrow[rr, bend left] \arrow[d] & A \wedge (pf)^*C \arrow[u] \arrow[r] \arrow[d] & B \wedge C \arrow[u] \arrow[d] \\
D \arrow[r, "s"'] \arrow[rr, bend left, "\widehat{g}"] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[r, "ps"] \arrow[rr, bend left, "p\widehat{g}"] & X \arrow[r, "pf"] & Y
\end{tikzcd}$$

where the arrow u over ps is given by cartesianity of $(pf)^C$ and h is given by the universal property of the fibred product. It is clear that the arrow (h, s) is the only one satisfying the universal property required for the cartesianity. $\square$

Lemma 2.2.4. *The fibration $\hat{p}$ has finite fibred products.*

Proof. First of all we show that it has fibred terminal objects. Let A be an object in $\mathcal{E}$. Then the pair $(A, \mathrm{T}^p pA)$, where T^p is the fibred terminal object functor of p, is clearly a fibred terminal object over A. Indeed, the universal arrow $(A, B) \to (A, \mathrm{T}^p pA)$ is $(\mathrm{id}_A, \mathrm{id}_A \wedge u)$, where $u: B \to \mathrm{T}^p pA$ is the unique vertical arrow with respect to p. These objects determine a fibred terminal object functor $\mathrm{T}^{\hat{p}}: \mathcal{E} \to \mathcal{E}^p$, where $\mathrm{T}^{\hat{p}}A = (A, \mathrm{T}^p pA)$.

Now let $(A, B), (A, C)$ be objects in $\mathcal{E}^p$. Their fibred product is the pair $(A, B \wedge C)$: there are vertical arrows $(A, B \wedge C) \to (A, B)$ and $(A, B \wedge C) \to (A, C)$ whose second components are respectively the compositions $A \wedge (B \wedge C) \cong (A \wedge B) \wedge C \to A \wedge B$ and $A \wedge (B \wedge C) \cong (A \wedge C) \wedge B \to A \wedge C$. $\square$

Definition 2.2.5. The functor $\mathrm{C}^{\hat{p}}: \mathcal{E}^p \to \mathcal{E}$ maps objects (A, B) into $A \wedge B$, and acts on arrows (f, g) as the second projection.

Now we want to show that there is an adjunction $\mathrm{T}^{\hat{p}} \dashv \mathrm{C}^{\hat{p}}$ to prove that $\hat{p}$ is indeed an object in **FPLComp**.

Given A in $\mathcal{E}$ over X and (A, B) in $\mathcal{E}^p$, set $\eta_A := \mathrm{pr}_{A \wedge \mathrm{T}^p X}: A \wedge \mathrm{T}^p X \to A$ and $\epsilon_{(A,B)} := (\mathrm{pr}_A, \mathrm{pr}_{A \wedge B})$

$$\begin{tikzcd}
(A \wedge B) \wedge \mathrm{T}^p X \arrow[r, "\mathrm{pr}_{A\wedge B}"] \arrow[d, "\mathrm{pr}_{A\wedge B}"'] & A \wedge B \arrow[d, "\mathrm{pr}_A"] \\
A \wedge B \arrow[r, "\mathrm{pr}_A"'] & A
\end{tikzcd}$$

Proposition 2.2.6. *In the previous setting, the adjunction $\mathrm{T}^{\hat{p}} \dashv \mathrm{C}^{\hat{p}}$ holds. In particular, the fibration $\hat{p}$ has Lawvere comprehension.*

Proof. Let us start by showing that η and ϵ defined above are respectively the unit and the counit of an adjunction, namely, they are natural and the triangular identities hold. The naturality of both is trivial. Triangular identities are verified since $\mathrm{T}^{\hat{p}}$ is clearly right inverse to $\mathrm{C}^{\hat{p}}$ and also the latter maps the counit to an iso. $\square$

We can see that this construction extends to a 2-functor $\wedge_L$: **FPFib** $\to$ **FPLComp**.

Definition 2.2.7. Let $F: p \to q$ be a morphism in **FPFib**. We can define a LC-morphism $\hat{F}: \hat{p} \to \hat{q}$ by setting $\widehat{\hat{F}} := \overline{F}$, $\overline{\hat{F}}(A, B) := (\overline{F}A, \overline{F}B)$ and $\overline{\hat{F}}f := \gamma_{C,D} \circ (\overline{F}f) \circ \gamma_{A,B}^{-1}$, where $\gamma_{A,B}: \overline{F}(A \wedge B) \to (\overline{F}A) \wedge (\overline{F}B)$ is the isomorphism given by Remark 1.3.9. Moreover we define the natural iso $\hat{\theta}: \overline{\hat{F}} \circ \mathrm{T}^{\hat{p}} \Rightarrow \mathrm{T}^{\hat{q}} \circ \widehat{\hat{F}}$ by setting $\hat{\theta}_A := (\mathrm{id}_{\overline{F}A}, \mathrm{id}_{\overline{F}A} \wedge \theta_X)$, where A is over X and $\theta: \overline{F} \circ \mathrm{T}^p \Rightarrow \mathrm{T}^q \circ \widehat{F}$ is the natural iso preserving fibred terminal objects. It is invertible since both its components are.

Finally, given a 2-cell $\alpha: F \Rightarrow G$, we define a 2-cell $\hat{\alpha}: \hat{F} \Rightarrow \hat{G}$ by setting $\widehat{\hat{\alpha}} := \overline{\alpha}$ and $\overline{\hat{\alpha}_{(A,B)}} := (\alpha_A \wedge \alpha_B, \alpha_A)$.

It is straightforward to verify that these data combine to give a 2-functor $\wedge_L\colon \mathbf{FPFib} \to \mathbf{FPLComp}$.

Now we want to show that this construction is free: in particular, the functor $\wedge_L$ is left bi-adjoint to the forgetful 2-functor $\boldsymbol{U}_{FLC}\colon \mathbf{FPLComp} \to \mathbf{FPFib}$.

Definition 2.2.8. Let p be a fibration with finite fibred products together with the terminal object functor T^p. We define $\boldsymbol{\eta}_p$ in the following way: $\widehat{\boldsymbol{\eta}_p} := \mathrm{T}^p$, and $\overline{\boldsymbol{\eta}_p}$ sends an object A over X in the pair $(\mathrm{T}^p X, A)$, and an arrow $f\colon A \to B$ into the pair $(\mathrm{T}^p(pf), \mathrm{T}^p(pf) \wedge f)$.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\overline{\boldsymbol{\eta}_p}"] \arrow[d, "p"'] & \mathcal{E}^p \arrow[d, "\hat{p}"] \\ \mathcal{B} \arrow[r, "\widehat{\boldsymbol{\eta}_p}"'] & \mathcal{E} \end{tikzcd}$$

Furthermore, given a morphism $F\colon p \to q$ in **FPFib** together with the natural isomorphism $\theta\colon \mathrm{T}^q \circ \widehat{F} \Rightarrow \overline{F} \circ \mathrm{T}^p$, we define $\boldsymbol{\eta}_F$ by setting $\widehat{\boldsymbol{\eta}_F} := \theta$ and $\overline{\boldsymbol{\eta}_F}_A := (\theta_X, \mathrm{id}_{\overline{F}A})$.

Lemma 2.2.9. *$\boldsymbol{\eta}_p\colon p \to \hat{p}$ is a morphism in **FPFib**.*

Proof. We only need to show that $\overline{\boldsymbol{\eta}_p}$ preserves finite fibred products. It does by the definition of fibred products of $\hat{p}$ given in the proof of Lemma 2.2.4. $\square$

Proposition 2.2.10. *$\boldsymbol{\eta}\colon \mathbf{Id}_{\boldsymbol{FPFib}} \Rightarrow (\boldsymbol{U}_{FLC} \circ \wedge_L)$ is a pseudo-natural transformation.*

Proof. By Definition 1.1.16 we only need to show that given $F, G\colon p \to q$ morphisms in **FPFib** and a 2-cell $\beta\colon F \Rightarrow G$ one has $(\boldsymbol{U}_{FLC} \circ \wedge_L)\beta * \boldsymbol{\eta}_F = \boldsymbol{\eta}_G * \beta$. By definition of the 2-category **FPFib** it is enough to check the equalities $\widehat{(\boldsymbol{U}_{FLC} \circ \wedge_L)\beta} * \widehat{\boldsymbol{\eta}_F} = \widehat{\boldsymbol{\eta}_G} * \widehat{\beta}$ and $\overline{(\boldsymbol{U}_{FLC} \circ \wedge_L)\beta} * \overline{\boldsymbol{\eta}_F} = \overline{\boldsymbol{\eta}_G} * \overline{\beta}$. These are easy to show. $\square$

Definition 2.2.11. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration together with the Lawvere comprehension functor C^p and a fixed cleavage. We define $\boldsymbol{\epsilon}_p$ in the following way: first, we set $\widehat{\boldsymbol{\epsilon}_p} := \mathrm{C}^p$. Then, given an object (A, B) in $\mathcal{E}^p$ we consider the projection of the counit indexed by the first component, $p\epsilon_A$, and we set $\overline{\boldsymbol{\epsilon}_p}(A, B) := (p\epsilon_A)^* B$; given a morphism $g\colon (A, B) \to (A', B')$, we set $\overline{\boldsymbol{\epsilon}_p} g$ to be the arrow defined in the following diagram:

$$
\begin{tikzcd}
A \wedge B \arrow[r, "\overline{g}"] \arrow[d] & A' \wedge B' \arrow[d] \\
A \arrow[r, "\widehat{g}"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A & \\
TCA \arrow[ur, "\epsilon_A"] & A \wedge B \arrow[u] \arrow[d] \arrow[dr, "\overline{g}"] & \\
(p\epsilon_A)^* B \arrow[u] \arrow[ur, dashed] \arrow[r, "(p\epsilon_A)^B"] \arrow[d, dashed, "\overline{\boldsymbol{\epsilon}_p} g"] & B & A' \wedge B' \arrow[dl] \\
(p\epsilon_{A'})^* B \arrow[r, "(p\epsilon_{A'})^{B'}"] & B' &
\end{tikzcd}
$$

$$
\begin{tikzcd}
CA \arrow[r, "p\epsilon_A"] \arrow[d, "C\widehat{g}"'] & pA \arrow[d, "p\widehat{g}"] \\
CA' \arrow[r, "p\epsilon_{A'}"'] & pA'
\end{tikzcd}
$$

where we used the universal properties of the fibred product $A' \wedge B'$ and the cartesianity of $(p\epsilon_{A'})^{B'}$.

$$
\begin{tikzcd}
\mathcal{E}^p \arrow[r, "\overline{\boldsymbol{\epsilon}_p}"] \arrow[d, "\hat{p}"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{E} \arrow[r, "\widehat{\boldsymbol{\epsilon}_p}"'] & \mathcal{B}
\end{tikzcd}
$$

Furthermore, given a morphism $F: p \to q$ together with the natural isomorphism $\alpha: \mathrm{C}^q \circ \overline{F} \Rightarrow \widehat{F} \circ \mathrm{C}^p$ obtained as the mate of the inverse of $\theta: \overline{F} \circ \mathrm{T}^p \Rightarrow \mathrm{T}^q \circ \widehat{F}$, we define $\boldsymbol{\epsilon}_F$ by setting $\widehat{\boldsymbol{\epsilon}_F} := \alpha$ and $\overline{\boldsymbol{\epsilon}_F}_{(A,B)} := \gamma_{A,B} \circ \beta_{p\epsilon^p_A}$, where $\beta_{p\epsilon^p_A,B}: (\overline{F}((p\epsilon^p_A)^*B)) \to (\widehat{F}p\epsilon^p_A)^*\overline{F}B$ is an instance of the arrow defined in Remark 1.2.28, and $\gamma_{A,B}$ is the only arrow determined by cartesianity in the diagram

$$
\begin{tikzcd}
\overline{F}((p\epsilon^p_A)^*B) \arrow[r, "\beta_{p\epsilon^p_A,B}"] & (\widehat{F}p\epsilon^p_A)^*\overline{F}B \arrow[d, dashed, "\gamma_{A,B}"'] \arrow[dr, "(\widehat{F}p\epsilon^p_A)^{\overline{F}B}"] & \\
 & (q\epsilon^q_{\overline{F}A})^*\overline{F}B \arrow[r, "(q\epsilon^q_{\overline{F}A})^{\overline{F}B}"'] & \overline{F}B
\end{tikzcd}
$$

$$
\begin{tikzcd}
\widehat{F}\mathrm{C}^p A \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, "\widehat{F}p\epsilon^p_A"] & \\
\mathrm{C}^q\overline{F}A \arrow[r, "q\epsilon^q_{\overline{F}A}"'] & \widehat{F}pA = q\overline{F}A
\end{tikzcd}
$$

Proposition 2.2.12. $\boldsymbol{\epsilon}_p: \hat{p} \to p$ *is a LC-morphism.*

Proof. First of all it is a fibration morphism: $\overline{\boldsymbol{\epsilon}_p}$ is a functor since it is defined using a universal property, and $p \circ \overline{\boldsymbol{\epsilon}_p} = \widehat{\boldsymbol{\epsilon}_p} \circ \hat{p}$ by construction.

Now consider A in $\mathcal{E}$ over X. Since reindexing preserves terminals we have that the canonical arrow $\phi: \overline{\boldsymbol{\epsilon}_p}\mathrm{T}^{\hat{p}}A \to \mathrm{T}^p\mathrm{C}^pA$ is invertible.

It is not hard to see that the mate of ϕ^{-1} is invertible again. $\square$

Proposition 2.2.13. $\boldsymbol{\epsilon}\colon (\wedge_L \circ \boldsymbol{U}_{FLC}) \to \mathbf{Id}_{\boldsymbol{LComp}}$ *is a pseudo-natural transformation.*

Proof. By Definition 1.1.16 we only need to show that, given a 2-cell $\boldsymbol{\lambda}\colon F \to G$, one has $\boldsymbol{\lambda} * \boldsymbol{\epsilon}_F = \boldsymbol{\epsilon}_G * (\wedge_L \circ \boldsymbol{U}_{FLC})\boldsymbol{\lambda}$. By definition of **LComp** it is enough to check the equalities $\widehat{\boldsymbol{\lambda}} * \widehat{\boldsymbol{\epsilon}_F} = \widehat{\boldsymbol{\epsilon}_G} * \widehat{(\wedge_L \circ \boldsymbol{U}_{FLC})\boldsymbol{\lambda}}$ and $\overline{\boldsymbol{\lambda}} * \overline{\boldsymbol{\epsilon}_F} = \overline{\boldsymbol{\epsilon}_G} * \overline{(\wedge_L \circ \boldsymbol{U}_{FLC})\boldsymbol{\lambda}}$. These are easy to show. $\square$

To show that this are indeed unit and counit of a bi-adjunction, one still needs to show triangular identities.

Theorem 2.2.14. *The 2-functor* $\wedge_L$ *is left bi-adjoint to the forgetful 2-functor* $\boldsymbol{U}_{FLC}$.

Proof. Let p be a fibration with finite fibred products and consider the diagram

$$\begin{tikzcd}
\mathcal{E}^p \arrow[r, "\overline{\boldsymbol{\eta}_p}"] \arrow[d, "\hat{p}"'] \arrow[rr, bend left=50, "\mathrm{id}_{\mathcal{E}^p}"] & \mathcal{E}^{\hat{p}} \arrow[r, "\overline{\boldsymbol{\epsilon}_{\hat{p}}}"] \arrow[d, "\hat{\hat{p}}"'] \arrow[u, phantom, "\zeta^p\Uparrow"] & \mathcal{E}^p \arrow[d, "\hat{p}"'] \\
\mathcal{E} \arrow[r, "\widehat{\boldsymbol{\eta}_p}"] \arrow[rr, bend right=50, "\mathrm{id}_{\mathcal{E}}"'] & \mathcal{E}^p \arrow[r, "\widehat{\boldsymbol{\epsilon}_{\hat{p}}}"] \arrow[d, phantom, "\lambda^p\Downarrow"] & \mathcal{E}
\end{tikzcd}$$

where, whenever A is an object in $\mathcal{E}$ over X, λ^p_A is the unique vertical iso $\lambda^p_A\colon \mathrm{T}X \wedge A \to A$ and $\zeta^p_{(A,B)} := (\lambda^p_A, \lambda^p_A \wedge \delta_B^{-1})$, with $\delta_B\colon B \to (\mathrm{id}_X)^*B$ the unique arrow defined in Remark 1.2.18. It is only matter of tedious calculations to check the naturality of λ^p and ζ^p, and then the naturality of λ and ζ as p varies in **FPFib**.

Now let p be a fibration with Lawvere's comprehension. Let ϵ and η be respectively counit and unit of the terminal-comprehension adjunction. Then by Proposition 2.1.4 we have that $p\epsilon_{\mathrm{T}p}$ is a natural iso. This implies that also the family $\alpha := \{(p\epsilon_{\mathrm{T}pA})^A\}$ of cartesian liftings of $p\epsilon_{\mathrm{T}pA}$ at A is a natural iso.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] & B \arrow[d] \\
\mathrm{T}pA \arrow[r, "h"'] & \mathrm{T}pB
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{T}pA & \\
\mathrm{TCT}pA \arrow[ur, "\epsilon_{\mathrm{T}pA}"] & A \arrow[u] \arrow[d, "\mathrm{id}_A"] \arrow[dr, "g"] & \\
(p\epsilon_{\mathrm{T}pA})^*A \arrow[u] \arrow[ur, dashed, "\alpha_A"] \arrow[r, "\alpha_A"] \arrow[d, dashed] & A & B \arrow[dl, "\mathrm{id}_B"] \\
(p\epsilon_{\mathrm{T}pB})^*B \arrow[r, "\alpha_B"] & B &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{CT}pA \arrow[r, "p\epsilon_{\mathrm{T}pA}"] \arrow[d, "\mathrm{C}h"'] & p\mathrm{T}pA \arrow[d, "ph"] \\
\mathrm{CT}pB \arrow[r, "p\epsilon_{\mathrm{T}pB}"'] & p\mathrm{T}pB
\end{tikzcd}$$

Then we have the diagram

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "\overline{\boldsymbol{\eta}_p}"] \arrow[d, "p"'] \arrow[rr, bend left, "\mathrm{id}_{\mathcal{E}}"] & \mathcal{E}^p \arrow[r, "\overline{\boldsymbol{\epsilon}_p}"] \arrow[d, "\hat{p}"'] \arrow[u, phantom, "\alpha\Uparrow"] & \mathcal{E} \arrow[d, "p"'] \\
\mathcal{B} \arrow[r, "\widehat{\boldsymbol{\eta}_p}"] \arrow[rr, bend right, "\mathrm{id}_{\mathcal{B}}"'] & \mathcal{E} \arrow[r, "\widehat{\boldsymbol{\epsilon}_p}"] \arrow[d, phantom, "\eta^{-1}=p\epsilon_T\Downarrow"] & \mathcal{B}
\end{tikzcd}
$$

The two diagrams clearly express the triangular identities required for the bi-adjunction, so using Proposition 2.2.10 and Proposition 2.2.13 we conclude by Proposition 1.1.20 . □

Remark 2.2.15. Let $p: \mathcal{E} \to \mathcal{B}$ be a faithful fibration with finite fibred products. Then $\hat{p}$ is faithful: given $f: A \to C$ in $\mathcal{E}$, we have that an arrow $g: (A, B) \to (C, D)$ is mapped to f if and only if $\widehat{g} = f$ and $\overline{g}: A \wedge B \to C \wedge D$ is over pf. By faithfulness of p, we conclude that there is at most one such $\overline{g}$.

Proposition 2.2.16. *Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with finite fibred products, and $\widehat{F}: \mathcal{C} \to \mathcal{B}$ be a functor. Then the functor $\wedge_L$ applied to the pullback of p and $\widehat{F}$ is again a pullback.*

Proof. We observed in Proposition 1.3.14 that the pullback of p and $\widehat{F}$ is a 1-cell $F: F^\star p \to p$ in **FPFib**. Given a category $\mathcal{X}$ together with two functors $S: \mathcal{X} \to \mathcal{E}^p$ and $R: \mathcal{X} \to \mathcal{C} \underset{p,F}{\times} \mathcal{E}$ such that $\hat{p} \circ S = \overline{F} \circ (\widehat{F^\star p})$, there is a unique functor $U: \mathcal{X} \to \mathcal{E}^{F^\star p}$ such that $S = \overline{\widehat{F}} \circ U$ and $R = \widehat{F^\star p}$. Unicity is given by commutativity requirements, and one verifies that the unique possible choice is indeed a functor.

$$
\begin{tikzcd}
\mathcal{X} \arrow[dr, dashed, "U"] \arrow[drr, bend left, "S"] \arrow[ddr, bend right, "R"'] & & \\
& \mathcal{E}^{F^\star p} \arrow[r, "\overline{\widehat{F}}"] \arrow[d, "\widehat{F^\star p}"'] & \mathcal{E}^p \arrow[d, "\hat{p}"] \\
& \mathcal{C} \underset{p,F}{\times} \mathcal{E} \arrow[r, "\overline{F}"] \arrow[d, "F^\star p"'] & \mathcal{E} \arrow[d, "p"] \\
& \mathcal{C} \arrow[r, "\widehat{F}"'] & \mathcal{B}
\end{tikzcd}
$$

□

We just verified that the free construction we provided preserves change of base.

Example 2.2.17. Let $\mathcal{B}$ be a category with finite products, and consider the terminal fibration $\mathbf{!}_{\mathcal{B}}: \mathcal{B} \to \mathbf{1}$ defined in Example 1.2.5. Since it has finite fibred product we can apply the 2-functor $\wedge_L$ to it, obtaining a fibration $\widehat{\mathbf{!}_{\mathcal{B}}}: \mathcal{E}^{\mathbf{!}_{\mathcal{B}}} \to \mathcal{B}$. It is very easy to see that the fibration $\widehat{\mathbf{!}_{\mathcal{B}}}$ is isomorphic (in **FPLComp**) to the simple fibration $\mathrm{s}_{\mathcal{B}}: \mathrm{s}(\mathcal{B}) \to \mathcal{B}$ defined in Example 1.3.12.

Remark 2.2.18. Example 2.2.17 can be generalized: given any fibration with finite fibred products $p\colon \mathcal{E} \to \mathcal{B}$, one can consider the completion $\hat{p}\colon \mathcal{E}^p \to \mathcal{E}$ and an object X in $\mathcal{B}$. The fiber over X is isomorphic to the pullback of p and the constant functor $X\colon \mathbf{1} \to \mathcal{B}$.

$$\begin{tikzcd}
\mathcal{E}_X \arrow[r, hook] \arrow[d, "!"'] & \mathcal{E} \arrow[d, "p"] \\
\mathbf{1} \arrow[r, hook, "X"'] & \mathcal{B}
\end{tikzcd}$$

By Proposition 2.2.16 applying the 2-functor $\wedge_L$ to this 1-cell yields a pullback. Its left leg $\hat{!}$ is isomorphic to the simple fibration $\mathrm{s}_{\mathcal{E}_X}$ on $\mathcal{E}_X$. So it appears how the completion acts fiberwise as the simple fibration construction.

$$\begin{tikzcd}
\mathcal{E}^! \arrow[r, hook] \arrow[d, "\hat{!}"'] & \mathcal{E}^p \arrow[d, "\hat{p}"] \\
\mathcal{E}_X \arrow[r, hook] & \mathcal{E}
\end{tikzcd}$$

Example 2.2.19. Let $\mathcal{B}$ be a category with finite products, and consider the fibration dom defined in Example 1.2.6. This has finite fibred products: a fibred terminal object over X is given by the unique arrow $X \to \mathrm{T}$ and a fibred product of $f\colon X \to Y$ and $g\colon X \to Z$ is given by $\langle f, g\rangle\colon X \to Y \times Z$. So we can apply the Lawvere completion $\wedge_L$ to it to get a fibration $\widehat{\mathrm{dom}}\colon \mathcal{E}^{\mathrm{dom}} \to \mathcal{B}^2$ with Lawvere comprehension. The objects in the total category are pairs of arrows (f, g) with the same domain $f\colon X \to Y$ and $g\colon X \to Z$. An arrow $(f, g) \to (f', g')$ is a triple of arrows (h, t, s) making the following diagram commute:

$$\begin{tikzcd}
& X' \arrow[rr, "\langle f',g'\rangle"] \arrow[dd] & & Y' \times Z' \arrow[dd, "\pi_{Y'}"] \\
X \arrow[ur, "h"] \arrow[rr, "\langle f,g\rangle"] \arrow[dd] & & Y \times Z \arrow[ur, "s"'] \arrow[dd, "\pi_Y"] & \\
& X' \arrow[rr, "f'"] & & Y' \\
X \arrow[ur, "h"] \arrow[rr, "f"'] & & Y \arrow[ur, "t"'] &
\end{tikzcd}$$

The comprehension of a pair (f, g) is then their pairing $\langle f, g\rangle\colon X \to Y \times Z$.

Example 2.2.20. Consider the fibration $U\colon \mathcal{Mod} \to \mathcal{Ring}$ of Example 1.2.4. Applying the completion yields the fibration $\hat{U}\colon \mathcal{E}^U \to \mathcal{Mod}$. Objects in $\mathcal{E}^U$ are pairs (M, N) of modules over the same ring A. Vertical arrows $(M, N) \to (M, K)$ correspond to A-module homomorphisms $f\colon M \oplus N \to K$. In particular, given arrows $f\colon M \oplus N \to K$ and $g\colon M \oplus K \to P$, their composition is given by $g \circ (\pi_M \oplus f)\colon M \oplus N \to P$.

$$M \oplus N \xrightarrow{\langle \pi_M, f\rangle} M \oplus K \xrightarrow{\quad g \quad} P$$

Example 2.2.21. Consider the fibration $\mathcal{K}\colon \mathcal{Real} \to \mathcal{Set}$ of Example 1.2.25. It has finite fibred products, so we can apply $\wedge_L$ to it getting a fibration $\hat{\mathcal{K}}\colon \mathcal{E}^{\mathcal{K}} \to$

$\mathcal{R}eal$. Objects in $\mathcal{E}^{\mathcal{K}}$ are pairs (a,b) of functions $a,b\colon X \to \mathcal{P}(\mathbb{N})$. There is a (unique) arrow $(a,b) \to (c,d)$ over $g\colon a \to c$ if and only if $a \le c \circ |g|$ and $a \wedge b \le d \circ |g|$. In particular for any morphism $(a,b) \to (c,d)$ we have the existence of two partial recursive functions $\phi, \psi\colon \mathbb{N} \to \mathbb{N}$ such that for each $x \in X$ the set $a(x)$ is a subset of $\phi^{-1}[c(|g|(x))]$ and $(a \wedge b)(x) \subseteq \psi^{-1}[d(|g|(x))]$.

$$\begin{tikzcd} X \arrow[r, "|g|"] \arrow[d, "a"'] \arrow[dr, phantom, "\subseteq"] & Y \arrow[d, "c"] \\ \mathcal{P}(\mathbb{N}) & \mathcal{P}(\mathbb{N}) \arrow[l, "\mathcal{P}(\phi)=\phi^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "|g|"] \arrow[d, "a\wedge b"'] \arrow[dr, phantom, "\subseteq"] & Y \arrow[d, "d"] \\ \mathcal{P}(\mathbb{N}) & \mathcal{P}(\mathbb{N}) \arrow[l, "\mathcal{P}(\psi)=\psi^{-1}"] \end{tikzcd}$$

Chapter 3

Jacobs comprehension

Jacobs introduces comprehension categories in [7] to study type dependencies from a categorical point of view. Since a declaration of type can only come together with a context, one sees that a natural way to think of types is as fibred over contexts: from a type declaration $\Gamma \vdash \sigma\colon \mathrm{Type}$, one gets the context Γ. Moreover, one would also like to be able to perform context extension, i.e. to pass from a judgement of type $\Gamma \vdash \sigma\colon \mathrm{Type}$ to the extended context $\Gamma, x\colon \sigma$. Finally, one may also link these two with a projection $\Gamma, x\colon \sigma \to \Gamma$ which forgets the new type σ. All this structure is essentially captured in the definition of comprehension category (Definition 3.1.1).

3.1 Comprehension categories: definition and first properties

In this section we provide the definition of comprehension category, together with examples, first of all the syntactic one coming from a dependent type theory. In the end we also define the 2-category of comprehension categories (Definition 3.1.5).

Definition 3.1.1 ([7, Def. 4.1]). A **comprehension category** is a fibration $p\colon \mathcal{E} \to \mathcal{B}$ together with a functor $\chi^p\colon \mathcal{E} \to \mathcal{B}^2$ such that $\mathrm{cod} \circ \chi^p = p$ and that χ^p preserves cartesian arrows, i.e. f cartesian in $\mathcal{E}$ implies $\chi^p f$ is a pullback in $\mathcal{B}$. The functor χ^p is called **comprehension functor**.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\chi^p"] \arrow[d, "p"'] & \mathcal{B}^2 \arrow[dl, "\mathrm{cod}"] \\ \mathcal{B} & \end{tikzcd}$$

For simplicity we will omit the fibration index whenever it will be clear from the context.

A **full comprehension category** is a comprehension category $p\colon \mathcal{E} \to \mathcal{B}$ such that the comprehension functor χ is full and faithful.

This terminology is strictly correlated to the fibration of types that we discussed in Example 1.2.9.

Example 3.1.2. Consider the term model fibration defined in Example 1.2.9. The functor $\chi\colon \mathcal{E} \to \mathcal{B}^2$ is described by $\chi(\Gamma \vdash \sigma\colon \mathrm{Type}) := (\Gamma, x\colon \sigma) \to \Gamma$, so it sends a judgement of type in a context to the projection from the extended context to the old one. Explicitly, it is the list of variables of Γ, that is a list of terms in the extended context. With this definition we have a comprehension category (see Example 1.2.9).

Let us notice that a section of χA is, by definition, a list of terms in context Γ such that its postcomposition with χA is the identity. So this list must consist of all the variables from Γ, plus a term of type A (again in context Γ). Hence sections of χA correspond to terms of type A.

Furthermore, given a type A and a type B in context Γ, one can consider the weakening of B in the extended context $\Gamma, x\colon A$, and the morphism over χA whose second component is just the variable of type B. It is easy to check that this morphism is cartesian.

A vertical arrow $(\Gamma \vdash \sigma\colon \mathrm{Type}) \to (\Gamma \vdash \tau\colon \mathrm{Type})$ is a term $N\colon \tau$ in context $\Gamma, x\colon \sigma$. We call it a proof term since it represents a proof that τ follows from σ: given any term of type σ in context Γ, we can substitute it in the proof term to get a term of type τ in context Γ. This operation is semantically represented by the composition in the total category.

Guided by the syntactic example, we will often use the following terminology. We will call **types** objects of the total category, and **terms of type** A the sections of the comprehension χA of A. Then we will call **proof terms** vertical arrows in $\mathcal{E}$, and a proof term is said **global** if its domain is terminal in the fiber.

Moreover one can consider a cartesian lifting $\mathrm{w}_A B\colon \mathrm{w}_A^* B \to B$ of χA at B. We call $\mathrm{w}_A^* A$ the **weakening of B along A**.

Consider then the pullback $\chi \mathrm{w}_A A$. We call **generic element of type** A the unique arrow g_A given by the universal property of the pullback on the pair of identities $\mathrm{id}_{\mathrm{C}A}$.

$$\begin{tikzcd}
\mathrm{C}A \arrow[dr, dashed, "\mathrm{g}_A"] \arrow[drr, bend left, "\mathrm{id}_{\mathrm{C}A}"] \arrow[ddr, bend right, "\mathrm{id}_{\mathrm{C}A}"'] & & \\
& \mathrm{Cw}_A^* A \arrow[r, "\chi \mathrm{w}_A^* A"] \arrow[d, "\mathrm{Cw}_A A"] & \mathrm{C}A \arrow[d, "\chi A"] \\
& \mathrm{C}A \arrow[r, "\chi A"'] & X
\end{tikzcd}$$

Example 3.1.3. The fibration cod of Example 1.2.7, together with the identity $\mathrm{Id}_{\mathcal{B}^2}$, is trivially a full comprehension category. A generalization to this is given by taking a family of arrows closed under pullback and considering the full subcategory of $\mathcal{B}^2$ on these arrows. A particular case of this is when the family of arrows is the family of the monos. In this case the subcategory corresponds to the category of subobjects of $\mathcal{B}$.

Definition 3.1.4. Given two comprehension categories $p\colon \mathcal{E} \to \mathcal{B}$ and $q\colon \mathcal{E}' \to \mathcal{B}'$, a **morphism of comprehension categories** from p to q consists of a fibration morphism $F\colon p \to q$ together with a natural isomorphism $\alpha\colon (\chi^q \circ \overline{F}) \Rightarrow (\widehat{F}^2 \circ \chi^p)$ such that $\operatorname{cod}\alpha = \mathrm{i}_{\widehat{F}\circ p}$.

Given two morphisms of comprehension categories $(F,\alpha): p \to q$ and $(G,\beta): q \to s$, their composition is given by $(G \circ F, \beta * \alpha)$, where $\beta * \alpha = (\widehat{G}^2\alpha) \circ (\beta\overline{F})$ is the whiskering.

$$
\begin{tikzcd}
 & \mathcal{B}^2 \arrow[rr, "\widehat{F}^2"] \arrow[dl, bend right, "\mathrm{cod}"'] & & {\mathcal{B}'}^2 \arrow[rr, "\widehat{G}^2"] \arrow[dl, bend right, "\mathrm{cod}"'] & & {\mathcal{B}''}^2 \arrow[dl, bend left, "\mathrm{cod}"] \\
\mathcal{E} \arrow[ur, "\chi^p"] \arrow[rr, "\overline{F}"] \arrow[d, "p"'] & & \mathcal{E}' \arrow[ur, "\chi^q"] \arrow[rr, "\overline{G}"] \arrow[d, "q"] \arrow[ul, phantom, "\Leftarrow\alpha"] & & \mathcal{E}'' \arrow[ur, "\chi^s"'] \arrow[d, "s"] \arrow[ul, phantom, "\Leftarrow\beta"] & \\
\mathcal{B} \arrow[rr, "\widehat{F}"'] & & \mathcal{B}' \arrow[rr, "\widehat{G}"'] & & \mathcal{B}'' &
\end{tikzcd}
$$

So a morphism between comprehension categories is a fibration morphism that preserves comprehensions up to (a specified) iso.

Definition 3.1.5. Let $F, G: p \to q$ together with α, β be morphisms of comprehension categories. A 2-cell of comprehension categories $(F,\alpha) \Rightarrow (G,\beta)$ is a 2-cell $\phi: F \Rightarrow G$ in **Fib** such that $\chi^q\overline{\phi} \circ \alpha^{-1} = \beta^{-1} \circ \widehat{\phi}^2\chi^p$.

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "\overline{F}"] \arrow[r, bend right, "\overline{G}"'] \arrow[r, phantom, "\Downarrow\widehat{\phi}"] \arrow[d, "p"'] & \mathcal{E}' \arrow[d, "q"] \\
\mathcal{B} \arrow[r, bend left, "\widehat{F}"] \arrow[r, bend right, "\widehat{G}"'] \arrow[r, phantom, "\Downarrow\overline{\phi}"] & \mathcal{B}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}^2 \arrow[r, bend left, "\widehat{F}^2"] \arrow[dr, phantom, "\Downarrow\alpha^{-1}"] & {\mathcal{B}'}^2 \\
\mathcal{E} \arrow[u, "\chi^p"] \arrow[r, bend left, "\overline{F}"] \arrow[r, bend right, "\overline{G}"'] \arrow[r, phantom, "\Downarrow\widehat{\phi}"] & \mathcal{E}' \arrow[u, "\chi^q"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}^2 \arrow[r, bend left, "\widehat{F}^2"] \arrow[r, bend right, "\widehat{G}^2"'] \arrow[r, phantom, "\Downarrow\widehat{\phi}^2"] \arrow[dr, phantom, "\Downarrow\beta^{-1}"] & {\mathcal{B}'}^2 \\
\mathcal{E} \arrow[u, "\chi^p"] \arrow[r, bend right, "\overline{G}"'] & \mathcal{E}' \arrow[u, "\chi^q"']
\end{tikzcd}
$$

We denote with **JComp** the 2-category of comprehension categories.

3.2 From Lawvere to Jacobs

In this section we exploit the differences between fibrations with Lawvere comprehension and comprehension categories. In particular, we show that every fibration with Lawvere comprehension yields a comprehension category in Theorem 3.2.1. Furthermore, we show that this assignation extends to a 2-functor $\boldsymbol{LJ}$: **LComp** $\to$ **JComp**, and we provide a characterization for its essential image in Theorem 3.2.4. Afterwards, we also show in Example 3.2.9 that the comprehension category structure over a fibration is not unique. In particular we consider the family fibration of pointed sets, used by Jacobs as an example of a comprehension category which has not Lawvere comprehension, and we equip it with a comprehension structure that turns it into a fibration with Lawvere comprehension.

Theorem 3.2.1 ([7, Def. 4.12])**.** *Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere comprehension. Then (p, χ^p) is a comprehension category, where $\chi^p: \mathcal{E} \to \mathcal{B}^2$ is defined by $\chi^p(A) := p\epsilon^p_A$, with ϵ^p the counit of the comprehension-terminal adjunction. Furthermore this assignation extends to a 2-functor **LJ**: **LComp** $\to$ **JComp**.*

Proof. Of course $\mathrm{cod} \circ \chi^p = p$. So we only need to verify that if $f: A \to B$ is cartesian, then χf is a pullback in $\mathcal{B}$. Consider a pair of arrows $g: Z \to \mathrm{C}B$ and $h: Z \to X$, where $X = pA$, such that $\chi B \circ g = pf \circ h$. The transpose

$\epsilon_B \circ \mathrm{T}g\colon \mathrm{T}Z \to B$ of g is over $\chi B \circ g = pf \circ h$, so by cartesianity of f there is a unique $s\colon \mathrm{T}Z \to A$ over h such that $f \circ s = \epsilon_B \circ \mathrm{T}g$. This yields a unique arrow $s^{\#}\colon Z \to \mathrm{C}A$ by taking the transpose of s. Furthermore we have the following

$$\mathrm{C}f \circ s^{\#} = \mathrm{C}(f \circ s) \circ \eta_Z = \mathrm{C}(\epsilon_B \circ \mathrm{T}g) \circ \eta_Z = g$$

$$\chi A \circ s^{\#} = p(\epsilon_A \circ \mathrm{TC}s \circ \mathrm{T}\eta_Z) = p(s \circ \epsilon_{\mathrm{T}Z} \circ \mathrm{T}\eta_Z) = p(s) = h$$

For the first equation we used the characterization of transposes via unit and counit, and for the second we used also the naturality of the counit on s.

$$\begin{tikzcd}
\mathrm{TCT}Z \arrow[rr, "\epsilon_{\mathrm{T}Z}"] \arrow[dr, "\mathrm{TC}s"'] & & \mathrm{T}Z \arrow[rr, "\mathrm{T}g"] \arrow[dr, dashed, "s"'] & & \mathrm{TC}B \arrow[dr, "\epsilon_B"] & \\
& \mathrm{TC}A \arrow[rr, "\epsilon_A"'] & & A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[drr, bend left, "h"] \arrow[dr, dashed, "s^{\#}"] \arrow[ddr, bend right, "g"'] & & \\
& \mathrm{C}A \arrow[r, "\chi A"] \arrow[d, "\mathrm{C}f"'] & X \arrow[d, "pf"] \\
& \mathrm{C}B \arrow[r, "\chi B"'] & Y
\end{tikzcd}$$

Now, let $F\colon p \to q$ together with $\theta\colon \overline{F} \circ \mathrm{T}^p \Rightarrow \mathrm{T}^q \circ \widehat{F}$ be a 1-cell in **LComp**. In order to show that F is a morphism in **JComp** as well we only need to define a natural isomorphism $\alpha\colon (\widehat{F}^2 \circ \chi^p) \Rightarrow (\chi^q \circ \overline{F})$ such that $\operatorname{cod}\alpha = \mathrm{i}_{\widehat{F}p}$. Let A be an object in $\mathcal{E}$ over X and consider the square

$$\begin{tikzcd}
\mathrm{T}^q\widehat{F}\mathrm{C}^pA \arrow[r, "\theta^{-1}_{\mathrm{C}^pA}"] \arrow[d, "\mathrm{T}^q(\theta^{-1})^{\#}_A"'] & \overline{F}\mathrm{T}^p\mathrm{C}^pA \arrow[d, "\overline{F}\epsilon^p_A"] \\
\mathrm{T}^q\mathrm{C}^q\overline{F}A \arrow[r, "\epsilon^q_{\overline{F}A}"'] & \overline{F}A
\end{tikzcd}$$

It commutes by Proposition 1.1.6. Applying q to this gives a commutative square in $\mathcal{B}'$ whose top side is the identity, since $\operatorname{cod}\theta = \mathrm{i}_{\mathrm{Id}_{\mathcal{B}'}}$. Then we can set $\alpha_A := ((\theta^{-1})^{\#}_A, \mathrm{id}_{\widehat{F}X})$. It is a natural iso since both its components are invertible.

$$\begin{tikzcd}
\widehat{F}\mathrm{C}^pA \arrow[r, "\mathrm{id}_{\widehat{F}\mathrm{C}^pA}"] \arrow[d, "(\theta^{-1})^{\#}_A"'] & \widehat{F}\mathrm{C}^pA \arrow[d, "\widehat{F}p\epsilon^p_A"] \\
\mathrm{C}^q\overline{F}A \arrow[r, "q\epsilon^q_{\overline{F}A}"'] & \widehat{F}X
\end{tikzcd}$$

Finally its action on the 2-cells is given by the identity. In fact, by Remark 2.1.13 one has that 2-cells of **LComp** preserve comprehensions and terminals. It is not hard to see that they satisfy the coherence required. ☐

Remark 3.2.2. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere comprehension. The square obtained by applying p to the naturality square of the counit is a pullback in $\mathcal{B}$, since χ^p preserves cartesian arrows.

Remark 3.2.3. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with Lawvere comprehension. Then the terms of p corresponds bijectively to global proof terms via transposition. In fact, given A in $\mathcal{E}$ over X and a section $f\colon X \to \mathrm{C}A$ of $\chi A\colon \mathrm{C}A \to X$, its transpose is $f^{\#} = \epsilon_A \circ \mathrm{T}f\colon \mathrm{T}X \to A$. This is a global proof term since it is vertical (f is a section) and it is from a terminal object.

$$\begin{tikzcd}
\mathrm{T}X \arrow[r, "\mathrm{T}f"'] \arrow[rr, bend left, "f^{\#}"] & \mathrm{TC}A \arrow[r, "\epsilon_A"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left, "\mathrm{id}_X"] & \mathrm{C}A \arrow[r, "\chi A"'] & X
\end{tikzcd}$$

Furthermore, if we transpose a global proof term $g\colon \mathrm{T}Y \to B$, we get $g^{\#} = \mathrm{C}G \circ \eta_Y\colon Y \to \mathrm{C}B$. Its postcomposition with χB is the identity: consider the naturality square of the counit on g. Applying p to it yields the equality $\chi B \circ \mathrm{C}g = \chi \mathrm{T}Y$. We observed in Proposition 2.1.4 that the unit η_Y is invertible, and that its inverse is given by $p\epsilon_{\mathrm{T}Y} = \chi \mathrm{T}Y$, thus proving that $g^{\#}$ is a section of χB.

$$\begin{tikzcd}
\mathrm{TCT}Y \arrow[r, "\epsilon_{\mathrm{T}Y}"] \arrow[d, "\mathrm{TC}g"'] & \mathrm{T}Y \arrow[d, "g"] \\
\mathrm{TC}B \arrow[r, "\epsilon_B"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"'] \arrow[rr, bend left, "\mathrm{id}_Y"] & \mathrm{CT}Y \arrow[r, "\chi \mathrm{T}Y"'] \arrow[d, "\mathrm{C}g"'] & Y \arrow[d, "\mathrm{id}_Y"] \\
& \mathrm{C}B \arrow[r, "\chi B"'] & Y
\end{tikzcd}$$

Theorem 3.2.4. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a comprehension category together with a terminal object functor T. Then p can be extended to a fibration with Lawvere comprehension if and only if the following conditions hold:*

1. *Given an object X in $\mathcal{B}$, there is a section $s_X\colon X \to \mathrm{CT}X$ of the comprehension $\chi \mathrm{T}X$;*

2. *Given A over X and a section $t\colon X \to \mathrm{C}A$ of the comprehension χA, there exist a unique vertical arrow $t^{\#}\colon \mathrm{T}X \to A$ such that $\mathrm{C}t^{\#} \circ s_X = t$.*

$$\begin{tikzcd}
A \\
\mathrm{T}X \arrow[u, dashed, "t^{\#}"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "t"] \arrow[dr, "s_X"'] & \mathrm{C}A \arrow[r, "\chi A"] & X \\
& \mathrm{CT}X \arrow[u, "\mathrm{C}t^{\#}"] \arrow[ur, "\chi \mathrm{T}X"'] &
\end{tikzcd}$$

Proof. One implication follows by Proposition 2.1.4 and Remark 3.2.3. For the converse, suppose that the conditions hold. We want to show that there is an adjunction $\mathrm{C} \vdash \mathrm{T}$. We start by defining the natural transformation $\eta \colon \mathrm{Id}_{\mathcal{B}} \to \mathrm{CT}$ whose components are the sections $\eta_X := s_X$. Now we can define the counit $\epsilon \colon \mathrm{TC} \to \mathrm{Id}_{\mathcal{E}}$. First, fix a cleavage of p and consider the generic element of type A, g_A. It is by definition a section of the comprehension $\chi \mathrm{w}_A^* A$, so by hypothesis we get a unique vertical arrow $\mathrm{g}_A^{\#} \colon \mathrm{TC}A \to \mathrm{w}_A^* A$ such that $\mathrm{C}\mathrm{g}_A^{\#} \circ s_X = \mathrm{g}_A$. Finally, we define $\epsilon_A := \mathrm{w}_A A \circ \mathrm{g}_A^{\#}$.

$$
\begin{tikzcd}
\mathrm{TC}A \arrow[d, dashed, "\mathrm{g}_A^{\#}"'] \arrow[dr, "\epsilon_A"] & \\
\mathrm{w}_A^* A \arrow[r, "\mathrm{w}_A A"'] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{C}A \arrow[dr, dashed, "\mathrm{g}_A"] \arrow[drr, bend left, "\mathrm{id}_{\mathrm{C}A}"] \arrow[ddr, bend right, "\mathrm{id}_{\mathrm{C}A}"'] & & \\
& \mathrm{C}\mathrm{w}_A^* A \arrow[r, "\chi A"] \arrow[d, "\mathrm{C}\mathrm{w}_A A"] & \mathrm{C}A \arrow[d, "\chi A"] \\
& \mathrm{C}A \arrow[r, "\chi A"'] & X
\end{tikzcd}
$$

This definition does not depend on the particular choice of cleavage: by Remark 1.2.18 there is a unique vertical iso between two different choices of a cleavage, and its mediation with the different reindexing functors does not change the composition. Triangular identities are easy to show. For X in $\mathcal{B}$, we have that $\epsilon_{\mathrm{T}X} = \mathrm{T}\chi\mathrm{T}X$. Then one has $\epsilon_{\mathrm{T}X} \circ \mathrm{T}\eta_X = \mathrm{id}_{\mathrm{T}X}$ since η_X is a section of $\chi\mathrm{T}X$. Instead, for A in $\mathcal{E}$, we have that $\mathrm{C}\epsilon_A \circ \eta_{\mathrm{C}A} = \mathrm{id}_{\mathrm{C}A}$ by definition of ϵ_A. $\square$

Corollary 3.2.5. *Let $p \colon \mathcal{E} \to \mathcal{B}$ be a full comprehension category together with a terminal object functor* T *such that the comprehension functor χ preserves fibred terminal objects. Then it can be extended to a fibration with Lawvere comprehension.*

Proof. We want to use the characterization given in Theorem 3.2.4. First, given X in $\mathcal{B}$ the comprehension $\chi\mathrm{T}X$ has a section: since χ preserves fibred terminal objects, $\chi\mathrm{T}X$ is fibred terminal with respect to cod. Then there exist a unique vertical arrow $\mathrm{id}_X \to \chi X$, that corresponds exactly to a section s_X of $\chi\mathrm{T}X$.

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{id}X"] \arrow[d, dashed, "s_X"'] & X \arrow[d, dashed, "\mathrm{id}_X"] \\
\mathrm{CT}X \arrow[r, "\chi\mathrm{T}X"'] & X
\end{tikzcd}
$$

Secondly, given A over X and a section $t \colon X \to \mathrm{C}A$ of the comprehension χA, we know that $(t \circ \chi\mathrm{T}X) \circ s_X = t$. Furthermore, we have that $\chi A \circ (t \circ \chi\mathrm{T}X) = \chi\mathrm{T}X$, so $(t \circ \chi\mathrm{T}X, \mathrm{id}_X) \colon \chi\mathrm{T}X \to \chi A$ is a morphism in $\mathcal{B}^2$. Then there exist a unique $f \colon \mathrm{T}X \to A$ such that $\chi f = (t \circ \chi\mathrm{T}X, \mathrm{id}_X)$ since χ is full and faithful. We conclude by Theorem 3.2.4. $\square$

Corollary 3.2.6. *Consider the comprehension category given by a family of arrows closed under pullback in $\mathcal{B}$ (see Example 3.1.3). If we moreover suppose that the family contains the identities, then it can be extended to a fibration with Lawvere comprehension.*

Proof. This comprehension category satisfies the conditions of Corollary 3.2.5. $\square$

Example 3.2.7 ([8, Exs. 10.4.8])**.** Consider the category $\mathcal{S}et_*$ of pointed sets and the family fibration $\mathrm{Fam}_{\mathcal{S}et_*}\colon \mathrm{Fam}(\mathcal{S}et_*) \to \mathcal{S}et$ described in Example 1.2.8. This fibration, together with the functor $\chi\colon \mathrm{Fam}(\mathcal{S}et_*) \to \mathcal{S}et^2$ that maps $(I, \{X_i\}_{i\in I})$ to $\coprod_{i\in I} X_i \to I$, is a comprehension category (see [8]). It also has a fibred terminal object functor T, since $\mathcal{S}et_*$ has a terminal object $(\{*\}, *)$. We can see that this fibration cannot be extended to a fibration with Lawvere comprehension: although the first condition of the characterization holds, the second is not satisfied. Indeed, given a set I, one has that $\chi \mathrm{T}X$ is an isomorphism, so it has a section. But given an object $(I, \{X_i\}_{i\in I})$ in $\mathrm{Fam}(\mathcal{S}et_*)$ over I there is a unique vertical arrow $\mathrm{T}X \to (I, \{X_i\}_{i\in I})$, while in general there are different sections of $\coprod_{i\in I} X_i \to I$.

Proposition 3.2.8. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration with fibred zero-object 0^p. Then the triple $(p, 0^p, p)$ is a fibration with Lawvere comprehension.*

Proof. Since 0^p is a fibred zero-object we have both $p \dashv 0^p$ and $0^p \dashv p$. Furthermore 0^p is a section of p. $\square$

Example 3.2.9. Consider again the family fibration $\mathrm{Fam}_{\mathcal{S}et_*}\colon \mathrm{Fam}(\mathcal{S}et_*) \to \mathcal{S}et$. By Proposition 3.2.8 we have that the triple $(\mathrm{Fam}_{\mathcal{S}et_*}, \mathrm{T}, \mathrm{Fam}_{\mathcal{S}et_*})$ is a fibration with Lawvere comprehension. This example, together with Example 3.2.7 and Theorem 3.2.1, shows that it is possible to put different comprehension structures over the same fibration. In particular in this example we get a comprehension category by applying $\boldsymbol{LJ}$, while we showed that the comprehension category in Example 3.2.7 is not in the essential image of $\boldsymbol{LJ}$. This puts in evidence one fundamental difference between Lawvere comprehension and Jacobs comprehension: while there is at most one possible structure of Lawvere comprehension (up to iso) over a fibration p, there are possibly many different structures of Jacobs comprehension.

In Section 3.3 we will provide also an example of a comprehension category in which the first condition of the characterization does not hold.

3.3 The free comprehension category over a fibration

There is an obvious forgetful 2-functor $\boldsymbol{U}_J\colon \mathbf{JComp} \to \mathbf{Fib}$ which forgets the comprehension structure. Our aim is to provide a left bi-adjoint to this, in order to build the free comprehension category over an arbitrary fibration (see Theorem 3.3.28).

Consider the following pullback in $\mathcal{C}at$

$$\begin{tikzcd} \mathcal{E}^p \arrow[r] \arrow[d, "\hat{p}"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{FFP}(p) \arrow[r, "ffp(p)"'] & \mathcal{B} \end{tikzcd}$$

By Proposition 1.2.34 $\hat{p}$ is a fibration. We will show that this yields the free comprehension category over p.

Remark 3.3.1. Objects in $\mathcal{E}^p$ are pairs $((X, \vec{A}), A_{n+1})$ with the first element in $\mathcal{FFP}(p)$ (see Definition 1.4.1) and the second in $\mathcal{E}$ over X. The morphisms in this category are pairs of arrows (f, g) such that $pg = |f|$. Furthermore, $\hat{p}$ is the first projection.

Remark 3.3.2. As for the completion given in Chapter 2, this construction preserves faithfulness: if p is a faithful fibration, then $\hat{p}$ is faithful again. It also preserves fibrations with finite fibred products by Proposition 1.3.14. Moreover, it preserves pullbacks: if we apply it to a 1-cell in **Fib** that is a pullback, then the morphism between the free categories is a pullback again. This claim follows easily from the following property of pullbacks (Exercise 3.1.viii in [13]): consider a commutative rectangle

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

whose right-hand square is a pullback. Then the left-hand square is a pullback if and only if the composite rectangle is a pullback.

Notation. Let $g\colon \underline{m} \to \underline{n}$ be a function. We denote with $g+1$ the function $g+1\colon \underline{m+1} \to \underline{n+1}$ such that $(g+1)\restriction_{\underline{m}} = g$ and $(g+1)(m+1) = n+1$.

Lemma 3.3.3. *There is a functor $\chi^{\hat{p}}\colon \mathcal{E}^p \to \mathcal{FFP}(p)^2$ defined by the following data:*

- *It maps objects $Y = ((X, \vec{A}), A_{n+1})$ into $\chi^{\hat{p}}Y\colon (X, (\vec{A}, A_{n+1})) \to (X, \vec{A})$ given by*

$$|\chi^{\hat{p}}Y| = \mathrm{id}_X, \qquad \underline{\chi^{\hat{p}}Y} = \underline{n} \hookrightarrow \underline{n+1}, \qquad \widetilde{\chi^{\hat{p}}Y} = \{\mathrm{id}_{A_i}\}_{i\in\underline{n}}.$$

- *it maps arrows $f\colon Y \to Z$, where $f = (g, h)$ and $Z = ((X', \vec{B}), B_{m+1})$, into the square*

$$\begin{tikzcd} (X, (\vec{A}, A_{n+1})) \arrow[r, "f^+"] \arrow[d, "\chi^{\hat{p}}Y"'] & (X', (\vec{B}, B_{m+1})) \arrow[d, "\chi^{\hat{p}}Z"] \\ (X, \vec{A}) \arrow[r, "g"'] & (X', \vec{B}) \end{tikzcd}$$

where $f^+ := (|g|, \underline{g}+1, \widetilde{g} \sqcup h)$. Sometimes we will use the notation (g, f^+) for $\chi^{\hat{p}}f$, since the vertical sides of the square are clear by the context.

Proof. It is easy to see that the following square commutes.

$$\begin{tikzcd}
(X, (\vec{A}, A_{n+1})) \arrow[r, "f^+"] \arrow[d, "\chi^{\hat{p}} Y"'] & (X', (\vec{B}, B_{m+1})) \arrow[d, "\chi^{\hat{p}} Z"] \\
(X, \vec{A}) \arrow[r, "g"'] & (X', \vec{B})
\end{tikzcd}$$

Furthermore given a pair of composable arrows f, f' one has that $f^+ \circ f'^+ = (f \circ f')^+$ and $\mathrm{id}_Y^+ = \mathrm{id}_{\mathrm{dom}(\chi^{\hat{p}} Y)}$. This implies that $\chi^{\hat{p}} \colon \mathcal{E}^p \to \mathcal{FFP}(p)^2$ is a functor.

□

Proposition 3.3.4. *Let $p \colon \mathcal{E} \to \mathcal{B}$ be a fibration. Then $\hat{p}$ together with $\chi^{\hat{p}} \colon \mathcal{E}^p \to \mathcal{FFP}(p)^2$ is a comprehension category.*

Proof. By construction $\mathrm{cod} \circ \chi^{\hat{p}} = \hat{p}$.

Now we only need to show that $\chi^{\hat{p}}$ preserves cartesian arrows. Let $f = (f_1, f_2)$ be cartesian in $\mathcal{E}^p$. By Lemma 1.2.33 f_2 is cartesian. Consider two arrows $g \colon (X'', \vec{C}) \to (X, \vec{A})$ and $h \colon (X'', \vec{C}) \to (X', (\vec{B}, B_{m+1}))$ such that $f_2 \circ g = \chi^{\hat{p}} Z \circ h$.

Then one can define $u \colon (X'', \vec{C}) \to (X, (\vec{A}, A_{n+1}))$ as follows:

$|u| = |g|$;

$\underline{u} \colon \underline{n+1} \to \underline{k}$ defined by $\underline{u}(i) = \underline{g}(i)$ for $i \in \underline{n}$ and $\underline{u}(n+1) = \underline{h}(m+1)$;

$\widetilde{u} = \widetilde{g} \sqcup (\widetilde{h})_{m+1}$.

Clearly u is the unique arrow that makes the diagram below to commute, rendering $\chi^{\hat{p}} f$ a pullback.

$$\begin{tikzcd}
(X'', \vec{C}) \arrow[dr, dashed, "u"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& (X, (\vec{A}, A_{n+1})) \arrow[r, "f^+"] \arrow[d, "\chi^{\hat{p}} Y"'] & (X', (\vec{B}, B_{m+1})) \arrow[d, "\chi^{\hat{p}} Z"] \\
& (X, \vec{A}) \arrow[r, "f_2"'] & (X', \vec{B})
\end{tikzcd}$$

□

In this category reindexing along comprehensions are very well-behaved.

Remark 3.3.5. Let $p \colon \mathcal{E} \to \mathcal{B}$ be a fibration, B be in $\mathcal{E}$ over X and $((X, \vec{A}), A_{n+1})$ be in $\mathcal{E}^p$. Then the arrow $(\chi^{\hat{p}}((X, \vec{A}), A_{n+1}), \mathrm{id}_B)$ is cartesian by Lemma 1.2.33.

One can show that $\hat{p}$ is the free comprehension category over an arbitrary fibration p. In particular, one can extend this construction to a 2-functor $\wedge_J \colon \mathbf{Fib} \to \mathbf{JComp}$ that is left bi-adjoint to the forgetful 2-functor $\boldsymbol{U}_J \colon \mathbf{JComp} \to \mathbf{Fib}$.

Definition 3.3.6. Let $F \colon p \to p'$ be a morphism in **Fib**. Define $\widehat{\hat{F}} := \overline{\mathbf{ffp} F}$ and $\overline{\hat{F}}$ as the unique arrow given by the universal property of the pullback defining

$\mathcal{E}^q$, since the compositions $\mathit{ffp}(p') \circ \widehat{\hat{F}} \circ \hat{p} = p' \circ \overline{F} \circ \pi_2$.

$$
\begin{tikzcd}
\mathcal{E} \arrow[rrd, "\overline{F}"] & & \\
\mathcal{E}^p \arrow[u, "\pi_2"] \arrow[r, dashed, "\overline{\hat{F}}"] \arrow[d, "\hat{p}"'] \arrow[rd] & \mathcal{E}^q \arrow[r] \arrow[d, "\hat{q}"] & \mathcal{E}' \arrow[d, "q"] \\
\mathcal{FFP}(p) \arrow[r, "\widehat{\hat{F}}"'] \arrow[d, "\mathit{ffp}(p)"'] & \mathcal{FFP}(q) \arrow[r, "\mathit{ffp}(q)"] & \mathcal{B}' \\
\mathcal{B} \arrow[rru, "\widehat{F}"'] & &
\end{tikzcd}
$$

Now consider a 2-cell $\alpha\colon F \to G$ in **Fib**. Thanks to Remark 1.1.11 we can define $\widehat{\hat{\alpha}} := \overline{\mathbf{ffp}\alpha}$ and $\overline{\hat{\alpha}}$ as the unique 2-cell given by the universal property of the 2-pullback.

Lemma 3.3.7. *The assignations given in Definition 3.3.6 determine respectively a morphism of comprehension categories $\hat{F}\colon \hat{p} \to \hat{q}$ and a 2-cell $\hat{\alpha}\colon \hat{F} \to \hat{G}$.*

Proof. It is easy to see that $\overline{\hat{F}}$ preserves comprehensions strictly. In fact, let $f = (g, h)\colon ((X, \vec{A}), A_{n+1}) \to ((X', \vec{B}), B_{m+1})$ be an arrow in $\mathcal{E}^p$. Then $\chi^{\hat{q}} \circ \overline{\hat{F}}(f)$ is the following square:

$$
\begin{tikzcd}
(\widehat{F}X, (\overline{F}\vec{A}, \overline{F}A_{n+1})) \arrow[r, "\overline{\hat{F}}(f)^+"] \arrow[d, "\chi^{\hat{q}}\overline{\hat{F}}Y"] & (\widehat{F}X', (\overline{F}\vec{B}, \overline{F}B_{m+1})) \arrow[d, "\chi^{\hat{q}}\overline{\hat{F}}Z"] \\
(\widehat{F}X, \overline{F}\vec{A}) \arrow[r, "\widehat{\hat{F}}(g)"] & (\widehat{F}X', \overline{F}\vec{B})
\end{tikzcd}
$$

Instead, $\widehat{\hat{F}}^2 \circ \chi^{\hat{p}}(f)$ is the following:

$$
\begin{tikzcd}
(\widehat{F}X, (\overline{F}\vec{A}, \overline{F}A_{n+1})) \arrow[r, "\widehat{\hat{F}}(f^+)"] \arrow[d, "\widehat{\hat{F}}\chi^{\hat{p}}Y"] & (\widehat{F}X', (\overline{F}\vec{B}, \overline{F}B_{m+1})) \arrow[d, "\widehat{\hat{F}}\chi^{\hat{p}}Z"] \\
(\widehat{F}X, \overline{F}\vec{A}) \arrow[r, "\widehat{\hat{F}}(g)"] & (\widehat{F}X', \overline{F}\vec{B})
\end{tikzcd}
$$

First, $\chi^{\hat{q}}\overline{\hat{F}}Y = \widehat{\hat{F}}\chi^{\hat{p}}Y$ because every component is the same. Furthermore, $\overline{\hat{F}}(f)^+ = \widehat{\hat{F}}(f^+)$ again because every component is the same. This proves that $\hat{F}$ together with the identity natural transformation $\mathrm{i}_{\chi^{\hat{q}} \circ \overline{\hat{F}}}$ is a morphism of comprehension categories.

Moreover, it is easy to see also that $\hat{\alpha}\colon \hat{F} \to \hat{G}$ is a 2-cell in **Fib** and that the equality $\chi^{\hat{q}}\overline{\hat{\alpha}} = \widehat{\hat{\alpha}}^2 \chi^{\hat{p}}$ holds, proving that it is a 2-cell in **JComp**. $\square$

Remark 3.3.8. Explicitly, given $f\colon (X, \vec{A}) \to (X', \vec{B})$ in $\mathcal{FFP}(p)$ one has $\widehat{\hat{F}}(f)\colon (\widehat{F}X, \overline{F}\vec{A}) \to (\widehat{F}X', \overline{F}\vec{B})$ given by:

$|\widehat{\hat{F}}(f)| = \widehat{F}|f|$;

$\underline{\widehat{\hat{F}}(f)} = \underline{f}$;

$$\widetilde{\widehat{\overline{F}}(f)} = \{\overline{F}(\widetilde{f})_i\}_{i\in\underline{n}}.$$

Moreover, given $f = (g,h)\colon((X,\vec{A}),A_{n+1}) \to ((X',\vec{B}),B_{m+1})$ in $\mathcal{E}^p$, one has $\overline{\overline{F}}(f) = (\widehat{\overline{F}}(g),\overline{F}h)\colon((\widehat{F}X,\overline{F}\vec{A}),\overline{F}A_{n+1}) \to ((\widehat{F}X',\overline{F}\vec{B}),\overline{F}B_{n+1})$.

We are ready to see that $\wedge_J\colon \mathbf{Fib} \to \mathbf{JComp}$ is a 2-functor. The only thing we still need to show is that it well-behaves with the composition of 1-cells and 2-cells.

Proposition 3.3.9. *In the previous setting, $\wedge_J\colon \mathbf{Fib} \to \mathbf{JComp}$ is a 2-functor.*

Proof. In Lemma 3.3.7 we showed that maps 1-cells and 2-cells of **Fib** to 1-cells and 2-cells of **JComp**, respectively. The fact that $\wedge_J\colon \mathbf{Fib} \to \mathbf{JComp}$ is functorial (on both 1-cells and 2-cells) follows from the universal property of the 2-pullback. □

Example 3.3.10. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration, together with a fibred terminal object functor T. Then its completion $\hat{p}$ is an example of comprehension category with fibred terminal objects which does not satisfy the first condition of Theorem 3.2.4. In fact, given $(X,())$ in $\mathcal{FFP}(p)$ one has that $\chi^{\hat{p}}\mathrm{T}^{\hat{p}}(X,())$ does not have sections, since there are no functions $\underline{0} \to \underline{1}$.

Example 3.3.11. Let $\mathcal{B}$ be a category, and consider the terminal fibration of Example 1.2.5. We can apply the completion $\wedge_J$ to it to get a fibration $\hat{\mathbf{!}}_{\mathcal{B}}\colon \mathcal{E}^{\mathbf{!}_{\mathcal{B}}} \to \mathcal{FFP}(\mathbf{!}_{\mathcal{B}})$. Objects in $\mathcal{FFP}(\mathbf{!}_{\mathcal{B}})$ are finite lists of objects of $\mathcal{B}$. Arrows $\vec{A} \to \vec{B}$, where $\vec{A}$ has length n and $\vec{B}$ has length m, are pairs of a function $g\colon \underline{m} \to \underline{n}$ and a family of morphisms $\{f_i\colon A_{g(i)\to B_i}\}_{i\in\underline{m}}$. In particular, objects can be thought as formal finite products of objects in $\mathcal{B}$, and arrows as morphisms between the products in which each component factors through one projection. The objects in the total category corresponds to finite non-empty lists of objects of $\mathcal{B}$, and arrows correspond to a morphism in $\mathcal{FFP}(\mathbf{!}_{\mathcal{B}})$ between the lists without the last element, together with an arrow in $\mathcal{B}$ between the last elements. In this case we are considering objects as non-empty products and arrows as before, with the condition that the last component of the morphism factors through the last projection of the domain, and the other components factor through projections different from the last one.

The difference with Lawvere completion is in the definition of arrows: in this case we have that morphisms in the total category correspond to morphisms between the products which do not use the last component, plus a morphism between the last component. Instead, in Lawvere completion we can consider arrows depending on a parameter, so we can use all the factors of the product in the domain to go in the last component of the codomain product. In particular to obtain Lawvere completion we have to consider more arrows than we do for Jacobs completion. This reflects the fact that a fibration with Lawvere comprehension carries the structure of a comprehension category, while the converse is not always true.

3.3.1 Towards the bi-adjunction

Now we can begin to show that $\wedge_J\colon \mathbf{Fib} \to \mathbf{JComp}$ and $\boldsymbol{U}_{JC}\colon \mathbf{JComp} \to \mathbf{Fib}$ are a bi-adjoint pair. The first step to do this is defining the unit and the counit of the bi-adjunction.

Definition 3.3.12. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration. We define a fibration morphism $\boldsymbol{\eta}_p\colon p \to \boldsymbol{U}_{JC}\hat{p}$ by setting $\widehat{\boldsymbol{\eta}_p}\colon \mathcal{B} \to \mathcal{FFP}(p)$ as the fibred terminal object functor $\mathrm{T}^{\mathit{ffp}(p)}$, and $\overline{\boldsymbol{\eta}_p}\colon \mathcal{E} \to \mathcal{E}^p$ sending an object A to itself over its basis, i.e. to $((pA, ()), A)$. Its action on arrows is defined trivially.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\overline{\boldsymbol{\eta}_p}"] \arrow[d, "p"'] & \mathcal{E}^p \arrow[d, "\hat{p}"] \\
\mathcal{B} \arrow[r, "\widehat{\boldsymbol{\eta}_p}"'] & \mathcal{FFP}(p)
\end{tikzcd}$$

Now let $F\colon p \to q$ be a fibration morphism. We define $\boldsymbol{\eta}_F$ as the identity 2-cell $\mathrm{i}_{\boldsymbol{\eta}_q \circ F}$.

Remark 3.3.13. We could have defined $\boldsymbol{\eta}_p$ equivalently using the universal property of the 2-pullback applied to $\mathrm{T}^{\mathit{ffp}(p)}$ and $\mathrm{Id}_{\mathcal{E}}$. Furthermore, the 2-cell $\boldsymbol{\eta}_F$ is well defined, since $\widehat{\boldsymbol{\eta}_q} \circ \widehat{F} = \widehat{\hat{F}} \circ \widehat{\boldsymbol{\eta}_p}$ and $\overline{\boldsymbol{\eta}_q} \circ \overline{F} = \overline{\hat{F}} \circ \overline{\boldsymbol{\eta}_p}$.

Proposition 3.3.14. *$\boldsymbol{\eta}$ is a pseudo-natural transformation.*

Proof. Let $F\colon p \to q$ be a fibration morphism, and consider the diagram

$$\begin{tikzcd}
& \mathcal{E}' \arrow[rr, "\overline{\boldsymbol{\eta}_q}"] \arrow[dd, "q", near start] & & \mathcal{E}^q \arrow[dd, "\hat{q}"] \\
\mathcal{E} \arrow[ur, "\overline{F}"] \arrow[rr, "\overline{\boldsymbol{\eta}_p}", near end] \arrow[dd, "p"'] & & \mathcal{E}^p \arrow[ur, "\overline{\hat{F}}"'] \arrow[dd, "\hat{p}", near start] \arrow[u, phantom, "\Leftarrow \boldsymbol{\eta}_F"] & \\
& \mathcal{B}' \arrow[rr, "\widehat{\boldsymbol{\eta}_q}", near start] & & \mathcal{FFP}(q) \\
\mathcal{B} \arrow[ur, "\widehat{F}"] \arrow[rr, "\widehat{\boldsymbol{\eta}_p}"'] & & \mathcal{FFP}(p) \arrow[ur, "\widehat{\hat{F}}"'] \arrow[u, phantom, "\Leftarrow \widehat{\boldsymbol{\eta}_F}"] &
\end{tikzcd}$$

It is easy to see that $\boldsymbol{\eta}_q \circ F = \hat{F} \circ \boldsymbol{\eta}_p$, which implies that $\boldsymbol{\eta}_F$ is an invertible 2-cell. Furthermore, the coherence axiom required for the pseudo-naturality is automatically satisfied since $\boldsymbol{\eta}_F$ is the identical 2-cell. $\square$

We are now going to define the counit $\boldsymbol{\epsilon}$ of the bi-adjunction. This is delicate since we define the functors involved by induction on the length of the list on which they apply.

Definition 3.3.15. Let $\chi\colon \mathcal{E} \to \mathcal{B}^2$ be a comprehension category, and fix a cleavage of $p = \mathrm{cod} \circ \chi$. First, given an object $(X, \vec{A})$ with length n we define a family of arrows (and their domains) $c_k^0\colon \widehat{\boldsymbol{\epsilon}_p}(X, \vec{A}\restriction_k) \to X$ for any $k \leq n$ by induction on k:

k=0: $c_0^0 = \mathrm{id}_X\colon X \to X$;

k+1: Let A_{k+1}^* be the reindexing of A_{k+1} along c_k^0; in particular, let $i_{k+1}\colon A_{k+1}^* \to A_{k+1}$ be cartesian over c_k^0. Then $c_{k+1}^0 = c_k^0 \circ \chi A_{k+1}^*$.

Then one can define $c_k^i\colon \widehat{\boldsymbol{\epsilon}_p}(X, \vec{A}\restriction_k) \to \widehat{\boldsymbol{\epsilon}_p}(X, \vec{A}\restriction_i)$ for $i \leq k$ by induction on $k-i$:

k-i=0: $c_k^k = \mathrm{id}_{(X,\vec{A}\restriction_k)}$;

k-i+1: $c_k^{i-1} = \chi A_i^* \circ c_k^i$.

Let us notice that the second definition coincide with the first one if $i = 0$. Moreover, one has that arrows c_k^i well-behave under composition in the following sense.

Lemma 3.3.16. *For any $i \leq k \leq j$ the composition $c_k^i \circ c_j^k$ is equal to c_j^i.*

Proof. It is easy to see by induction. $\square$

Intuitively, the object $\widehat{\boldsymbol{\epsilon}}_p(X, \vec{A})$ is given by taking the reindexing of A_1 along the identity, then the reindexing of A_2 along the comprehension of (the reindexing of) A_1, and so on until we get to the domain of the comprehension of (the reindexing of) A_n.

Lemma 3.3.17. *Consider an arrow $f\colon (X, \vec{A}) \to (X', \vec{B})$. We define $\widehat{\boldsymbol{\epsilon}}_p f$ by induction on the length of $\vec{B}$:*

m=0: *$\widehat{\boldsymbol{\epsilon}}_p f = |f| \circ c_n^0\colon \widehat{\boldsymbol{\epsilon}}_p(X, \vec{A}) \to X'$;*

m+1: *Consider the square χi_m in $\mathcal{B}$: this is a pullback by the second property of comprehension categories. Then consider the diagram*

$$\begin{tikzcd}
 & & & \widehat{\boldsymbol{\epsilon}}_p(X, \vec{A}\restriction_{\underline{f}(m+1)}) \arrow[dl] \arrow[d, "c^{\underline{f}(m+1)-1}_{\underline{f}(m+1)}"] \\
\widehat{\boldsymbol{\epsilon}}_p(X, \vec{A}) \arrow[urrr, bend left, "c_n^{\underline{f}(m+1)}"] \arrow[dr, dashed, "\widehat{\boldsymbol{\epsilon}}_p f"'] \arrow[ddr, bend right, "\widehat{\boldsymbol{\epsilon}}_p h"'] & & \mathrm{dom}\cdot\chi A_{\underline{f}(m+1)} \arrow[d, "\mathrm{dom}\cdot\chi(\widetilde{f})_{m+1}"'] \arrow[dr] & \widehat{\boldsymbol{\epsilon}}_p(X, \vec{A}\restriction_{\underline{f}(m+1)-1}) \arrow[d, "c^0_{\underline{f}(m+1)-1}"] \\
 & \widehat{\boldsymbol{\epsilon}}_p(X', (\vec{B}, B_{m+1})) \arrow[r] \arrow[d, "\chi B^*_{m+1}"] & \mathrm{dom}\cdot\chi B_{m+1} \arrow[d, "\chi B_{m+1}"'] & X \arrow[dl, "|f|"'] \\
 & \widehat{\boldsymbol{\epsilon}}_p(X', \vec{B}) \arrow[r, "c^0_m"] & X' &
\end{tikzcd}$$

with $h = (\chi^{\hat{p}}((X', \vec{B}), B_{m+1}) \circ f)$ and $\widehat{\boldsymbol{\epsilon}}_p f$ is the unique arrow given by the universal property of the pullback. One just need to show that $c_m^0 \circ \widehat{\boldsymbol{\epsilon}}_p h = |f| \circ c_n^0$ in each step.

Proof. By induction on m, it suffices to show that $c_m^0 \circ \widehat{\boldsymbol{\epsilon}}_p f = |f| \circ c_n^0$, since this would imply that $c_m^0 \circ \widehat{\boldsymbol{\epsilon}}_p h = |f| \circ c_n^0$ in the inductive step.

m=0: $c_m^0 \circ \widehat{\boldsymbol{\epsilon}}_p f = |f| \circ c_n^0$ by definition;

m+1: by inductive hypothesis one has that $c_m^0 \circ \widehat{\boldsymbol{\epsilon}}_p h = |h| \circ c_n^0$. Clearly $|f| = |h|$, so one can define $\widehat{\boldsymbol{\epsilon}}_p f$ by the universal property of the pullback. Since $\chi B_{m+1}^* = c_{m+1}^m$, one has that

$$c_{m+1}^0 \circ \widehat{\boldsymbol{\epsilon}}_p f = c_m^0 \circ \widehat{\boldsymbol{\epsilon}}_p h = |h| \circ c_n^0 = |f| \circ c_n^0$$

proving the claim.

□

The definition of $\widehat{\epsilon_p}$ is clearly functorial, so we have a functor $\widehat{\epsilon_p}\colon \mathcal{FFP}(p) \to \mathcal{B}$.

Definition 3.3.18. Let $Y = ((X, \vec{A}), A_{n+1})$ be in $\mathcal{E}^p$. We define $\overline{\epsilon_p}(Y)$ as the reindexing of A_{n+1} along c^0_n. Given also $Z = ((X', \vec{B}), B_{m+1})$ and an arrow $f = (g, h)\colon Y \to X$, we define $\overline{\epsilon_p}f$ as the unique arrow given by cartesianity of the cartesian lifting of c^0_m at B_{m+1} over $\widehat{\epsilon_p}g$.

$$\begin{tikzcd}
\overline{\epsilon_p}Y \arrow[r] \arrow[d, dashed, "\overline{\epsilon_p}f"'] & A_{n+1} \arrow[d, "h"] \\
\overline{\epsilon_p}Z \arrow[r] & B_{m+1}
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\epsilon_p}(X, \vec{A}) \arrow[r, "c^0_n"] \arrow[d, "\widehat{\epsilon_p}g"'] & X \arrow[d, "|g|"] \\
\widehat{\epsilon_p}(X', \vec{B}) \arrow[r, "c^0_m"'] & X'
\end{tikzcd}$$

Proposition 3.3.19. *The pair of functors $\epsilon_p = (\overline{\epsilon_p}, \widehat{\epsilon_p})$ defined above is a fibration morphism $\epsilon_p\colon \hat{p} \to p$.*

Proof. We only need to show that $\overline{\epsilon_p}$ maps cartesian arrows to cartesian arrows. So consider a cartesian arrow $(f, g)\colon ((X, \vec{A}), A_{n+1}) \to ((Y, \vec{B}), B_{m+1})$. By Lemma 1.2.33 we have that $g\colon A_{n+1} \to B_{m+1}$ is cartesian (over $|f|$). Given then $h\colon \overline{\epsilon_p}((X, \vec{A}), A_{n+1}) \to A_{n+1}$ and $k\colon \overline{\epsilon_p}((Y, \vec{B}), B_{m+1}) \to B_{m+1}$ the cartesian liftings of, respectively, c^0_n and c^0_m, we have that $\overline{\epsilon_p}f$ is given by cartesianity of k. Then $c^0_m \circ \overline{\epsilon_p}f = |f| \circ c^0_n$ is cartesian by Lemma 1.2.11, and we conclude that $\overline{\epsilon_p}f$ is cartesian by Lemma 1.2.12. □

We are now about to show that ϵ_p is a morphism between comprehension categories. In particular, we will define the natural isomorphism $\alpha\colon (\widehat{\epsilon_p}^2 \circ \chi^{\hat{p}}) \Rightarrow (\chi^p \circ \overline{\epsilon_p})$ of Definition 3.1.4 to be the identity 2-cell. To do so, we need to show that the diagram below commutes.

$$\begin{tikzcd}
\mathcal{E}^p \arrow[r, "\overline{\epsilon_p}"] \arrow[d, "\chi^{\hat{p}}"'] & \mathcal{E} \arrow[d, "\chi^p"] \\
\mathcal{FFP}(p)^2 \arrow[r, "\widehat{\epsilon_p}^2"'] & \mathcal{B}^2
\end{tikzcd}$$

Remark 3.3.20. Given an arrow $f = (g, h)\colon Y = ((X, \vec{A}), A_{n+1}) \to Z = ((X', \vec{B}), B_{m+1})$, one has $\chi\overline{\epsilon_p}(f)\colon \chi A^*_{n+1} \to \chi B^*_{m+1}$, so it is the square

$$\begin{tikzcd}
\mathrm{dom}\cdot\chi A^*_{n+1} \arrow[r, "\mathrm{dom}\cdot\chi h^*"] \arrow[d, "\chi A^*_{n+1}"'] & \mathrm{dom}\cdot\chi B^*_{m+1} \arrow[d, "\chi B^*_{m+1}"] \\
\widehat{\epsilon_p}(X, \vec{A}) \arrow[r, "\widehat{\epsilon_p}g"'] & \widehat{\epsilon_p}(X', \vec{B})
\end{tikzcd}$$

Analogously, $\widehat{\boldsymbol{\epsilon}_p}^2\chi^{\hat{p}}(f)$ is the square

$$\begin{tikzcd}
\widehat{\boldsymbol{\epsilon}_p}(X,(\vec{A},A_{n+1})) \arrow[r, "\widehat{\boldsymbol{\epsilon}_p}f^{+}"] \arrow[d, "\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y"'] & \widehat{\boldsymbol{\epsilon}_p}((X',(\vec{B},B_{m+1})) \arrow[d, "\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Z"] \\
\widehat{\boldsymbol{\epsilon}_p}(X,\vec{A}) \arrow[r, "\widehat{\boldsymbol{\epsilon}_p}g"'] & \widehat{\boldsymbol{\epsilon}_p}(X',\vec{B})
\end{tikzcd}$$

Now we only need to prove that $\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y = \chi A^*_{n+1}$ and that $\operatorname{dom}\cdot\chi h^* = \widehat{\boldsymbol{\epsilon}_p}f^{+}$. These will be shown in the following lemmas.

Lemma 3.3.21. *Let* $Y = ((X,\vec{A}),A_{n+1})$ *in* $\mathcal{E}^p$. *Then* $\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y = c^n_{n+1}$.

Proof. By induction on n.

n=0: By definition, $\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y = |\chi^{\hat{p}}Y| \circ c^0_{n+1} = c^0_1$;

n+1: Let $Z = ((X,\vec{A}\restriction_n),A_{n+1})$. By inductive hypothesis $\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Z = c^n_{n+1}$. By definition $\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y$ is the only arrow defined by the universal property of the pullback

$$\begin{tikzcd}
 & & & & \widehat{\boldsymbol{\epsilon}_p}(X,\vec{A}) \arrow[dl] \arrow[d, "c^n_{n+1}"] \\
\widehat{\boldsymbol{\epsilon}_p}(X,(\vec{A},A_{n+2})) \arrow[urrrr, bend left, "c^{n+1}_{n+2}"] \arrow[drr, dashed, "\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y"] \arrow[ddrr, bend right, "\widehat{\boldsymbol{\epsilon}_p}h"'] & & & \operatorname{dom}\cdot\chi A_{n+1} \arrow[d, "\mathrm{id}_{\operatorname{dom}\cdot\chi A_{n+1}}"'] \arrow[dr] & \widehat{\boldsymbol{\epsilon}_p}(X,\vec{A}\restriction_n) \arrow[d, "c^0_n"] \\
 & & \widehat{\boldsymbol{\epsilon}_p}(X,\vec{A}) \arrow[r] \arrow[d, "\chi A^*_{n+1}"] & \operatorname{dom}\cdot\chi A_{n+1} \arrow[d, "\chi A_{n+1}"'] & X \arrow[dl, "\mathrm{id}_X"'] \\
 & & \widehat{\boldsymbol{\epsilon}_p}(X,\vec{A}\restriction_n) \arrow[r, "c^0_n"] & X &
\end{tikzcd}$$

where $h = \chi^{\hat{p}}Z \circ \chi^{\hat{p}}Y$. So $\widehat{\boldsymbol{\epsilon}_p}h = \widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Z \circ \widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y = c^n_{n+1} \circ \chi^{\hat{p}}Y$. Since $\chi A^*_{n+1} = c^n_{n+1}$ one has that c^{n+1}_{n+2} makes the left triangle commute. It clearly makes also the right triangle to commute since the arrows are the same, so by the universal property of the pullback we have that $\widehat{\boldsymbol{\epsilon}_p}\chi^{\hat{p}}Y = c^{n+1}_{n+2}$.

$\square$

Lemma 3.3.22. *Let* $f = (g,h)\colon ((X,\vec{A}),A_{n+1}) \to ((X',\vec{B}),B_{m+1})$ *be an arrow in* $\mathcal{E}^p$. *Then* $\operatorname{dom}\cdot\chi h^* = \widehat{\boldsymbol{\epsilon}_p}f^{+}$.

Proof.

$$\begin{tikzcd}
& & & & \widehat{\boldsymbol{\epsilon}_p}(X,(\vec{A},A_{n+1})) \arrow[dl] \arrow[d, "c^{n}_{n+1}"] \\
\widehat{\boldsymbol{\epsilon}_p}(X,(\vec{A},A_{n+1})) \arrow[urrrr, bend left=10, "c^{n+1}_{n+1}"] \arrow[drr, dashed, "\widehat{\boldsymbol{\epsilon}_p} f^+"'] \arrow[ddrr, bend right, "\widehat{\boldsymbol{\epsilon}_p}(\chi^{\hat{p}} Z\circ f^+)"'] & & & \mathrm{dom}\cdot\chi A_{n+1} \arrow[d, "\mathrm{dom}\cdot\chi h"'] \arrow[dr] & \widehat{\boldsymbol{\epsilon}_p}(X,\vec{A}) \arrow[d, "c^0_n"] \\
& & \widehat{\boldsymbol{\epsilon}_p}(X',(\vec{B},B_{m+1})) \arrow[r] \arrow[d, "\chi B^*_{m+1}"] & \mathrm{dom}\cdot\chi B_{m+1} \arrow[d, "\chi B_{m+1}"'] & X \arrow[dl, "|g|"'] \\
& & \widehat{\boldsymbol{\epsilon}_p}(X',\vec{B}) \arrow[r, "c^0_m"] & X' &
\end{tikzcd}$$

Since $\mathrm{dom}\cdot\chi h^*$ makes the right triangle to commute, we only need to show that $\widehat{\boldsymbol{\epsilon}_p}(\chi^{\hat{p}} Z\circ f^+) = \chi B^*_{m+1}\circ(\mathrm{dom}\cdot\chi h^*)$. But one has $\widehat{\boldsymbol{\epsilon}_p}(\chi^{\hat{p}} Z\circ f^+) = \widehat{\boldsymbol{\epsilon}_p}(g\circ\chi^{\hat{p}} Y) = \widehat{\boldsymbol{\epsilon}_p}(g)\circ\chi A^*_{n+1} = \chi B^*_{m+1}\circ(\mathrm{dom}\cdot\chi h^*)$, where the first and the third equalities hold by commutativity of the two diagrams in Remark 3.3.20. Then by the universal property of the pullback one has that $\widehat{\boldsymbol{\epsilon}_p} f^+ = \mathrm{dom}\cdot\chi h^*$. □

Proposition 3.3.23. *The fibration morphism $\boldsymbol{\epsilon}_p$, together with the identity 2-cell, is a morphism in* ***JComp***.

Proof. It is enough to show $(\widehat{\boldsymbol{\epsilon}_p}^{\,2}\circ\chi^{\hat{p}}) = (\chi^p\circ\overline{\boldsymbol{\epsilon}_p})$. This is a straightforward consequence of Remark 3.3.20, Lemma 3.3.21 and Lemma 3.3.22. □

Definition 3.3.24. Let $F\colon p\to q$ together with $\alpha\colon(\widehat{F}^2\circ\chi^p)\Rightarrow(\chi^q\circ\overline{F})$ be a morphism in **JComp**, and $((X,\vec{A}),A_{n+1})$ an object in $\mathcal{E}^p$, and fix a cleavage of p and q, respectively. We define simultaneously $(\widehat{\boldsymbol{\epsilon}_F})_{(X,\vec{A})}$ and $(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}),A_{n+1})}$ by induction on the length n of $\vec{A}$. For $1\le k\le n$, consider the families of cartesian arrows $i_k\colon A^*_k\to A_k$ over c^0_{k-1} and $j_k\colon(\overline{F}A_k)^*\to\overline{F}A_k$ cartesian over d^0_{k-1}, where we denote with d^i_j the maps defined in Definition 3.3.15 w.r.t. q.

For $n=0$, we set $(\widehat{\boldsymbol{\epsilon}_F})_{(X,())} := \mathrm{id}_{\widehat{F}X}$ and $(\overline{\boldsymbol{\epsilon}_F})_{((X,()),A_1)}$ as the unique vertical arrow obtained by cartesianity. Notice that $(\overline{\boldsymbol{\epsilon}_F})_{((X,()),A_1)}$ is over $(\widehat{\boldsymbol{\epsilon}_F})_{(X,())}$. The latter is trivially invertible. The former is invertible since $\overline{F}$ preserves cartesianity.

$$\begin{tikzcd}
\overline{F}A_1^* \arrow[d, dashed, "(\overline{\boldsymbol{\epsilon}_F})_{((X,()),A_{n+1})}"'] \arrow[dr, "\overline{F}i_1"] & \\
(\overline{F}A_1)^* \arrow[r, "j_1"'] & \overline{F}A_1
\end{tikzcd}$$

$$\widehat{F}X \xrightarrow{\mathrm{id}_{\widehat{F}X}} \widehat{F}X$$

For $n+1$, we set $(\widehat{\boldsymbol{\epsilon}_F})_{(X,\vec{A})} := \mathrm{C}^q(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}\restriction_n),A_{n+1})}\circ\psi_{A^*_{n+1}}$, where $\psi\colon\widehat{F}\mathrm{C}^p\Rightarrow\mathrm{C}^q\overline{F}$ is the whiskering $\mathrm{dom}\,\alpha$. It is iso since it is composition of two invertible morphisms ($(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}\restriction_n),A_{n+1})}$ is iso by inductive hypothesis). Then we set $(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}),A_{n+2})}$ as the unique arrow over $(\widehat{\boldsymbol{\epsilon}_F})_{(X,\vec{A})}$ given by cartesianity. Again, this is invertible because $\overline{F}$ preserves cartesian arrows.

$$\begin{tikzcd}
\overline{F}A^*_{n+2} \arrow[d, dashed, "(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}),A_{n+2})}"'] \arrow[dr, "\overline{F}i_{n+2}"] & \\
(\overline{F}A_{n+2})^* \arrow[r, "j_{n+2}"'] & \overline{F}A_{n+2}
\end{tikzcd}$$

$$\begin{tikzcd}
& \widehat{F}\mathrm{C}^pA^*_{n+1} \arrow[dl, "\psi_{A^*_{n+1}}"'] \arrow[dd, "(\widehat{\boldsymbol{\epsilon}_F})_{(X,\vec{A})}"'] \arrow[dr, "\widehat{F}c^0_{n+1}"] & \\
\mathrm{C}^q\overline{F}A^*_{n+1} \arrow[dr, "\mathrm{C}^q(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}\restriction_n),A_{n+1})}"'] & & \widehat{F}X \\
& \mathrm{C}^q(\overline{F}A_{n+1})^* \arrow[ur, "d^0_{n+1}"'] &
\end{tikzcd}$$

Let us show that the downside diagram commutes, allowing us to use cartesianity. By Lemma 3.3.16 and using the definition of c^n_{n+1}, we know that $c^0_n \circ \chi^p A^*_{n+1} = c^0_{n+1}$, and analogously $d^0_n \circ \chi^q(\overline{F}A_{n+1})^* = d^0_{n+1}$. Furthermore, we have $\chi^q\overline{F}A^*_{n+1} \circ \psi_{A^*_{n+1}} = \widehat{F}\chi^p A^*_{n+1}$ since it is a component of α, and $\chi^q(\overline{F}A^*_{n+1}) \circ \mathrm{C}^q(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}\restriction_n),A_{n+1})} = (\widehat{\boldsymbol{\epsilon}_F})_{(X,\vec{A}\restriction_n)} \circ \chi^q\overline{F}A^*_{n+1}$ since it is image of $(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}\restriction_n),A_{n+1})}$ under χ^q. Finally, we have $d^0_n \circ (\widehat{\boldsymbol{\epsilon}_F})_{(X,\vec{A}\restriction_n)} = \widehat{F}c^0_n$ by inductive hypothesis. These equalities let us conclude that the diagram below commutes.

$$\begin{tikzcd}
\widehat{F}\mathrm{C}^pA^*_n \arrow[d, "(\widehat{\boldsymbol{\epsilon}_F})_{(X,\vec{A}\restriction_n)}"'] \arrow[dr, "\widehat{F}c^0_n"] & \widehat{F}\mathrm{C}^pA^*_{n+1} \arrow[l, "\widehat{F}\chi^pA^*_{n+1}"'] \arrow[r, "\psi_{A^*_{n+1}}"] \arrow[d, "\widehat{F}c^0_{n+1}"] & \mathrm{C}^q\overline{F}A^*_{n+1} \arrow[d, "\mathrm{C}^q(\overline{\boldsymbol{\epsilon}_F})_{((X,\vec{A}\restriction_n),A_{n+1})}"] \arrow[ll, bend right, "\chi^q\overline{F}A^*_{n+1}"'] \\
\mathrm{C}^q(\overline{F}A_n)^* \arrow[r, "d^0_n"'] & \widehat{F}X & \mathrm{C}^q(\overline{F}A_{n+1})^* \arrow[l, "d^0_{n+1}"] \arrow[ll, bend left, "\chi^q(\overline{F}A_{n+1})^*"]
\end{tikzcd}$$

Proposition 3.3.25. *$\boldsymbol{\epsilon}$ is a pseudo-natural transformation.*

Proof. Let $(F,G)\colon p \to q$ together with $\alpha\colon(\widehat{F}^2 \circ \chi^p) \Rightarrow (\chi^q \circ \overline{F})$ be a morphism in **JComp**, and consider the following diagram:

$$\begin{tikzcd}
& \mathcal{E}^q \arrow[rr, "\overline{\boldsymbol{\epsilon}_q}"] \arrow[dd, "\hat{q}"'] & & \mathcal{E}' \arrow[dd, "q"] \\
\mathcal{E}^p \arrow[ur, "\overline{\hat{F}}"] \arrow[rr, "\overline{\boldsymbol{\epsilon}_p}"] \arrow[dd, "\hat{p}"'] & & \mathcal{E} \arrow[ur, "\overline{F}"'] \arrow[dd, "p"'] \arrow[u, phantom, "\Uparrow \overline{\boldsymbol{\epsilon}_F}"] & \\
& \mathcal{FFP}(q) \arrow[rr, "\widehat{\boldsymbol{\epsilon}_q}"] & & \mathcal{B}' \\
\mathcal{FFP}(p) \arrow[ur, "\widehat{F}"] \arrow[rr, "\widehat{\boldsymbol{\epsilon}_p}"'] & & \mathcal{B} \arrow[ur, "\widehat{F}"'] \arrow[u, phantom, "\Uparrow \widehat{\boldsymbol{\epsilon}_F}"] &
\end{tikzcd}$$

We have that $\boldsymbol{\epsilon}_F$ is an invertible 2-cell by construction. With tedious calculations can be shown that the coherence axiom required for the pseudo-naturality is satisfied. $\square$

Proposition 3.3.26. *Let $p\colon \mathcal{E} \to \mathcal{B}$ together with $\chi^p\colon \mathcal{E} \to \mathcal{B}^2$ be a comprehension category, and consider the diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\overline{\boldsymbol{\eta}_p}"'] \arrow[d, "p"'] \arrow[rr, bend left, "\mathrm{Id}_{\mathcal{E}}"] & \mathcal{E}^p \arrow[r, "\overline{\boldsymbol{\epsilon}_p}"'] \arrow[d, "\hat{p}"] \arrow[u, phantom, "\Uparrow\overline{\alpha_p}"] & \mathcal{E} \arrow[d, "p"'] \\
\mathcal{B} \arrow[r, "\widehat{\boldsymbol{\eta}_p}"] \arrow[rr, bend right, "\mathrm{Id}_{\mathcal{B}}"'] & \mathcal{FFP}(p) \arrow[r, "\widehat{\boldsymbol{\epsilon}_p}"] \arrow[d, phantom, "\Downarrow\widehat{\alpha_p}"] & \mathcal{B}
\end{tikzcd}$$

where $\widehat{\alpha_p} := \mathrm{i}_{\mathrm{Id}_{\mathcal{B}}}$ and $\overline{\alpha_p} := \delta^{-1}$, with $\delta\colon \mathrm{Id}_{\mathcal{E}} \Rightarrow \mathrm{id}^*$ the natural isomorphism of Remark 1.2.18. Then $\alpha\colon (\wedge_J \circ \boldsymbol{U}_{JC}) \Rightarrow \mathbf{Id}_{\boldsymbol{JComp}}$ is an invertible modification.

Proof. First, we need to show that $\widehat{\boldsymbol{\epsilon}_p} \circ \widehat{\boldsymbol{\eta}_p} = \mathrm{Id}_{\mathcal{B}}$. This is just a straightforward consequence of the definitions of $\widehat{\boldsymbol{\eta}_p}$ and the base case of $\widehat{\boldsymbol{\epsilon}_p}$.

Afterwards, it is enough to show that $\delta^{-1}\colon \overline{\boldsymbol{\epsilon}_p} \circ \overline{\boldsymbol{\eta}_p} \Rightarrow \mathrm{Id}_{\mathcal{E}}$, since δ^{-1} is trivially invertible. But this is again obvious by their definition: $\overline{\boldsymbol{\epsilon}_p \boldsymbol{\eta}_p} A = \overline{\boldsymbol{\epsilon}_p}((pA, ()), A) = (c_0^0)^* A = \mathrm{id}^*_{(pA)} A$.

Finally, we need to show the naturality of α with respect to the 1-cells. $\square$

Proposition 3.3.27. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a fibration, and consider the diagram*

$$\begin{tikzcd}
\mathcal{E}^p \arrow[r, "\overline{\hat{\boldsymbol{\eta}}_p}"] \arrow[d, "\hat{p}"'] \arrow[rr, bend left, "\mathrm{Id}_{\mathcal{E}^p}"] & \mathcal{E}^{\mathcal{E}^p} \arrow[r, "\overline{\boldsymbol{\epsilon}_{\hat{p}}}"] \arrow[d, "\hat{\hat{p}}"] \arrow[u, phantom, "\Uparrow\overline{\beta_p}"] & \mathcal{E}^p \arrow[d, "\hat{p}"] \\
\mathcal{FFP}(p) \arrow[r, "\widehat{\hat{\boldsymbol{\eta}}_p}"'] \arrow[rr, bend right, "\mathrm{Id}_{\mathcal{FFP}(p)^2}"'] & \mathcal{FFP}(\hat{p}) \arrow[r, "\widehat{\boldsymbol{\epsilon}_{\hat{p}}}"'] \arrow[d, phantom, "\Downarrow\widehat{\beta_p}"] & \mathcal{FFP}(p)
\end{tikzcd}$$

where $\beta_p := \mathrm{id}_{\hat{p}}$. Then β is an invertible modification.

Proof. We only need to show that $\boldsymbol{\epsilon}_{\hat{p}} \circ \hat{\boldsymbol{\eta}}_p = \mathrm{id}_{\hat{p}}$. This is a straightforward consequence of the definition of $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$ and of Remark 3.3.5.

$\square$

Theorem 3.3.28. *The 2-functor $\wedge_J$ is left bi-adjoint to the 2-functor $\boldsymbol{U}_{JC}$.*

Proof. It is straightforward using Proposition 1.1.20, the pseudo-naturality of unit and counit proved in Proposition 3.3.14 and Proposition 3.3.25 and the triangular identities proved in Proposition 3.3.26 and Proposition 3.3.27. $\square$

Conclusions

Our work was mainly focused on building the completions for, respectively, Lawvere and Jacobs comprehension. In Section 3.3 we achieved the result for an arbitrary fibration. Instead, in Section 2.2 we achieved it only with the further assumption of finite fibred products in the fibration.

Furthermore in Section 3.2 we gave a characterization of the comprehension categories that are in the essential image of $\boldsymbol{LJ}$.

The diagram below recaps the actual situation.

$$
\begin{tikzcd}
\mathbf{FPLComp} \arrow[r, "\boldsymbol{U}_{FLC}"'] \arrow[d, "\boldsymbol{U}_{FPC}"'] \arrow[r, phantom, shift left=3, "\perp"] & \mathbf{FPFib} \arrow[l, bend right, "\wedge_L"'] \arrow[d, "\boldsymbol{U}_{FP}"'] \arrow[d, phantom, shift left=2, "\vdash"] \\
\mathbf{LComp} \arrow[r, "\boldsymbol{U}_{LC}"] \arrow[d, "\boldsymbol{LJ}"'] & \mathbf{Fib} \arrow[u, bend right, "\mathbf{ffp}"'] \arrow[dl, bend left, "\wedge_J"] \\
\mathbf{JComp} \arrow[ur, "\boldsymbol{U}_{JC}"] \arrow[ur, phantom, shift right=3, "\dashv"] &
\end{tikzcd}
$$

This thesis leaves some interesting questions that we did not investigate. For example one may try to build a left bi-adjoint to the forgetful 2-functors $\boldsymbol{U}_{LC}$ and $\boldsymbol{LJ}$.

Another direction could be studying the monadicity of the constructions we provided. If this was the case one could answer the question “Is every fibration with Lawvere comprehension (resp. comprehension category) a quotient of a free one?”. Consequently, one may also want investigate distributive laws that may occur between our completions and other free constructions for fibrations. Such distributive laws enable us to lift the completions to 2-categories with more structure than ours: if for example $\wedge_J$ and $\mathbf{ffp}$ satisfied a distributive law, then we would have automatically a completion for finite fibred products that preserves Jacobs comprehensions.

Bibliography

[1] Jean Bénabou. Fibered categories and the foundations of naive category theory. *The Journal of Symbolic Logic*, 50(1):10–37, 1985.

[2] Francis Borceux. *Handbook of categorical algebra: volume 1, Basic category theory*, volume 1. Cambridge University Press, 1994.

[3] G. Coraglia and J. Emmenegger. A 2-categorical analysis of context comprehension. *Theory and Applications of Categories*, 41(42):1476–1512, 2024.

[4] Thomas Ehrhard. A categorical semantics of constructions. In *Proceedings. Third Annual Symposium on Logic in Computer Science*, pages 264–273, 1988.

[5] Alexandre Grothendieck and Michèle Raynaud. Revêtements étales et groupe fondamental (SGA 1). *arXiv:math/0206203*, 2002.

[6] J. M. E. Hyland, P. T. Johnstone, and A. M. Pitts. Tripos theory. *Mathematical Proceedings of the Cambridge Philosophical Society*, 88(2):205–232, 1980.

[7] Bart Jacobs. Comprehension categories and the semantics of type dependency. *Theoretical Computer Science*, 107(2):169–207, 1993.

[8] Bart Jacobs. *Categorical logic and type theory.* Elsevier, 1999.

[9] Niles Johnson and Donald Yau. *2-dimensional categories.* Oxford University Press, USA, 2021.

[10] G Max Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, pages 75–103. Springer, 2006.

[11] F. William Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pages 1–14. Amer. Math. Soc., Providence, R.I., 1970.

[12] Paul-André Melliès and Nicolas Rolland. Comprehension and quotient structures in the language of 2-categories. *arXiv preprint arXiv:2005.10015*, 2020.

[13] Emily Riehl. *Category theory in context.* Courier Dover Publications, 2017.

[14] Thomas Streicher. Fibered Categories à la Jean Bénabou. manuscript, 2020.